On Optimal Control of Quasi-Linear Elliptic Equation with Variable $p(x)$-Laplacian

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***Abstract*—We consider an optimal control problem for quasi-linear elliptic equation containing the p-Laplacian with variable exponent $p = p(x)$. The exponent $p(x)$ are used as the controls in $L^1(\Omega)$. The optimal control problem is to minimize the discrepancy between a given distribution $y_d \in L^\alpha(\Omega)$ and the current system state $y \in W_0^{1,p(\cdot)}(\Omega)$ by choosing an appropriate exponent $p(x)$. Basing on the perturbation theory of extremal problems, we study the existence of optimal pairs and propose the ways for relaxation of the original optimization problem.**

***Index Terms*—optimal control, $p(x)$-Laplacian, Lavrientiev phenomenon, existence theorem.**

I. Setting of Optimal Control Problem

Let Ω be a bounded open connected subset of $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega$. Let $p : \Omega \to \mathbb{R}$ be a measured real-valued scalar function such that $1 < \alpha \leq p(x) \leq \beta < +\infty$ for almost all $x \in \Omega$. Let $f \in L^\infty(\Omega)^N$, $p_d \in L^2(\Omega)$, and $y_d \in L^\alpha(\Omega)$ be given distributions. We consider the following optimal control problem (OCP):

$$\text{Minimize} \quad \left\{ J(p,y) = \int_\Omega |y(x) - y_d(x)|^\alpha \, dx \right.$$
$$\left. + \gamma \int_\Omega |\nabla y(x)|^{p(x)} \, dx + \int_\Omega |p(x) - p_d(x)|^2 \, dx \right\} \tag{1}$$

subject to the constraints

$$- \operatorname{div} \left(|\nabla y|^{p(x)-2} \nabla y \right) = - \operatorname{div} f, \quad x \in \Omega, \tag{2}$$

$$y = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

$$p \in \mathbb{P}_{ad} = \left\{ p \in L^2(\Omega) \; : \; 1 < \alpha \leq p(x) \leq \beta, \quad \text{a.e. in} \quad \Omega \right\}, \tag{4}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$.

To the best of the authors' knowledge, the existence of solutions for the optimal control problem (1)–(4) remains an open question. Only very few articles deal with distributed or boundary optimal control problems for the systems of similar type (see, for instance, [2], [6], [8] and the references therein). There are several reasons for this:

- it is unknown whether the set of admissible solutions to the problem (1)–(4) is weakly closed in the corresponding functional space;
- we have no a priori estimates for the weak solutions (in the sense of Minty) to the boundary value problem (2)–(3) [9], [8];
- the asymptotic behaviour of a minimizing sequence to the cost functional (1) is unclear in general;
- the optimal control problem (1)–(4) is ill-posed and requires some relaxation (see, for instance, [1], [2], [5], [7], [10]).

To see these and other characteristic features of the optimization problem (1)–(4) more clearly, we introduce the well-known notions of solutions for nonlinear elliptic problems with variable exponent and discuss how the equation (2) can be interpreted.

To begin with, we note that if the exponent p is constant, then the Dirichlet boundary value problem (2)–(3) is well-posed in the classical Sobolev space $W_0^{1,p}(\Omega)$. For the variable measurable exponent, we look for the solution of this problem in the Sobolev-Orlicz space

$$W_0^{1,p(\cdot)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(x)} \, dx < +\infty \right\} \tag{5}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)^N}.$$

Here, $L^{p(\cdot)}(\Omega)^N$ stands for the set of all measurable vector-valued functions $f : \Omega \to \mathbb{R}^N$ such that

$$\rho_p(f) := \int_\Omega |f(x)|^{p(x)} \, dx < +\infty, \tag{6}$$

and $L^{p(\cdot)}(\Omega)^N$ is endowed with the so-called Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf \left\{ \lambda > 0 \; : \; \rho_p(\lambda^{-1} f) \leq 1 \right\}.$$

It is well-known that, unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega)$. Hence, with variable exponent $p = p(x)$ ($1 < \alpha \leq p \leq \beta$) it can be associated another Sobolev space,

$$H = H_0^{1,p(\cdot)}(\Omega) \;\; \text{as the closure of the set} \;\; C_0^\infty(\Omega) \;\; \text{in} \; W_0^{1,p(\cdot)}(\Omega)\text{-norm}.$$

Since the identity $W = H$ is not always valid, it makes sense to say that an exponent $p(x)$ is regular if $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.

II. Preliminaries

Definition 1. *A function $y \in W_0^{1,p(\cdot)}(\Omega)$ is said to be a weak solution to the boundary value problem* (2)–(3)*, if*

$$\int_\Omega |\nabla y|^{p(x)-2} \left(\nabla y, \nabla \varphi \right) \, dx = \int_\Omega (f, \nabla \varphi) \, dx \; \forall \, \varphi \in C_0^\infty(\Omega). \tag{7}$$

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Here, $(\cdot,\cdot)$ *stands for the scalar product in* $\mathbb{R}^N$.

Since we can lose the density of the set $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ for some (irregular) variable exponents $p(x)$, it follows that a weak solution to the problem (2)–(3) is not unique, in general. Moreover, the set of all weak solutions in not necessary convex in spite of the fact that the operator $\mathcal{A} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, given by the equality

$$(\mathcal{A}u, v) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u, \nabla\varphi \right) dx, \quad \forall\, v \in W_0^{1,p(\cdot)}(\Omega), \tag{8}$$

is strictly monotone. The question as to whether this set is weakly closed in $W_0^{1,p(\cdot)}(\Omega)$ remains open. At the same time, the following result is well-known [13].

