MATRIX-VALUED L^1-OPTIMAL CONTROLS IN THE COEFFICIENTS OF LINEAR ELLIPTIC PROBLEMS

PETER I. KOGUT* AND GÜNTER LEUGERING†

Abstract. We consider optimal control problems for linear degenerate elliptic equations with mixed boundary conditions. In particular, we take the matrix-valued coefficients $A(x)$ of such systems as controls in $L^1(\Omega;\mathbb{R}^{\frac{N(N+1)}{2}})$. One of the important features of the admissible controls is the fact that eigenvalues of the coefficient matrices may vanish in Ω. Equations of this type may exhibit non-uniqueness of weak solutions. Using the concept of convergence in variable spaces and following the direct method in the Calculus of variations, we establish the solvability of this optimal control problem in the class of weak admissible solutions.

Key words. Degenerate elliptic equations, control in coefficients, weighted Sobolev spaces, Lavrentieff phenomenon, direct method in the Calculus of Variations.

1. Introduction. Material optimization is an emerging field in the engineering context of design of advanced materials. The notions of advanced materials and meta-materials have recently evolved where desired, possibly counterintuitive, material properties are realized via systematic model-based optimization of material parameters. Often, such an inverse engineering approach leads to micro-structures, where mathematical optimization indicates singular behavior for the material parameters. This is particularly true for optical meta-materials in the context of cloaking. Following the exploration of electromagnetic cloaking on the base physics by Pendry [22] and Leonhard [23], a subject that has become a major branch of modern physics, the mathematical theory of cloaking has been established by Uhlmann, Lassas and coworkers (see the review article [26] and e.g. [25, 24]). The references given are by no means complete and rather exemplary in nature. The notion of transformational optics has been developed that allows, based on differential geometry, to construct Riemann metrics with special features, such that objects are "hidden". This is a question typically posed in the context of inverse problems: given a set of data on the boundary, as inputs and measured outputs, is it possible to reconstruct "objects" represented by, say, different material properties? If we can provide a situation, represented by a Riemann metric, where this question can be answered in the negative sense, we deal with non-identifiable objects. In the language of electro-magneto-dynamics, this means that objects can then be invisible. The studies of Uhlmann et.al.[24] strongly indicate that the corresponding Riemann metrics which are represented as coefficient matrices in elliptic systems, exhibit singular behavior along the object to be cloaked. Indeed, eigenvalues of that matrix may vanish or tend to infinity. Several other physical phenomena related to equilibrium of continuous media modeled by elliptic problems concern media which are "perfect" insulators or "perfect" conductors (see [11]) necessitate eigenvalues of the matrix A either to vanish somewhere or to be unbounded. These circumstances appearing in modern technologies are the major motivation for the paper.

The aim of this work is to study the existence of optimal controls in the matrix-valued coefficients associated with a linear elliptic equation and mixed boundary con-

*Department of Differential Equations, Dnipropetrovsk National University, Gagarin av., 72, 49010 Dnipropetrovsk, Ukraine (p.kogut@i.ua)

†Department Mathematik, Lehrstuhl für Angewandte Mathematik II Universität Erlangen-Nürnberg Martensstr.3 D-91058 Erlangen, Germany (Guenter.Leugering@am.uni-erlangen.de). Research funded by the DFG-cluster CE315: Engineering of Advanced Materials

dition. The controls are taken as the matrix of the coefficients in the main part of the elliptic operator. The most important feature of such controls is the fact that eigenvalues of the matrix A may either vanish on subsets with zero Lebesgue measure or be unbounded. In this case the precise answer for the question of existence or non-existence of optimal solutions heavily depends on the class of admissible controls chosen. The main questions are: what is the right setting of the optimal control problem in terms of the coefficient matrices? Her we will show that a certain class of L^1-controls in the matrix coefficients is appropriate in order to admit degeneracy at least on thin sets. In connection with this question we ask for the right class of admissible solutions to the above problem. Using the direct method in the Calculus of variations, we discuss the solvability of this optimal control problem in a class of weak admissible solutions. It should be emphasized that in contrast to [8], we do not make use of any relaxations of the degeneration for the original optimal control problem.

To be more specific, in this paper we deal with an optimal control problem in the coefficient-matrix for boundary value problems of the form

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) = f & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_D, \\ \frac{\partial y}{\partial \nu_A} = g & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ are given functions, the boundary of Ω consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$, and A is a measurable positive-semidefinite square symmetric matrix on a bounded open domain Ω in $\mathbb{R}^N$.

Even though numerous articles (see, for instance, [2, 7, 9, 12, 20, 21, 29] and references therein) are devoted to variational and non variational approaches to problems related to (1.1), only few deal with optimal control problems for degenerate partial differential equations (see for example [4, 5, 6, 15, 16]). This can be explained by several reasons. Firstly, boundary value problem (1.1) for locally integrable matrix-valued function A may exhibit non-uniqueness of weak solutions, as well as other surprising consequences. So, in general, the mapping $A \mapsto y(A)$ can be multivalued. One cannot expect that for every admissible data $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, and $A \in L^1(\Omega; \mathbb{R}^{N\times N})$, problem (1.1) admits a weak solution. Besides, for every admissible control function A, the weak solutions to the boundary value problem (1.1) belong to the corresponding weighted Sobolev space $W^{1,2}(\Omega, A\,dx)$. In addition, even if the elliptic equation is non-degenerate, i.e. admissible controls $A(x)$ are such that

$$\beta\|\xi\|^2_{\mathbb{R}^N} \geq \xi\cdot A(x)\xi \geq \alpha\|\xi\|^2_{\mathbb{R}^N} \quad \xi\in\mathbb{R}^N$$

with $\alpha > 0$, the optimal control problems in the coefficients may not have any solution (see for instance [19]).

In spite of the fact that the original boundary value problem is ill-posed in general, we show that the corresponding extremal problem has a practical sense and is indeed well-posed. This problem is, thus, yet another example for the difference between well-posedness for optimal control problems for systems with distributed parameters and partial differential equations. See the monograph by the authors [17] for a discussion and further examples.