Theorem 1. *If the domain* $\Omega \subset \mathbb{R}^N$ *is sufficiently smooth and the constant* β *in* (4) *is such that*

$$\beta < \frac{\alpha(N-1)}{N-1-\alpha} \ \text{ for } \alpha < N-1, \ \text{ and } \quad \beta < +\infty, \ \text{ for } \alpha \geq N-1,$$

then the Dirichlet problem (2)–(3) *has a weak solution* $y \in W_0^{1,p(\cdot)}(\Omega)$ *satisfying the energy inequality*

$$\int_\Omega |\nabla y|^{p(x)}\, dx \leq \int_\Omega (f, \nabla y)_{\mathbb{R}^N}\, dx \tag{9}$$

The main idea of the proof of Theorem 1 is based on the fact that some weak solutions to the Dirichlet problem (2)–(3) can be attained through C^1-regularization of the exponent $p = p(x)$ or through some approximation of operator $\mathcal{A}$ using its perturbation by $\varepsilon\Delta_\beta$-Laplacian. Here, by attainability of a weak solution $y \in W_0^{1,p(\cdot)}(\Omega)$, we mean the existence of a sequence $\{y_\varepsilon\}_{\varepsilon>0}$, where y_ε are the solutions of 'more regular' boundary value problems, such that $y_\varepsilon \to y$ in some appropriate topology as ε tends to zero. However, because of the fact that the energy inequality (9) can be strict for some irregular variable exponents $p(x)$, it is unknown whether each weak solution to the Dirichlet problem (2)–(3) can be attained in such way.

Let $p(x)$ be an irregular exponent and let V be an arbitrary intermediate space between H and W, i.e. $H \subseteq V \subseteq W$.

Definition 2. *A function* $y \in V$ *is a* V*-solution of the problem* (2)–(3)*, or its variational solution, if the integral identity* (7) *holds for any test function* $\varphi \in V$.

Using the strict monotonicity of the nonlinear operator $\mathcal{A} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ (see (8)), it is easy to show that a V-solution exists and it is unique. Moreover, since in the case of V-solutions, the test function φ in (7) can be taken equal to the solution, it leads us to the energy equality

$$\int_\Omega |\nabla y|^{p(x)}\, dx = \int_\Omega (f, \nabla y)_{\mathbb{R}^N}\ dx. \tag{10}$$

Theorem 2. *Let* V *be an arbitrary intermediate space between* H *and* W*. Then for any* $f \in L^\infty(\Omega)^N$ *there exists a unique* V*-solution to the boundary value problem* (2)–(3) *and it satisfies the energy equality* (10).

The converse statement is also true.

Proposition 1. *A weak solution in the sense of Definition 1 is variational if and only if the energy equality* (10) *holds.*

Indeed, in this case we can take V as the smallest closed subspace containing $C_0^\infty(\Omega)$ and the solution itself. For $V = H$, we speak of H-solutions.

Another definition of a weak solution to (2)–(3) can be stated as follows.

Definition 3. *A function* $y \in W_0^{1,p(\cdot)}(\Omega)$ *is said to be a weak solution in the sense of Minty to the boundary value problem* (2)–(3)*, if the integral inequality*

$$\int_\Omega |\nabla\varphi|^{p(x)-2} (\nabla\varphi, \nabla\varphi - \nabla y)\, dx \geq \int_\Omega (f, \nabla\varphi - \nabla y)\, dx \tag{11}$$

holds true for all $\varphi \in C_0^\infty(\Omega)$.

It follows from this definition that the set of weak solutions in the sense of Minty is convex and closed. However, the relations between Definitions 1 and 2 are very intricate for a general exponent $p(\cdot) \in \mathbb{P}_{ad}$. At least we can not assert that each of the Minty's weak solutions satisfies the integral identity (7) or vise versa. In the last section we describe the case where the three concepts of the weak solutions coincide.

As a result, the variational formulation of the optimal control problem (1)–(4) can be stated in different forms and this depends on the choice of the corresponding set of solutions. In view of this, we indicate the following sets of admissible pairs to the problem (1)–(4):

$$\Xi_w = \left\{ \begin{array}{c} (p, y) \in \mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega), \\ y \text{ and } p \text{ are related by integral} \\ \text{identity (7) for all } \varphi \in C_0^\infty(\Omega) \end{array} \right\}, \tag{12}$$

$$\Xi_V = \left\{ \begin{array}{c} (p, y) \in \mathbb{P}_{ad} \times V, \\ \exists \text{ an intermediate space } V \text{ such that} \\ H_0^{1,p(\cdot)}(\Omega) \subseteq V \subseteq W_0^{1,p(\cdot)}(\Omega) \\ \text{and } y \text{ and } p \text{ are related by integral} \\ \text{identity (7) for all } \varphi \in V \end{array} \right\}, \tag{13}$$

$$\Xi_M = \left\{ \begin{array}{c} (p, y) \in \mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega), \\ y \text{ and } p \text{ are related by Minty} \\ \text{inequality (11)} \\ \text{for all } \varphi \in C_0^\infty(\Omega) \end{array} \right\} \tag{14}$$

However, because of the Lavrentieff effect, it may happen that the corresponding minimization problems

$$\left\langle \inf_{(p,y)\in\Xi_w} J(p, y) \right\rangle, \ \left\langle \inf_{(p,y)\in\Xi_V} J(p, y) \right\rangle, \quad \text{and } \left\langle \inf_{(p,y)\in\Xi_M} J(p, y) \right\rangle \tag{15}$$

are essentially different, in general. In particular, it means that optimal pairs to the problems (15) can be different as well.

Thus, the main question we are going to answer in this paper is about solvability of optimal control problem (1)–(4) with respect to the different choice of the set of admissible solutions. To the best knowledge of the authors, the existence of optimal pairs to the problems (15) has not been studied in the literature.