The proof of existence of optimal matrix-valued controls requires a considerable set of preparations. In order to provide an orientation for the reader, we provide an outline of the article. In section 2 we introduce notations and provide a concept for admissible matrix-valued controls. As for the existence proof, we need to consider pairs (A, u) consisting of the matrix-valued control $A(\cdot)$ and the state u. In order to be able to consider minimizing sequences, we need a concept of convergence of sequences of matrix-valued functions A and the corresponding solutions u. It is amply clear that degeneracy of the matrices will make it necessary to introduce weighted Sobolev spaces. As the weights turn out to be exactly these matrices, we need a concept of Sobolev spaces with varying measures. This will be considered in section 3. In section 4, we establish properties of such sequences and consider sequences of pairs (A_n, u_n) and their weak convergence. This enables us in section 5 to formulate the optimal control problem (5.8). The final section 6 presents the existence result. Clearly, once existence of optimal solutions is guaranteed, one would like to know about optimality conditions. In this context, this is still a challenging task, and we don't know how to establish those in full generality at this moment.

2. Notation and Preliminaries. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary. We assume that the boundary of Ω consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$ with Dirichlet boundary conditions on Γ_D, and Neumann boundary conditions on Γ_N. Let the sets Γ_D and Γ_N have positive $(N-1)$-dimensional measures. Let χ_E be the characteristic function of a subset $E \subset \Omega$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

Let $C_0^\infty(\mathbb{R}^N; \Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N)\ :\ \varphi = 0 \text{ on } \Gamma_D\right\}$. We define the Banach space $W^{1,1}(\Omega; \Gamma_D)$ as the closure of $C_0^\infty(\mathbb{R}^N; \Gamma_D)$ in the classical Sobolev space $W^{1,1}(\Omega)$. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(E)$.

Symmetric matrices with degenerate eigenvalues. We denote by $\mathbb{S}^N := \mathbb{R}^{\frac{N(N+1)}{2}}$ the set of all symmetric matrices $\xi = [\xi_{ij}]_{i,j=1}^N$, $(\xi_{ij} = \xi_{ji})$. We suppose that $\mathbb{S}^N$ is endowed with the Euclidian scalar product $\xi \cdot \eta = \mathrm{tr}(\xi\,\eta) = \xi_{ij}\eta_{ij}$ and with the corresponding Euclidian norm $\|\xi\|_{\mathbb{S}^N} = (\xi \cdot \xi)^{1/2}$. Let

$$L^1(\Omega)^{\frac{N(N+1)}{2}} = L^1\big(\Omega; \mathbb{S}^N\big)$$

be the space of integrable functions whose values are symmetric matrices.

Let $\alpha \in \mathbb{R}$ be a fixed positive value. Let $\zeta_{ad} : \Omega \to [0, \alpha]$ be a given function satisfying the properties

$$\zeta_{ad} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \notin L^\infty(\Omega).$$

Let Ψ_* be a nonempty compact subset of $L^1(\Omega)$ such that for any $\zeta_* \in \Psi_*$ the following conditions hold true

$$\zeta_{ad}(x) < \zeta_*(x) \ \text{ a.e. in } \ \Omega, \tag{2.1}$$

$$\zeta_* : \Omega \to \mathbb{R}^1_+ \ \text{ is smooth function along the boundary } \ \partial\Omega, \tag{2.2}$$

$$\zeta_* = \alpha \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

Remark 2.1. *In this setting, degeneracy of the coefficient matrices is controlled by ζ_{ad} which can exhibit degenerate behavior on sets of Lebesgue measure zero. This is the case for the cloaking applications mentioned in the introduction, where degeneracy takes place along the boundary of a subset of Ω.*

By $\mathfrak{M}_\alpha^\beta(\Omega)$ we denote the set of all matrices $A(x) = [a_{i\,j}(x)] \in \mathbb{S}^N$ such that

$$A(x) \leq \beta(x) I \quad \text{a. e. in } \ \Omega, \tag{2.4}$$

$$\exists\, \zeta_* \in \Psi_* \text{ s.t. } \ \zeta_* I \leq A(x) \quad \text{a. e. in } \ \Omega. \tag{2.5}$$

Here $\beta \in L^1(\Omega)$ is a given function such that $\beta(x) > 0$ a.e. in Ω, I is the identity matrix in $\mathbb{R}^{N\times N}$, and (2.4)–(2.5) should be considered in the sense of quadratic forms. Therefore, (2.4)–(2.5) imply the following inequalities:

$$\text{if} \;\; A \in L^1(\Omega;\mathbb{S}^N), \;\text{then} \;\; \|A(x)\|_{L^1(\Omega;\mathbb{S}^N)} \le \|\beta\|_{L^1(\Omega)} < +\infty, \tag{2.6}$$

$$\zeta_*(x)\|\xi\|^2_{\mathbb{R}^N} \le (A(x)\xi,\xi)_{\mathbb{R}^N} \quad \text{a. e. in } \;\Omega, \;\forall\,\xi \in \mathbb{R}^N. \tag{2.7}$$

Remark 2.2. *For every measurable matrix-valued function $A : \Omega \to \mathbb{S}^N$ we can define the corresponding collection of its eigenvalues $\{\lambda_1^A,\dots,\lambda_N^A\}$ (in fact eigenvalue functions), where each $\lambda_k^A = \lambda_k^A(x)$ is counted with its multiplicity. Then, in view of the properties* (2.5) *of the class Ψ_* and the Rayleigh quotient, we have: $(\lambda_k^A)^{-1} \in L^1(\Omega)$ for all $k = 1,\dots,N$. It means that, in general, eigenvalues of matrices $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ cannot be strictly separated from zero on Ω (in the sense of almost everywhere) by a positive constant. Because of this, these matrices are sometime referred to as matrices with degenerate spectrum. In the sequel, properties* (2.1)–(2.7) *play a central role in definition of the class of admissible controls for the control object* (1.1).