A. On Orlicz and Sobolev – Orlicz Spaces

We note that class $L^{p(\cdot)}(\Omega)^N$ is a reflexive separable Banach space with respect to both the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1} f) \leq 1\right\}$$

and the Orlicz norm

$$\|f\|_{O^{p(\cdot)}(\Omega)^N} = \sup\left\{ \int_\Omega (f, g)\ dx \ : \ \int_\Omega |g|^{p'(x)}\, dx \leq 1 \right\},$$

where $p'(x) = \frac{p(x)}{p(x)-1}$ is the conjugate exponent.

In what follows, we make use of the following well-known results [13], [8]:

- If $f \in L^{p(\cdot)}(\Omega)^N$ and $g \in L^{p'(\cdot)}(\Omega)^N$, then $(f,g) \in L^1(\Omega)$ and
$$\int_\Omega (f,g)\, dx \le 2\|f\|_{L^{p(\cdot)}(\Omega)^N}\|g\|_{L^{p'(\cdot)}(\Omega)^N}; \qquad (16)$$
- The following estimates
$$\|f\|_{L^\alpha(\Omega)^N} \le (1+|\Omega|)^{1/\alpha}\|f\|_{L^{p(\cdot)}(\Omega)^N}, \qquad (17)$$
$$\forall f \in L^p(\cdot)(\Omega)^N, \qquad (18)$$
$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le (1+|\Omega|)^{1/\beta'}\|f\|_{L^\beta(\Omega)^N}, \qquad (19)$$
$$\beta' = \frac{\beta}{\beta-1}, \quad \forall f \in L^\beta(\Omega)^N. \qquad (20)$$
are valid;
- If a sequence $\{f_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p(\cdot)}(\Omega)^N$ and $f_k \rightharpoonup f$ in $L^\alpha(\Omega)^N$ as $k\to\infty$, then $f \in L^{p(\cdot)}(\Omega)^N$ and $f_k \rightharpoonup f$ in $L^{p(\cdot)}(\Omega)^N$, i.e.
$$\lim_{k\to\infty}\int_\Omega (f_k,\varphi)\, dx = \int_\Omega (f_k,\varphi)\, dx, \quad \forall \varphi \in L^{p'(\cdot)}(\Omega)^N;$$
- Let $f, f_k \in L^{p(\cdot)}(\Omega)^N$ for $k = 1, 2, \dots$ Then the following statements are equivalent to each other:
$$(i)\ \lim_{k\to\infty}\|f_k - f\|_{L^{p(\cdot)}(\Omega)^N} = 0;$$
$$(ii)\ \lim_{k\to\infty}\int_\Omega |f_k - f|^{p(x)}\, dx = 0;$$
$$(iii)\ f_k \to f \ \text{ in } \ \Omega \ \text{ in measure and}$$
$$\lim_{k\to\infty}\int_\Omega |f_k|^{p(x)}\, dx = \int_\Omega |f|^{p(x)}\, dx;$$
- Let $p(\cdot) \in \mathbb{P}_{ad}$ and $y(\cdot) \in W_0^{1,p(x)}(\Omega)$ be a given distributions. Let $F = F(\varphi)$, where
$$F(\varphi) = |\nabla\varphi|^{p(x)-2}(\nabla\varphi, \nabla\varphi - \nabla y).$$
Then $\varphi \overset{\mathcal{F}:}{\mapsto} F(\varphi)$ is the mapping $W_0^{1,p(x)}(\Omega) \mapsto L^1(\Omega)$;
- $\mathcal{F} : W_0^{1,p(x)}(\Omega) \mapsto L^1(\Omega)$ is a continuous mapping.

III. Some Auxiliary Results and Properties of the Sets of Admissible Solutions

The sets of admissible solutions Ξ_w, Ξ_V, and Ξ_M, which are defined in (12)–(14), possess drastically different properties in general.

We begin this section with the case when the sets Ξ_w, Ξ_V, and Ξ_M describe the same collection of admissible pairs to the OCP (1)–(4).

Proposition 2. *Assume that the set of admissible controls $\mathbb{P}_{ad}$ is specified as follows: $p \in \mathbb{P}_{ad}$ if and only if the following conditions*
$$|p(x)-p(y)| \le \omega(|x-y|),\ \forall x, y, \in \Omega,\ |x-y| \le 1/2,$$
$$\omega(t) = k_0/\ln(|t|^{-1}),\ 1 < \alpha \le p(x) \le \beta \text{ in } \Omega \qquad (21)$$
hold true with a given constant $k_0 > 0$. Then the sets Ξ_w, Ξ_V, and Ξ_M coincide.

Proof. First of all, we note that if $p = p(x)$ is an admissible exponent, then $p = p(x)$ is a continuous function in $\overline{\Omega}$ with the same logarithmic modulus of continuity $\omega(t) = \frac{k_0}{\ln(1/t)}$. As a result we have: the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ for each $p \in \mathbb{P}_{ad}$.

We divide the proof into three steps. Step 1. Let us show that $\Xi_w \subseteq \Xi_V$. Let $(p,y) \in \Xi_w$ be an arbitrary pair. It is worth to notice that such choice is always possible because Ξ_w is a nonempty set. Indeed, if we set $p(x) = \beta$, then the boundary value problem
$$-\operatorname{div}\left(|\nabla y|^{\beta-2}\nabla y\right) = -\operatorname{div} f, \quad x \in \Omega, \qquad (22)$$
$$y = 0 \ \text{ on } \ \partial\Omega \qquad (23)$$
is well-posed and it admits a unique weak solution $y_\beta \in W_0^{1,\beta}(\Omega)$ satisfying the integral identity (7) for all $\varphi \in C_0^\infty(\Omega)$. Hence, $(\beta, y_\beta) \in \Xi_w$ and $\Xi_w \ne \emptyset$ follows.