To each matrix function $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ we will associate two weighted Sobolev spaces:

$$W_A(\Omega;\Gamma_D) = W(\Omega;\Gamma_D;A\,dx) \;\;\text{and}\;\; H_A(\Omega;\Gamma_D) = H(\Omega;\Gamma_D;A\,dx),$$

where $W_A(\Omega;\Gamma_D)$ is the set of functions $y \in W^{1,1}(\Omega;\Gamma_D)$ for which the norm

$$\|y\|_A = \Big(\int_\Omega \big(y^2 + (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}\,\big)\,dx\Big)^{1/2} \tag{2.8}$$

is finite, and $H_A(\Omega;\Gamma_D)$ is the closure of $C_0^\infty(\Omega;\Gamma_D)$ in $W_A(\Omega;\Gamma_D)$. Note that due to the inequality (2.7) and estimates

$$\int_\Omega |y|\,dx \le \Big(\int_\Omega |y|^2\,dx\Big)^{1/2}|\Omega|^{1/2} \le C\|y\|_A, \tag{2.9}$$

$$\begin{aligned}\int_\Omega \|\nabla y\|_{\mathbb{R}^N}\,dx &\le \Big(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\zeta_*\,dx\Big)^{1/2}\Big(\int_\Omega \zeta_*^{-1}\,dx\Big)^{1/2}\\ &\le C\Big(\int_\Omega (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}\,dx\Big)^{1/2} \le C\|y\|_A,\end{aligned} \tag{2.10}$$

the space $W_A(\Omega;\Gamma_D)$ is complete with respect to the norm $\|\cdot\|_A$. It is clear that $H_A(\Omega;\Gamma_D) \subset W_A(\Omega;\Gamma_D)$, and $W_A(\Omega;\Gamma_D)$, $H_A(\Omega;\Gamma_D)$ are Hilbert spaces. If the eigenvalues $\{\lambda_1^A,\dots,\lambda_N^A\}$ of $A : \Omega \to \mathbb{S}^N$ are bounded between two positive constants, then it is easy to verify that $W_A(\Omega;\Gamma_D) = H_A(\Omega;\Gamma_D)$. However, for a "typical" weight-matrix $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in $W_A(\Omega;\Gamma_D)$. Hence the identity $W_A(\Omega;\Gamma_D) = H_A(\Omega;\Gamma_D)$ is not always valid (for the corresponding examples in the case when $A(x) = \rho(x)I$, we refer to [10, 27]). This is an example of the so-called Lavrentieff gap-phenomenon. We remark that the classical Lavrentieff phenomenon is associated with the minimization problem

$$\inf J(W_0^{1,p}(\Omega)) := \inf_{v\in W_0^{1,p}(\Omega)} J(v),\; p \in [1,\infty],$$

where

$$-\infty \le \inf J(W_0^{1,1}(\Omega)) \le \inf J(W_0^{1,\infty}(\Omega)) < \infty,$$

but $\inf J(W_0^{1,1}(\Omega)) \neq \inf J(W_0^{1,\infty}(\Omega))$. See also [21], where various extensions to elliptic problems are discussed. As we will have to deal with minimizing sequences of admissible matrix-valued functions, we need to establish an appropriate concept of convergence. This concept will be based on weighted Sobolev spaces introduced above. As the minimizing sequences will then correspond to sequences of matrix-valued measures, we will need the concept of varying spaces (see also [17]). This will be the subject of the next section. In order to prepare the setting, we need some more definitions and results.

Weak Compactness Criterion in $L^1(\Omega;\mathbb{S}^N)$. Throughout the paper we will often use the concept of weak and strong convergence in $L^1(\Omega;\mathbb{S}^N)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a bounded sequence of matrices in $L^1(\Omega;\mathbb{S}^N)$. We recall that $\{A_n\}_{n\in\mathbb{N}}$ is called equi-integrable on Ω, if for any $\delta>0$ there is a $\tau=\tau(\delta)$ such that $\int_S \|A_n\|_{\mathbb{S}^N}\,dx<\delta$ for every measurable subset $S\subset\Omega$ of Lebesgue measure $|S|<\tau$. Then the following assertions are equivalent for $L^1(\Omega;\mathbb{S}^N)$-bounded sequences (Dunford-Pettis, [13]):

(i) a sequence $\{A_k\}_{k\in\mathbb{N}}$ is weakly convergent in $L^1(\Omega;\mathbb{S}^N)$;

(ii) the sequence $\{A_k\}_{k\in\mathbb{N}}$ is equi-integrable.

THEOREM 2.1 ([13]). *If a sequence $\{A_k\}_{k\in\mathbb{N}}\subset L^1(\Omega;\mathbb{S}^N)$ is equi-integrable and $A_k\to A$ almost everywhere in Ω then $A_k\to A$ in $L^1(\Omega;\mathbb{S}^N)$.*

Functions with Bounded Variation. Let $f:\Omega\to\mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$\int_\Omega |Df| = \sup\Big\{ \int_\Omega f \operatorname{div}\varphi\,dx \ : \ \varphi=(\varphi_1,\dots,\varphi_N)\in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)|\le 1 \text{ for } x\in\Omega\Big\},$$

where $\operatorname{div}\varphi=\sum_{i=1}^N \frac{\partial\varphi_i}{\partial x_i}$.

According to the Radon-Nikodym theorem, if $\int_\Omega|Df|<+\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f\in[L^1(\Omega)]^N$ and a measure D_sf, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N\lfloor\Omega$ restricted to Ω, such that

$$Df=\nabla f\mathcal{L}^N\lfloor\Omega + D_sf.$$

DEFINITION 2.2. *A function $f\in L^1(\Omega)$ is said to have a bounded variation in Ω if $\int_\Omega|Df|<+\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation.*

Under the norm $\|f\|_{BV(\Omega)}=\|f\|_{L^1(\Omega)}+\int_\Omega|Df|$, $BV(\Omega)$ is a Banach space. The following compactness result for BV-functions is well-known:

PROPOSITION 2.3. *[1, p.17] Uniformly bounded sets in BV-norm are relatively compact in $L^1(\Omega)$.*

For our further analysis, we need the following concept of waek convergence for BV-functions.