By definition of the set Ξ_w and the arguments of the density, the validity of the integral identity (7), which is written down for the chosen pair (p,y), can be extended to the test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. Then, putting $\varphi = y$ in (7), we immediately arrive at the energy equality (10). Hence, $(p,y) \in \Xi_V$ and, therefore, $\Xi_w \subseteq \Xi_V$.

Step 2. At this step we show that $\Xi_V \subseteq \Xi_M$. Let $(\widehat{p}, \widehat{y}) \in \Xi_V$ be an arbitrary pair. Let V be the smallest closed subspace of $W_0^{1,\widehat{p}(\cdot)}(\Omega)$ containing $C_0^\infty(\Omega)$ and the solution $\widehat{y}$ itself. By density of $C_0^\infty(\Omega)$ in $W_0^{1,\widehat{p}(\cdot)}(\Omega)$, it follows that $V = W_0^{1,\widehat{p}(\cdot)}(\Omega)$. As a result, (13) implies that
$$\int_\Omega |\nabla\widehat{y}|^{\widehat{p}(x)-2}(\nabla\widehat{y}, \nabla\varphi)\, dx = \int_\Omega (f, \nabla\varphi)\, dx, \qquad (24)$$
for all $\varphi \in W_0^{1,\widehat{p}(\cdot)}(\Omega)$. Using the strict monotonicity of operator $\mathcal{A} : W_0^{1,\widehat{p}(\cdot)}(\Omega) \to \left(W_0^{1,\widehat{p}(\cdot)}(\Omega)\right)^*$, given by the equality (8), we have
$$\begin{aligned} 0 &\le \int_\Omega \left(|\nabla v|^{\widehat{p}(x)-2}\nabla v - |\nabla\widehat{y}|^{\widehat{p}(x)-2}\nabla\widehat{y}, \nabla v - \nabla\widehat{y}\right) dx \\ &= \int_\Omega |\nabla v|^{\widehat{p}(x)-2}(\nabla v, \nabla v - \nabla\widehat{y})\, dx \\ &\quad - \int_\Omega |\nabla\widehat{y}|^{\widehat{p}(x)-2}(\nabla\widehat{y}, \nabla v - \nabla\widehat{y})\, dx \\ &\overset{\text{by (24)}}{=} \int_\Omega |\nabla v|^{\widehat{p}(x)-2}(\nabla v, \nabla v - \nabla\widehat{y})\, dx \\ &\quad - \int_\Omega (f, \nabla v - \nabla\widehat{y})\, dx, \end{aligned} \qquad (25)$$
where $\varphi = v - \widehat{y}$ and v is an arbitrary element of $W_0^{1,\widehat{p}(\cdot)}(\Omega)$. Hence,
$$\int_\Omega |\nabla v|^{\widehat{p}(x)-2}(\nabla v, \nabla v - \nabla\widehat{y})\, dx \ge \int_\Omega (f, \nabla v - \nabla\widehat{y})\, dx,$$
$\forall v \in W_0^{1,\widehat{p}(\cdot)}(\Omega)$ and we arrive at the Minty relation (11). Thus, $(\widehat{p}, \widehat{y}) \in \Xi_M$.

Step 3. It remains to show that $\Xi_M \subseteq \Xi_w$. Let $(\widetilde{p}, \widetilde{y}) \in \Xi_M$ be a fixed pair. We can apply the so-called Minty trick. Namely, we can take any $\varphi \in W_0^{1,\widehat{p}(\cdot)}(\Omega)$ as a test function in the Minty inequality
$$\int_\Omega |\nabla\varphi|^{\widetilde{p}(x)-2}(\nabla\varphi, \nabla\varphi - \nabla\widetilde{y})\, dx \ge \int_\Omega (f, \nabla\varphi - \nabla\widetilde{y})\, dx, \qquad (26)$$
and, after taking $\varphi = \widetilde{y} \pm v$ with $v \in C_0^\infty(\Omega)$ and $t > 0$ in (26), we can pass to the limit in this relation as $t \to 0$. This

yields

$$\pm \int_\Omega |\nabla \widetilde{y} \pm t\nabla v|^{\widetilde{p}(x)-2} (\nabla \widetilde{y} \pm t\nabla v, \nabla v)\, dx \geq \pm \int_\Omega (f, \nabla v)\, dx, \quad \forall\, t > 0 \quad (27)$$

and, therefore, after the limit passage as $t \to 0$, we finally obtain

$$\int_\Omega |\nabla \widetilde{y}|^{\widetilde{p}(x)-2} (\nabla \widetilde{y}, \nabla v)\, dx = \int_\Omega (f, \nabla v)\, dx, \quad \forall\, v \in C_0^\infty(\Omega).$$

Thus, $(\widetilde{p}, \widetilde{y}) \in \Xi_w$, and this concludes the proof.

As follows from this result, the sets Ξ_w, Ξ_V, and Ξ_M coincide under rather restrictive assumptions on the class of admissible exponents which exclude the appearance of the Lavrentieff effect.