DEFINITION 2.4. *[1, p.17] A sequence $\{f_k\}_{k=1}^\infty\subset BV(\Omega)$ weakly converges to some $f\in BV(\Omega)$, and we write $f_k\rightharpoonup f$ iff the two following conditions hold: $f_k\to f$ strongly in $L^1(\Omega)$, and $Df_k\rightharpoonup Df$ weakly-$*$ in the space of Radon measures $M(\Omega;\mathbb{R}^N)$, i.e.*

$$\lim_{k\to\infty}\int_\Omega(\varphi,Df_k)_{\mathbb{R}^N}=\int_\Omega(\varphi,Df)_{\mathbb{R}^N}\qquad\forall\varphi\in C_0(\mathbb{R}^N)^N, \tag{2.11}$$

In the proposition below we give a compactness result related to this convergence, together with lower semicontinuity (see [1] and [14], Theorem 1.9):

PROPOSITION 2.5. *[1, p.18] Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying $\sup_{k\in\mathbb{N}}\int_\Omega |Df_k| < +\infty$. Then*

(i) $f \in BV(\Omega)$ *and* $\displaystyle\int_\Omega |Df| \le \liminf_{k\to\infty}\int_\Omega |Df_k|$;

(ii) $f_k \rightharpoonup f$ *in* $BV(\Omega)$.

3. $\mathbb{S}^N$-Valued Radon Measures and Weak Convergence in Variable L^2-Spaces. By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $M_+(\Omega)$. According to the Riesz theory, each Radon measure $\mu \in M_+(\Omega)$ can be interpreted as an element of the dual of the space $C_0(\Omega)$ of all continuous functions with compact support. Let $M(\Omega;\mathbb{S}^N)$ denote the space of all $\mathbb{S}^N$-valued Borel measures. Then $\mu = [\mu_{ij}] \in M(\Omega;\mathbb{S}^N) \Leftrightarrow \mu_{ij} \in C_0'(\Omega)$, $i,j = 1,\dots,N$.

Let μ and the sequence $\{\mu_k\}_{k\in\mathbb{N}}$ be matrix-valued Radon measures. We say that $\{\mu_k\}_{k\in\mathbb{N}}$ weakly-$*$ converges to μ in $M(\Omega;\mathbb{S}^N)$ if

$$\lim_{k\to\infty}\int_\Omega \varphi\cdot d\mu_k = \int_\Omega \varphi\cdot d\mu \quad \forall\,\varphi \in C_0(\Omega;\mathbb{S}^N).$$

A typical example of such measures is

$$d\mu_k = A_k(x)\,dx, \quad d\mu = A(x)\,dx, \tag{3.1}$$

$$\text{where}\quad A_k, A \in \mathfrak{M}_\alpha^\beta(\Omega)\cap L^1(\Omega;\mathbb{S}^N) \quad\text{and}\quad A_k \rightharpoonup A \quad\text{in}\quad L^1(\Omega;\mathbb{S}^N), \tag{3.2}$$

$$\text{or}\quad A_k, A \in \mathfrak{M}_\alpha^\beta(\Omega)\cap L^\infty(\Omega;\mathbb{S}^N) \quad\text{and}\quad A_k \overset{*}{\rightharpoonup} A \quad\text{in}\quad L^\infty(\Omega;\mathbb{S}^N). \tag{3.3}$$

As we will see later (see Lemma 4.3), the sets $\mathfrak{M}_\alpha^\beta(\Omega)\cap L^1(\Omega;\mathbb{S}^N)$ are sequentially closed with respect to strong convergence in $L^1(\Omega;\mathbb{S}^N)$.

In this section we suppose that the measures μ and $\{\mu_k\}_{k\in\mathbb{N}}$ are defined by (3.1)–(3.3) and $\mu_k \overset{*}{\rightharpoonup} \mu$ in $M(\Omega;\mathbb{S}^N)$. Further, we will use $L^2(\Omega, A\,dx)^N$ to denote the Hilbert space of measurable vector-valued functions $f \in \mathbb{R}^N$ on Ω such that

$$\|f\|_{L^2(\Omega,A\,dx)^N} = \Big(\int_\Omega (f, A(x)f)_{\mathbb{R}^N}\,dx\Big)^{1/2} < +\infty.$$

As follows from estimate (2.10) any vector-valued function of $L^2(\Omega, A\,dx)^N$ is Lebesgue integrable on Ω.

We say that a sequence $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty}\int_\Omega (v_k, A_k(x)v_k)_{\mathbb{R}^N}\;dx < +\infty.$$

DEFINITION 3.1. *A bounded sequence $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is weakly convergent to a function $v \in L^2(\Omega, A\,dx)$ in the variable space $L^2(\Omega, A_k\,dx)^N$ if*

$$\lim_{k\to\infty}\int_\Omega (\varphi, A_k(x)v_k)_{\mathbb{R}^N}\;dx = \int_\Omega (\varphi, A(x)v)_{\mathbb{R}^N}\;dx \quad \forall\,\varphi \in C_0^\infty(\Omega)^N. \tag{3.4}$$

The main property concerning the weak convergence in $L^p(\Omega, d\mu_\varepsilon)$ can be expressed as follows (see for comparison [28]):

Proposition 3.2. *If a sequence* $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ *is bounded and the condition* (3.2) *holds true, then it contains a weakly convergent subsequence in* $L^2(\Omega, A_k\,dx)^N$.