Definition 4. *We say that the Lavrentieff phenomenon is inherent in the OCP* (1)–(4) *if there is a gap between two constrained minimization problems*

$$\left\langle \inf_{(p,y)\in \Xi_w} J(p,y) \right\rangle \quad \textit{and} \quad \left\langle \inf_{(p,y)\in \Xi_V} J(p,y) \right\rangle, \quad (28)$$

namely, there exist two pairs $(\widetilde{p}^0, \widetilde{y}^0) \in \Xi_w$ *and* $(\widehat{p}^0, \widehat{y}^0) \in \Xi_V$ *such that*

$$J\left(\widetilde{p}^0, \widetilde{y}^0\right) = \inf_{(p,y)\in \Xi_w} J(p,y) < \inf_{(p,y)\in \Xi_V} J(p,y) = J\left(\widehat{p}^0, \widehat{y}^0\right). \quad (29)$$

It is interesting to note that solutions of the problems (28), in general, are different in the sense of smoothness provided Lavrentieff effect takes a place. In particular, the optimal state $\widetilde{y}^0$ cannot belong to the space V and, hence, to $H_0^{1,p(\cdot)}(\Omega)$. In view of this, we can indicate a few characteristic properties of the sets Ξ_w, Ξ_V, and Ξ_M that will be useful later on.

Proposition 3. *For a given set of admissible controls* $\mathbb{P}_{ad}$ *the following statements hold:*

(i) the inclusions $\Xi_V \subset \Xi_w$ and $\Xi_V \subset \Xi_M$ are valid;
(ii) the sets Ξ_w, Ξ_V, and Ξ_M are nonempty;
(iii) Ξ_M is a convex set with respect to y;
(iv) the set Ξ_M is sequentially closed in the following sense: if $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi_M$ is a sequence of pairs such that $p_k(x) \to p(x)$ a.e. in Ω, $y_k \rightharpoonup y$ in $W_0^{1,\alpha}(\Omega)$ as $k \to \infty$, and $y \in W_0^{1,p(\cdot)}(\Omega)$, then $(p, y) \in \Xi_M$.

Proof. The validity of assertions (i)–(ii) can be easily established following the similar arguments as in the proof of Proposition 2 Let us show that Ξ_M is a convex set with respect to y. Let (p, y_1) and (p, y_2) be arbitrary pairs of Ξ_M. Then, for each $\lambda \in [0, 1]$ and $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \left(|\varphi|^{p(x)-2} \nabla\varphi, \nabla\varphi - (\lambda\nabla y_1 + (1-\lambda)\nabla y_2)\right) dx \\
&= \lambda \int_\Omega \left(|\varphi|^{p(x)-2} \nabla\varphi, \nabla\varphi - \nabla y_1\right) dx \\
&+ (1-\lambda) \int_\Omega \left(|\varphi|^{p(x)-2} \nabla\varphi, \nabla\varphi - \nabla y_2\right) dx \\
&\overset{\text{by (11)}}{\geq} \lambda \int_\Omega (f, \nabla\varphi - \nabla y_1)\, dx \\
&+ (1-\lambda) \int_\Omega (f, \nabla\varphi - \nabla y_2)\, dx \\
&= \int_\Omega (f, \nabla\varphi - (\lambda\nabla y_1 + (1-\lambda)\nabla y_2))\, dx.
\end{aligned}$$

Hence, $(p, \lambda y_1 + (1-\lambda)y_2) \in \Xi_M$, i.e. $\lambda y_1 + (1-\lambda)y_2$ is a weak solution in the sense of Minty of the boundary value problem (2)–(3).

It remains to show that Ξ_M is a closed set in the sense of convergence (iv). Let $\{(p_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence such that $(p_k, y_k) \in \Xi_M$ for all $k \in \mathbb{N}$ and properties (iv) hold with some distributions $p \in \mathbb{P}_{ad}$ and $y \in W_0^{1,p(\cdot)}(\Omega)$. Our aim is to show that $(p, y) \in \Xi_M$. With that in mind, we note that, in view of the estimate

$$\begin{aligned}
\int_E |\nabla y_k|\, dx &\leq |E|^{1/\alpha'} \left(\int_E |\nabla y_k|^\alpha\, dx\right)^{1/\alpha} \\
&\leq |E|^{1/\alpha'} \left(\int_\Omega |\nabla y_k|^\alpha\, dx\right)^{1/\alpha} \\
&\leq |E|^{1/\alpha'} \sup_{k\in\mathbb{N}} \|y_k\|_{W_0^{1,\alpha}(\Omega)} \leq C|E|^{1/\alpha'}
\end{aligned}$$

where $E \neq \emptyset$ is an arbitrary measurable subset of Ω, the sequence $\{\nabla y_k\}_{k\in\mathbb{N}}$ is equi-integrable, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_B |\nabla y_k|\, dx < \varepsilon$$

holds for all $k \in \mathbb{N}$ and all Borel sets $B \subset \Omega$ with $|B| < \delta$. Let $\varphi \in C_0^\infty(\Omega)$ be a test function. Then, setting up

$$\psi_k := |\nabla\varphi|^{p_k(x)-2} \nabla\varphi \quad \text{and} \quad \psi := |\nabla\varphi|^{p(x)-2} \nabla\varphi = \psi,$$

we see that $\psi_k \to \psi$ almost everywhere in Ω as $k \to \infty$ and

$$|\psi_k| = |\nabla\varphi|^{p_k(x)-1} \leq \|\varphi\|_{C(\overline{\Omega})}^{\beta-1} + 1 = C^* \quad \text{a.e. in } \Omega \ \ \forall\, k \in \mathbb{N}.$$

Therefore, $\psi_k \to \psi$ in $L^1(\Omega)$ by Lebesgue dominated theorem and

$$\begin{aligned}
&\int_\Omega \left(|\varphi|^{p_k(x)-2} \nabla\varphi, \nabla\varphi - \nabla y_k\right) dx \\
&= \int_\Omega (\psi, \nabla\varphi - \nabla y_k)\, dx \\
&+ \int_\Omega (\psi_k - \psi, \nabla\varphi - \nabla y_k)\, dx = I_1 + I_2,
\end{aligned}$$

where, by definition of the weak convergence in $L^\alpha(\Omega)^N$, we have

$$\lim_{k\to\infty} I_1 = \lim_{k\to\infty} \int_\Omega (\psi, \nabla\varphi - \nabla y_k)\, dx = \int_\Omega (\psi, \nabla\varphi - \nabla y)\, dx,$$

and

$$\begin{aligned}
I_2 &\leq \int_\Omega |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\, dx \\
&= \int_{\{|\nabla\varphi - \nabla y_k| \geq n\}} |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\, dx \\
&+ \int_{\{|\nabla\varphi - \nabla y_k| < n\}} |(\psi_k - \psi, \nabla\varphi - \nabla y_k)|\, dx \\
&\leq 2C^* \int_{\{|\nabla\varphi - \nabla y_k| \geq n\}} |\nabla\varphi - \nabla y_k|\, dx \\
&+ n \int_\Omega |\psi_k - \psi|\, dx
\end{aligned}$$

for any fixed $n \in \mathbb{R}_+$. Hence, for a given $\varepsilon > 0$ there exist indices $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for all $k > k_0$ and $n > n_0$ we have

$$\int_{\{|\nabla\varphi-\nabla y_k|\geq n\}} |\nabla\varphi - \nabla y_k| \, dx < \frac{\varepsilon}{4C^*}$$

by equi-integrability of $\{\nabla\varphi - \nabla y_k\}_{k\in\mathbb{N}}$,

$$\int_\Omega |\psi_k - \psi| \, dx < \frac{\varepsilon}{2n}$$

by the strong convergence $\psi_k \to \psi$ in $L^1(\Omega)$. Thus, $I_2 < \varepsilon$ for k large enough. Hence,

$$\lim_{k\to\infty} \int_\Omega \left(|\varphi|^{p_k(x)-2} \nabla\varphi, \nabla\varphi - \nabla y_k \right) dx = \int_\Omega \left(|\varphi|^{p(x)-2} \nabla\varphi, \nabla\varphi - \nabla y \right) dx.$$

Taking into account this fact and the weak convergence $\nabla y_k \rightharpoonup \nabla y$ in $L^\alpha(\Omega)^N$, we can pass to the limit in the integral inequality

$$\int_\Omega |\nabla\varphi|^{p_k(x)-2} (\nabla\varphi, \nabla\varphi - \nabla y_k) \, dx \geq \int_\Omega (f, \nabla\varphi - \nabla y_k) \, dx$$

as k tends to ∞. As a result, we get the following: the limit pair (p, y) belongs to the set $\mathbb{P}_{ad} \times W_0^{1,p(\cdot)}(\Omega)$ and satisfies the inequality (11). Hence, $(p, y) \in \Xi_M$. The proof is complete.

Proposition 4. *Assume that the set $\mathbb{P}_{ad}$ is given as in (4). Then*

$$\emptyset \neq \Xi_V \subset \Xi_M \quad \text{and} \quad \Xi_M \setminus \Xi_V \neq \emptyset. \tag{30}$$

At the end of this section, it is worth to notice that the relationship between Ξ_w, Ξ_V, and Ξ_M is very intricate problem, in general. In particular, it is unknown whether $\Xi_M \subseteq \Xi_w$ or vice versa. We even can not assert that if $(p, y) \in \Xi_M$ and this pair is related by the energy equality (10), then $(p, y) \in \Xi_V$. It also remains an open question about the standard topological properties of Ξ_w such as compactness, closedness and etc. Moreover, as was shown in [**?**, pp.107-112], the set Ξ_w is not convex, in general. Thus, in contrast to the standard situation, where non-uniqueness is possible in classical monotone problems, it usually comes from the missing of strict convexity of the corresponding operator, whereas the solution set is convex and closed. In the case of boundary value problem (2)–(3), the corresponding operator $\mathcal{A} : W_0^{1,\widehat{p}(\cdot)}(\Omega) \to \left(W_0^{1,\widehat{p}(\cdot)}(\Omega)\right)^*$ is strictly monotone. So, non-uniqueness and non-convexity are of completely different nature.

IV. On Solvability of Optimal Control Problem

We discuss the following relaxed form to the original optimal control problem

$$\text{Minimize} \quad \Big\{ J(p, y) = \int_\Omega |y(x) - y_d(x)|^\alpha \, dx + \gamma \int_\Omega |\nabla y(x)|^{p(x)} \, dx + \int_\Omega |p(x) - p_d(x)|^2 \, dx + \int_\Omega |Dp| \Big\} \tag{31}$$

subject to the constraints

$$-\operatorname{div}\left(|\nabla y|^{p(x)-2} \nabla y \right) = -\operatorname{div} f, \quad x \in \Omega, \tag{32}$$

$$y = 0 \quad \text{on} \quad \partial\Omega, \tag{33}$$

$$p \in \widehat{\mathbb{P}}_{ad} = \{ p \in BV(\Omega) \ : \ 1 < \alpha \leq p(x) \leq \beta \}, \tag{34}$$

where by $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\|p\|_{BV(\Omega)} = \|p\|_{L^1(\Omega)} + \int_\Omega |Dp| = \|p\|_{L^1(\Omega)} + \sup \left\{ \int_\Omega p \operatorname{div}\varphi \, dx \ : \ \varphi \in C_0^1(\Omega; \mathbb{R}^N), \ |\varphi(x)| \leq 1 \right\}$$

is finite. For motivation of BV-choice for the set of admissible controls, we refer to [3], [4], [11], [12].

We introduce the set of admissible solutions to the OCP (31)–(34) as follows:

$$\widehat{\Xi}_M = \left\{ \begin{array}{c} p \in \widehat{\mathbb{P}}_{ad}, \ y \in W_0^{1,p(\cdot)}(\Omega), \\ (p, y) \text{ is related by Minty inequality (10).} \end{array} \right\} \tag{35}$$

It is clear that $J(p, y) < +\infty$ for all $(p, y) \in \widehat{\Xi}_M$.