Proof. Having set $L_k(\varphi) = \int_\Omega (\varphi, A_k(x)v_k)_{\mathbb{R}^N}\,dx \;\forall\, \varphi \in C_0^\infty(\Omega)^N$ and making use the Hölder inequality, we get

$$\begin{aligned}
|L_k(\varphi)| &\le \left(\int_\Omega |A_k^{1/2} v_k|^2_{\mathbb{R}^N}\,dx\right)^{1/2} \left(\int_\Omega |A_k^{1/2}\varphi|^2_{\mathbb{R}^N}\,dx\right)^{1/2} \\
&= \left(\int_\Omega (v_k, A_k v_k)_{\mathbb{R}^N}\,dx\right)^{1/2} \left(\int_\Omega (\varphi, A_k\varphi)_{\mathbb{R}^N}\,dx\right)^{1/2} \\
&\le C\left(\int_\Omega (\varphi, A_k\varphi)_{\mathbb{R}^N}\,dx\right)^{1/2} \le C\left(\int_\Omega \beta(x)\|\varphi\|^2_{\mathbb{R}^N}\,dx\right)^{1/2} \\
&\le C\|\varphi\|_{C(\Omega;\mathbb{R}^N)}\|\beta\|^{1/2}_{L^1(\Omega)} \quad \forall\, k\in\mathbb{N}. \qquad (3.5)
\end{aligned}$$

Since the set $C_0^\infty(\Omega)^N$ is separable with respect to the norm $\|\cdot\|_{C(\Omega;\mathbb{R}^N)}$ and $\{L_k(\varphi)\}_{k\in\mathbb{N}}$ is a uniformly bounded sequence of linear functionals, it follows that there exists a subsequence of positive numbers $\{k_j\}_{j=1}^\infty$ for which the limit (in the sense of point-by-point convergence)

$$\lim_{j\to\infty} L_{k_j}(\varphi) = L(\varphi) \qquad (3.6)$$

is well defined for every $\varphi \in C_0^\infty(\Omega)^N$. As a result, using (3.2), we have

$$|L(\varphi)| \le C \lim_{j\to\infty}\left(\int_\Omega (\varphi, A_{k_j}\varphi)_{\mathbb{R}^N}\,dx\right)^{1/2} = C\left(\int_\Omega (\varphi, A\varphi)_{\mathbb{R}^N}\,dx\right)^{1/2}.$$

Hence, $L(\varphi)$ is a continuous functional on $L^2(\Omega, A\,dx)^N$ admitting the following representation $L(\varphi) = \int_\Omega (\varphi, A(x)v)_{\mathbb{R}^N}\,dx$, where v is some element of $L^2(\Omega, A\,dx)^N$. Thus, taking into account Definition 3.1, v can be taken as the weak limit of

$$\left\{v_{k_j} \in L^2(\Omega, A_{k_j}\,dx)^N\right\}_{j\in\mathbb{N}}.$$

□

The following property of weak convergence in $L^2(\Omega, A_k\,dx)^N$ shows that the variable L^2-norm is lower semicontinuous with respect to weak convergence.

Proposition 3.3. *If the sequence* $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ *converges weakly to* $v \in L^2(\Omega, A\,dx)^N$ *and the condition* (3.2) *holds true, then*

$$\liminf_{k\to\infty}\int_\Omega (v_k, A_k(x)v_k)_{\mathbb{R}^N}\,dx \ge \int_\Omega (v, A(x)v)_{\mathbb{R}^N}\,dx. \qquad (3.7)$$

Proof. Indeed, we have

$$\frac{1}{2}\int_\Omega (v_k, A_k v_k)_{\mathbb{R}^N}\ dx = \frac{1}{2}\int_\Omega |A_k^{1/2} v_k|^2_{\mathbb{R}^N}\, dx$$
$$\geq \int_\Omega (\varphi, A_k v_k)_{\mathbb{R}^N}\ dx - \frac{1}{2}\int_\Omega (\varphi, A_k\varphi)_{\mathbb{R}^N}\ dx \quad \forall \varphi \in C_0^\infty(\Omega)^N,$$
$$\frac{1}{2}\liminf_{k\to\infty}\int_\Omega (v_k, A_k v_k)_{\mathbb{R}^N}\ dx \geq \int_\Omega (\varphi, Av)_{\mathbb{R}^N}\ dx - \frac{1}{2}\int_\Omega (\varphi, A\varphi)_{\mathbb{R}^N}\ dx.$$

Since the last inequality is valid for all $\varphi \in C_0^\infty(\Omega)^N$ and $C_0^\infty(\Omega)^N$ is a dense subset of $L^2(\Omega, A\,dx)^N$, it holds also true for $\varphi \in L^2(\Omega, A\,dx)^N$. So, taking $\varphi = v$, we arrive at (3.7). □

Definition 3.4. *A sequence $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is said to be strongly convergent to a function $v \in L^2(\Omega, A\,dx)^N$ if*

$$\lim_{k\to\infty}\int_\Omega (b_k, A_k(x)v_k)_{\mathbb{R}^N}\ dx = \int_\Omega (b, A(x)v)_{\mathbb{R}^N}\ dx \tag{3.8}$$

whenever $b_k \rightharpoonup b$ in $L^2(\Omega, A_k\,dx)^N$ as $k \to \infty$.

Remark 3.1. *Note that in the case $A_k \equiv A$, Definitions 3.1–3.4 leads to the usual notion of convergence in the weighted Hilbert space $L^2(\Omega, A\,dx)^N$.*

The following property of strong convergence in the variable $L^2(\Omega, A_k\,dx)^N$-spaces will be used later on.

Proposition 3.5. *Assume the condition* (3.2) *holds true. Then the weak convergence of a sequence $\left\{v_k \in L^2(\Omega, A_k\,dx)^N\right\}_{k\in\mathbb{N}}$ to $v \in L^2(\Omega, A\,dx)^N$ and*

$$\lim_{k\to\infty}\int_\Omega (v_k, A_k(x)v_k)_{\mathbb{R}^N}\ dx = \int_\Omega (v, A(x)v)_{\mathbb{R}^N}\ dx \tag{3.9}$$

are equivalent to strong convergence of $\{v_k\}_{k\in\mathbb{N}}$ in $L^2(\Omega, A_k\,dx)^N$ to $v \in L^2(\Omega, A\,dx)^N$.

Proof. It is easy to verify that strong convergence implies weak convergence and (3.9). Indeed, we use $b_k = \varphi \in C_0^\infty(\Omega)^N$ in (3.8) and then substitute $b_k = v_k$.

In view of Proposition 3.2, we may assume that there exist two values ν_1 and ν_2 such that (up to subsequences)

$$\lim_{k\to\infty}\int_\Omega (b_k, A_k(x)v_k)_{\mathbb{R}^N}\ dx = \nu_1, \quad \lim_{k\to\infty}\int_\Omega (b_k, A_k(x)b_k)_{\mathbb{R}^N}\ dx = \nu_2.$$

Using lower semicontinuity (3.7) and (3.9), we obtain

$$\lim_{k\to\infty}\int_\Omega (v_k + tb_k, A_k(x)(v_k + tb_k))_{\mathbb{R}^N}\ dx$$
$$= \lim_{k\to\infty}\int_\Omega (v_k, A_k(x)v_k)_{\mathbb{R}^N}\ dx + 2t\nu_1 + t^2\nu_2$$
$$\geq \int_\Omega (v + tb, A(x)(v + tb))_{\mathbb{R}^N}\ dx = \int_\Omega (v, A(x)v)_{\mathbb{R}^N}\ dx$$
$$+ 2t\int_\Omega (b, A(x)v)_{\mathbb{R}^N}\ dx + t^2\int_\Omega (b, A(x)b)_{\mathbb{R}^N}\ dx.$$

From this we conclude that

$$2t\nu_1 + t^2\nu_2 \geq 2t\int_\Omega (b, A(x)v)_{\mathbb{R}^N}\ dx + t^2\int_\Omega (b, A(x)b)_{\mathbb{R}^N}\ dx \quad \forall\, t \in \mathbb{R}^1.$$

Hence, $\nu_1 = \displaystyle\int_\Omega (b, A(x)v)_{\mathbb{R}^N}\ dx$. Thereby the strong convergence of the sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is established. □

4. Auxiliary Results. To begin with, we provide the following property of the set $\Psi_* \subset L^1(\Omega)$ defined in (2.1)–(2.3).

Lemma 4.1. *Let $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ be any sequence in Ψ_*. Then there is an element $\zeta_* \in L^1(\Omega)$ such that, within a subsequence of $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$, we have*

$$\zeta_{*,n} \to \zeta_* \ \ in \ \ L^1(\Omega), \quad \zeta_* \in \Psi_*, \tag{4.1}$$

$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \ in \ \ L^1(\Omega), \quad and \tag{4.2}$$

$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \ \ in\ variable\ space \ \ L^2(\Omega, \zeta_{*,n}\, dx). \tag{4.3}$$

Proof. Strong convergence in (4.1) is a direct consequence of the compactness property of Ψ_*. Hence, $\zeta_* \in \Psi_*$ and we may assume that $\zeta_{*,n}^{-1} \to \zeta_*^{-1}$ almost everywhere in Ω. Since $\zeta_{*,n} \to \zeta_*$ in $L^1(\Omega)$ and $\zeta_*^{-1} \leq \zeta_{ad}^{-1} \in L^1(\Omega)$, it follows that the sequence $\left\{\zeta_{*,n}^{-1}\right\}_{n\in\mathbb{N}}$ is equi-integrable. As a result, (4.2) immediately follows from Lebesgue's Theorem (see Theorem 2.1). As for (4.3), we make use the following observation. For any $\varphi \in C_0^\infty(\Omega)$, we have

$$\zeta_n\, dx \stackrel{*}{\rightharpoonup} \zeta\, dx \ \text{ in } \ \mathcal{M}_+(\Omega),$$

$$\int_\Omega \zeta_n^{-1}\varphi\zeta_n\, dx = \int_\Omega \varphi\, dx = \int_\Omega \zeta^{-1}\varphi\zeta\, dx.$$

Hence, $\zeta_n^{-1} \rightharpoonup \zeta^{-1}$ in $L^2(\Omega_T, \zeta_n\, dx)$ (see [28]). Moreover, strong convergence in (4.2) implies the relation

$$\lim_{n\to\infty}\int_\Omega \zeta_n^{-2}\zeta_n\, dx = \lim_{n\to\infty}\int_\Omega \zeta_n^{-1}\, dx = \int_\Omega \zeta^{-2}\zeta\, dx.$$

Therefore, $\zeta_n^{-1} \to \zeta^{-1}$ strongly in $L^2(\Omega, \zeta_n\, dx)$ by the properties of strong convergence in variable spaces. The proof is complete. □

Remark 4.1. *Note that the main assertion of Lemma 4.1 can fail, if in definition of the set Ψ_*, instead of condition (2.1), we admit the following one*

$$0 < \zeta_*(x) \leq \alpha \ a.e.\ in \ \ \Omega, \quad \zeta_*^{-1} \in L^1(\Omega). \tag{4.4}$$

Indeed, let Ω be the open ball in $\mathbb{R}^N$ with the center at 0 and radius 1, let $1 < \delta < N$, and let $\zeta_*(x) := \alpha\|x\|_{\mathbb{R}^N}^\delta$. Then it is easy to see that $\zeta_* \in L^1(\Omega)$ and $0 < \zeta_*(x) \leq \alpha$ for every $x \in \Omega\setminus 0$. Since $\zeta_*^{-1} = \alpha^{-1}\|x\|_{\mathbb{R}^N}^{-\delta}$ and $\delta \in (1, N)$, we have $\zeta_*^{-1} \in L^1(\Omega)$ and $\zeta_*^{-1} \notin L^\infty(\Omega)$. Moreover, ζ is smooth in $\overline{\Omega}\setminus 0$ and $\zeta_* = \alpha$ on $\partial\Omega$. This shows that the properties (2.2), (2.3), and (4.4) are satisfied.