We recall that a sequence $\{p_k\}_{k=1}^\infty$ converges weakly* to p in $BV(\Omega)$ if and only if the two following conditions hold: $p_k \to p$ strongly in $L^1(\Omega)$ and $Dp_k \overset{*}{\rightharpoonup} Dp$ weakly* in the space of Radon measures $\mathcal{M}(\Omega; \mathbb{R}^N)$. Moreover, if $\{p_k\}_{k=1}^\infty \subset BV(\Omega)$ converges strongly to some p in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Dp_k| < +\infty$, then

$$\begin{array}{l} (i) \ p \in BV(\Omega) \ \text{ and } \ \displaystyle\int_\Omega |Dp| \leq \liminf_{k\to\infty} \int_\Omega |Dp_k|; \\ (ii) \ p_k \overset{*}{\rightharpoonup} p \ \text{ in } \ BV(\Omega). \end{array} \tag{36}$$

We say that $(p^0, y^0) \in BV(\Omega) \times W_0^{1,p^0(\cdot)}(\Omega)$ is a Minty optimal solution to the problem (31)–(34) if

$$(p^0, y^0) \in \widehat{\Xi}_M \quad \text{and} \quad J(p^0, y^0) = \inf_{(p,y)\in\widehat{\Xi}_M} J(p, y).$$

Our main intention in this section is to show that the set of Minty optimal pairs is nonempty for the problem (31)–(34). With that in mind we make use of the direct method of Calculus of Variations.

To begin with, we note that the set of admissible controls $\widehat{\mathbb{P}}_{ad}$, given by (34), is nonempty, convex, it has an empty topological interior, and satisfies the inclusion $\widehat{\mathbb{P}}_{ad} \subset L^2(\Omega)$. Hence, all results of the previous section, concerning topological and algebraic properties of the sets Ξ_w, Ξ_V, and Ξ_M, remain valid. Moreover, it is worth to emphasize that $\widehat{\mathbb{P}}_{ad}$ is a sequentially closed set with respect to the weak* convergence in $BV(\Omega)$.

In what follows, we make use of a couple of auxiliary results which are crucial for our further analysis.

Lemma 1. *Let $\{p_k\}_{k\in\mathbb{N}} \subset \widehat{\mathbb{P}}_{ad}$ and $\left\{ y_k \in W_0^{1,p_k(\cdot)}(\Omega) \right\}_{k\in\mathbb{N}}$ be sequences such that $p_k \overset{*}{\rightharpoonup} p$ in $BV(\Omega)$, and $y_k \rightharpoonup y$ in $W_0^{1,\alpha}(\Omega)$. Then*

$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)} \, dx \geq \int_\Omega |\nabla y|^{p(x)} \, dx. \tag{37}$$

Lemma 2. *Let $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_M$ be a sequence such that*

$$\sup_{k\in\mathbb{N}} \left[\|p_k\|_{BV(\Omega)} + \int_\Omega |\nabla y_k|^{p_k(x)} \, dx \right] < +\infty. \tag{38}$$

Then, there is a pair $(p,y) \in \widehat{\Xi}_M$ *such that, up to a subsequence,* $p_k \overset{*}{\rightharpoonup} p$ *in* $BV(\Omega)$, $p_k(x) \to p(x)$ *almost everywhere in* Ω, *and* $y_k \rightharpoonup y$ *in* $W_0^{1,\alpha}(\Omega)$.

Proof. Taking into account condition (38) and estimate (41), we see that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,\alpha}(\Omega)$. Hence, by compactness properties of $BV(\Omega) \times W_0^{1,\alpha}(\Omega)$, there exists a subsequence of the sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}}$, still denoted by the same indices, and functions $p \in BV(\Omega)$ and $y \in W_0^{1,\alpha}(\Omega)$ such that

$$p_k \overset{*}{\rightharpoonup} p \text{ in } BV(\Omega), \quad y_k \rightharpoonup y \;\text{ in }\; W_0^{1,\alpha}(\Omega), \tag{39}$$

$$\text{and }\; p_k(x) \to p(x) \text{ a.e. in }\; \Omega. \tag{40}$$

Then by Lemma 1, we have

$$+\infty > \sup_{k\in\mathbb{N}} \int_\Omega |\nabla y_k|^{p_k(x)}\,dx \geq \liminf_{k\to\infty} \int_\Omega |\nabla y_k|^{p_k(x)}\,dx$$
$$\geq \int_\Omega |\nabla y|^{p(x)}\,dx \overset{\text{by (41)}}{\geq} \|y\|^\alpha_{W_0^{1,p(\cdot)}(\Omega)} - 1.$$

This estimate implies that $y \in W_0^{1,p(\cdot)}(\Omega)$. To conclude the proof, i.e. to show that the limit pair is related by the Minty inequality (11), it remains to use the property (iv) of Proposition 3.

We are now in a position to give the existence result for optimal pairs to the OCP (31)–(34).