Let us fix $x_0 \in \Omega$ with $\|x_0\|_{\mathbb{R}^N} = \frac{1}{2}$. We consider the following sequence $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ in $L^1(\Omega)$, where $\zeta_{*,n} = \zeta_*$ for $n \leq 2$ and

$$\zeta_{*,n}(x) = \begin{cases} \alpha\|x\|_{\mathbb{R}^N}^\delta & if \ \ \|x - x_0\|_{\mathbb{R}^N} \geq \frac{1}{n}, \\ \frac{\alpha}{n^N} & if \ \ \|x - x_0\|_{\mathbb{R}^N} < \frac{1}{n}. \end{cases} \qquad if \ \ n \geq 3.$$

Then each function $\zeta_{*,n}$ *satisfies the properties* (2.2), (2.3), *and* (4.4). *Indeed,* $\zeta_{*,n} \in L^1(\Omega)$ *and* $0 < \zeta_{*,n}(x) \le \alpha$ *for every* $x \in \Omega$. *Since*

$$\zeta_{*,n}^{-1}(x) = \begin{cases} \frac{1}{\alpha \|x\|_{\mathbb{R}^N}^{\delta}} & \text{if } \|x - x_0\|_{\mathbb{R}^N} \ge \frac{1}{n}, \\ \frac{n^N}{\alpha} & \text{if } \|x - x_0\|_{\mathbb{R}^N} < \frac{1}{n}. \end{cases} \quad \text{for all } \ n \ge 3,$$

it follows that $\zeta_{*,n}^{-1} \in L^1(\Omega)$ *and* $\zeta_{*,n}^{-1} \notin L^\infty(\Omega)$. *Moreover, the functions* $\zeta_{*,n}$ *are smooth near* $\partial\Omega$ *and* $\zeta_{*,n} = \alpha$ *on* $\partial\Omega$. *This shows that the properties* (2.2), (2.3), *and* (4.4) *are satisfied.*

It is clear that $\zeta_{*,n} \to \zeta_*$ *strongly in* $L^1(\Omega)$ *and pointwise a.e. in* Ω. *The problem is that the sequence* $\left\{\zeta_{*,n}^{-1}\right\}_{n\in\mathbb{N}}$ *does not converge to* ζ_*^{-1} *strongly in* $L^1(\Omega)$. *Indeed, it is the case when the sequence* $\left\{\zeta_{*,n}^{-1}\right\}_{n\in\mathbb{N}}$ *is not equi-integrable. As a result, we have*

$$\int_\Omega \left|\zeta_{*,n}^{-1} - \zeta_*^{-1}\right| dx = \int_{B(x_0, \frac{1}{n})} \left| \frac{n^N}{\alpha} - \frac{1}{\alpha \|x\|_{\mathbb{R}^N}^{\delta}} \right| dx \to \alpha^{-1}\omega_N \quad \text{as} \quad n \to \infty,$$

where $B(x_0, \frac{1}{n})$ *is an open ball with center at* x_0 *and radius* $\frac{1}{n}$, *while* ω_N *is the Lebesgue measure of the unit ball in* $\mathbb{R}^N$.

For our further analysis, we make use of the following concept.

Definition 4.2. *We say that a bounded sequence*

$$\left\{(A_n, u_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\right\}_{n\in\mathbb{N}} \tag{4.5}$$

w-converges to $(A, u) \in L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega)$ *as* $n \to \infty$ *(in symbols,* $(A_n, y_n) \xrightarrow{w} (A, y)$*) if*

$$A_n \to A \quad \text{ in } \ L^1(\Omega; \mathbb{S}^N), \tag{4.6}$$

$$u_n \rightharpoonup u \quad \text{ in } \ L^2(\Omega), \tag{4.7}$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in the variable space} \ \ L^2(\Omega, A_n\, dx)^N. \tag{4.8}$$

In particular, as follows from this definition, if $(A_n, u_n) \stackrel{w}{\to} (A, u)$, then

$$\lim_{n\to\infty} \int_\Omega A_n \eta\, dx = \int_\Omega A\eta\, dx \quad \forall\, \eta \in L^\infty(\Omega; \mathbb{S}^N), \tag{4.9}$$

$$\lim_{n\to\infty} \int_\Omega u_n \lambda\, dx = \int_\Omega u\lambda\, dx \quad \forall\, \lambda \in L^2(\Omega), \tag{4.10}$$

$$\lim_{n\to\infty} \int_\Omega (\xi, A_n \nabla u_n)_{\mathbb{R}^N}\ dx = \int_\Omega (\xi, A\nabla u)_{\mathbb{R}^N}\ dx \quad \forall\, \xi \in C_0^\infty(\Omega)^N. \tag{4.11}$$

In order to motivate this definition, we give the following result.

Lemma 4.3. *Let* $\left\{(A_n, u_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\right\}_{n\in\mathbb{N}}$ *be a sequence such that*

(i) the sequence $\left\{u_n \in W_{A_n}(\Omega; \Gamma_D)\right\}_{n\in\mathbb{N}}$*is bounded, i.e.*

$$\sup_{n\in\mathbb{N}} \int_\Omega \left(u_n^2 + (\nabla u_n, A_n \nabla u_n)\right) dx < +\infty; \tag{4.12}$$

(ii) $\left\{A_n\right\}_{n\in\mathbb{N}} \subset \mathfrak{M}_\alpha^\beta(\Omega)$ *and there exists a matrix-valued function* $A(x) \in \mathbb{S}^N$ *such that*

$$A_n \to A \quad \text{and} \quad A_n^{-1} \to A^{-1} \quad \text{in} \ \ L^1(\Omega; \mathbb{S}^N) \ \ \text{as} \ \ n \to \infty. \tag{4.13}$$

Then, $A \in \mathfrak{M}_\alpha^\beta(\Omega) \cap L^1(\Omega;\mathbb{S}^N)$ and within a subsequence the original sequence is w-convergent. Moreover, each w-limit pair (A,u) belongs to the space $L^1(\Omega;\mathbb{S}^N) \times W_A(\Omega;\Gamma_D)$.