Theorem 2. *Let* $p_d \in L^2(\Omega)$, $y_d \in L^\alpha(\Omega)$, *and* $f \in L^\infty(\Omega)^N$ *be given functions. Then the optimal control problem* (31)–(34) *admits at least one solution in the sense of Minty.*

*Proof.*Since the set $\widehat{\Xi}_M$ is nonempty and the cost functional is bounded from below on $\widehat{\Xi}_M$, it follows that there exists a minimizing sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_M$ to the problem (31)–(34), i.e.

$$\inf_{(p,y)\in\widehat{\Xi}_M} J(p,y)$$
$$= \lim_{k\to\infty}\Bigg[\int_\Omega |y_k(x) - y_d(x)|^\alpha\,dx + \gamma\int_\Omega |\nabla y_k(x)|^{p_k(x)}\,dx$$
$$+ \int_\Omega |p_k(x) - p_d(x)|^2\,dx + \int_\Omega |Dp_k|\Bigg] < +\infty.$$

Hence, in view of estimate

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N} - 1 \leq \int_\Omega |f(x)|^{p(x)}\,dx \leq \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N} + 1, \tag{41}$$

which holds for every $f \in L^{p(\cdot)}(\Omega)^N$, and definition of the set $\widehat{\mathbb{P}}_{ad}$, the sequence $\{(p_k, y_k)\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)\times W_0^{1,\alpha}(\Omega)$. From Lemma 2 we deduce the existence of a subsequence, which is denoted in the same way, and a pair $(p^*, y^*) \in \widehat{\Xi}_M$ such that $p_k \overset{*}{\rightharpoonup} p^*$ in $BV(\Omega)$, $p_k(x) \to p^*(x)$ almost everywhere in Ω, and $y_k \rightharpoonup y^*$ in $W_0^{1,\alpha}(\Omega)$. From these convergences and Sobolev embedding theorem, by compactness of the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\alpha(\Omega)$, we infer that

$$\lim_{k\to\infty}\int_\Omega |y_k - y_d|^\alpha\,dx = \int_\Omega |y^* - y_d|^\alpha\,dx$$
$$\liminf_{k\to\infty}\int_\Omega |Dp_k| \geq \int_\Omega |Dp^*| \text{ by (36)},$$
$$\liminf_{k\to\infty}\int_\Omega |\nabla y_k(x)|^{p_k(x)}\,dx \geq \int_\Omega |\nabla y^*(x)|^{p^*(x)}\,dx,$$
$$\lim_{k\to\infty}\int_\Omega |p_k - p_d|^2\,dx = \int_\Omega |p^* - p_d|^2\,dx,$$

where the last assertion is a direct consequence of the strong convergence $p_k \to p^*$ in $L^1(\Omega)$ and boundedness of this sequence in $L^\infty(\Omega)$. So,

$$J(p^*, y^*) \leq \inf_{(p,y)\in\widehat{\Xi}_M} J(p,y)$$

and, consequently, (p^*, y^*) is a Minty optimal solution of the OCP (31)–(34).

References

[1] E. Casas, P.I. Kogut and G. Leugering, "Approximation of Optimal Control Problems in the Coefficient for the p-Laplace Equation. I. Convergence Result," *SIAM J. Control Optim.*, vol. 54, no. 3, pp.1406–1422, 2016.

[2] C. D'Apice, U. De Maio, and P. Kogut, "An Indirect Approach to the Existence of Quasi-Optimal Controls in Coefficients for Multi-Dimensional Thermistor Problem," *SIAM Journal on Control and Optimization*, 2018, to be published..

[3] C. D'Apice, U. De Maio, and O. Kogut, "Optimal Control Problems in Coefficients for Degenerate Equations of Monotone Type: Shape Stability and Attainability Problems," *SIAM Journal on Control and Optimization*, vol. 50, no. 3, pp.1174–1199, 2012.

[4] C. D'Apice, U. De Maio, P.I. Kogut, and R. Manzo, "Solvability of an Optimal Control Problem in Coefficients for Ill-Posed Elliptic Boundary Value Problems," *Electronic Journal of Differential Equations*, vol. 2014, no. 166, pp. 1–23, 2014.

[5] T. Durante, O. P. Kupenko and R. Manzo, "On attainability of optimal controls in coefficients for system of Hammerstein type with anisotropic p-Laplacian," *Ricerche di Matematica*, vol.66, no. 2, pp.259-292, 2017.

[6] D. Hömberg, C. Meyer, J. Rehberg, and W. Ring, "Optimal control for the thermistor problem," *SIAM Journal on Control and Optimization*, vol. 49, no. 5, pp. 3449-3481, 2010.

[7] P.I. Kogut and O. P. Kupenko "On Optimal Control Problem for an Ill-Posed Strongly Nonlinear Elliptic Equation with p-Laplace Operator and L^1-Type of Nonlinearity," *Discrete and Continuous Dynamic Systems*. Series B, 2018, to be published.

[8] P.I. Kogut and P.I. Tkachenko P. I., "On optimal control problem for nonlinear elliptic equation with variable $p(x)$-Laplacian," *Visnyk DNU. Series: Modelling*, Dnipro:DNU, vol. 24, no. 8, pp. 71–92, 2016.

[9] P. I. Kogut, R. Manzo and A. O. Putchenko, "On approximate solutions to the Neumann elliptic boundary value problem with non-linearity of exponential type," *Boundary Value Problems*, vol. 2016, no. 1, pp. 1–32, 2016.

[10] O. P. Kupenko and R. Manzo, "Approximation of an optimal control problem in the coefficients for variational inequality with anisotropic p-Laplacian," *Nonlinear Differential Equations and Applications*, 23:35, 2016 https://doi.org/10.1007/s00030-016-0387-9.

[11] O. P. Kupenko and R. Manzo, "On optimal controls in coefficients for ill-posed non-linear elliptic Dirichlet boundary value problems," *Discrete and Continuous Dynamic Systems*. Series B, 2018, to be published.

[12] O. P. Kupenko and R. Manzo, "Shape stability of optimal control problems in coefficients for coupled system of Hammerstein type," *Discrete and Continuous Dynamical Systems Journal*, Series B, vol. 20, no. 9, pp.2967-2992, 2015.

[13] V. V. Zhikov, "On variational problems and nonlinear elliptic equations with nonstandard growth conditions," *Journal of Mathematical Sciences*, vol. 173, no. 5, pp.463-570, 2011.