Proof. We note that (4.12)–(4.13) and (2.9)–(2.10) immediately imply the boundedness of the original sequence in $L^1(\Omega;\mathbb{S}^N) \times W^{1,1}(\Omega;S)$. Moreover, due to (4.13), we have (see the suppositions (3.1)–(3.3) of Section 3):

$$d\mu_n := A_n\,dx \overset{*}{\rightharpoonup} A\,dx =: d\mu \ \text{ in } \ \mathcal{M}(\Omega;\mathbb{S}^N).$$

Thus, the compactness criterium for weak convergence in variable spaces (see Proposition 3.2) and (4.12) imply the existence of a pair $(u,v) \in L^2(\Omega)\times L^2(\Omega, A\,dx)^N$ such that, within a subsequence of $\{u_n\}_{n\in\mathbb{N}}$,

$$u_n \rightharpoonup u \ \text{ in } \ L^2(\Omega), \tag{4.14}$$

$$\nabla u_n \rightharpoonup v \ \text{ in the variable space } \ L^2(\Omega, A_n\,dx)^N. \tag{4.15}$$

Our aim is to show that $A \in \mathfrak{M}_\alpha^\beta(\Omega)$, $v = \nabla u$, and $u \in W_A(\Omega;\Gamma_D)$. It is clear that $A(x) \in \mathbb{S}^N$ and this matrix satisfies (2.4). Since $A_n \in \mathfrak{M}_\alpha^\beta(\Omega) \cap L^1(\Omega;\mathbb{S}^N)$ for all $n \in \mathbb{N}$, it follows that there is a sequence $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ in Ψ_* such that

$$\zeta_{*,n}(x)I \le A_n(x)I \le \beta(x)I \quad \text{a. e. in } \ \Omega, \quad \forall\, k \in \{1,\dots,N\}\,. \tag{4.16}$$

Then, by L^1-compactness of the set Ψ_*, there exists an element $\zeta_* \in \Psi_*$ such that $\zeta_{*,n} \to \zeta_*$ in $L^1(\Omega)$ as $n \to \infty$. Moreover, Lemma 4.1 implies strong convergence

$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \ \text{ in } \ L^1(\Omega), \tag{4.17}$$

and (2.1)–(2.3). Hence, passing to the limit in (4.16) as $n \to \infty$, we come to (2.5). Thus, $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ and the limit matrix $A(x) \in \mathbb{S}^N$ satisfies (2.6)–(2.7).

For our further analysis, we fix any test function $\varphi \in C_0^\infty(\Omega)^N$, and make use of the following equality

$$\int_\Omega \left(A_n^{-1}\varphi, A_n\psi\right)_{\mathbb{R}^N}\,dx = \int_\Omega (\varphi,\psi)_{\mathbb{R}^N}\,dx = \int_\Omega \left(A^{-1}\varphi, A\psi\right)_{\mathbb{R}^N}\,dx, \tag{4.18}$$

which is obviously true for each $\psi \in C_0^\infty(\Omega)^N$ and for all $n \in \mathbb{N}$. Since

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \left(A_n^{-1}\varphi, A_nA_n^{-1}\varphi\right)_{\mathbb{R}^N}\,dx &= \limsup_{n\to\infty}\int_\Omega \left(\varphi, A_n^{-1}\varphi\right)_{\mathbb{R}^N}\,dx\\ &\le \limsup_{n\to\infty}\int_\Omega \zeta_{*,n}^{-1}\|\varphi\|^2_{\mathbb{R}^N}\,dx \overset{\text{by (4.17)}}{=} \int_\Omega \zeta_*^{-1}\|\varphi\|^2_{\mathbb{R}^N}\,dx\\ &\le \|\varphi\|^2_{C(\Omega)^N}\|\zeta_*^{-1}\|_{L^1(\Omega)} < +\infty,\end{aligned}$$

it follows that the sequence $\left\{A_n^{-1}\varphi \in L^2(\Omega, A_n\,dx)^N\right\}_{n\in\mathbb{N}}$ is bounded. Consequently, combining this fact with (4.18), we conclude $A_n^{-1}\varphi \rightharpoonup A^{-1}\varphi$ in the variable space $L^2(\Omega, A_n\,dx)^N$ (see Definition 3.1). At the same time, strong convergence in (4.13) implies the relation

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \left(A_n^{-1}\varphi, A_nA_n^{-1}\varphi\right)_{\mathbb{R}^N}\,dx &= \lim_{n\to\infty}\int_\Omega \left(\varphi, A_n^{-1}\varphi\right)_{\mathbb{R}^N}\,dx\\ &= \int_\Omega \left(\varphi, A^{-1}\varphi\right)_{\mathbb{R}^N}\,dx = \int_\Omega \left(A^{-1}\varphi, AA^{-1}\varphi\right)_{\mathbb{R}^N}\,dx.\end{aligned}$$

Hence (see Proposition 3.5),

$$A_n^{-1}\varphi \to A^{-1}\varphi \ \text{ strongly in } \ L^2(\Omega, A_n\,dx)^N \quad \forall\,\varphi \in C_0^\infty(\Omega)^N. \tag{4.19}$$

Further, we note that for every measurable subset $K \subset \Omega$, the estimate

$$\begin{aligned}\int_K \|\nabla u_n\|_{\mathbb{R}^N}\,dx &\le \Big(\int_K \|\nabla u_n\|^2_{\mathbb{R}^N}\zeta_{*,n}\,dx\Big)^{1/2}\Big(\int_K \zeta_{*,n}^{-1}\,dx\Big)^{1/2}\\ &\le \Big(\int_\Omega (\nabla u_n, A_n(x)\nabla u_n)_{\mathbb{R}^N}\,dx\Big)^{1/2}\Big(\int_K \zeta_{*,n}^{-1}\,dx\Big)^{1/2} \le C\left(\int_K \zeta_{*,n}^{-1}\,dx\right)^{1/2}\end{aligned}$$

implies equi-integrability of the family $\{\|\nabla u_n\|_{\mathbb{R}^N}\}_{n\in\mathbb{N}}$. Hence, $\{\|\nabla u_n\|_{\mathbb{R}^N}\}_{n\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$, which means the weak compactness of the vector-valued sequence $\{\nabla u_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^N)$. As a result, by the properties of the strong convergence in variable spaces, we obtain

$$\begin{aligned}\int_\Omega (\xi, \nabla u_n)_{\mathbb{R}^N}\ dx &= \int_\Omega \big(A_n^{-1}\xi, A_n\nabla u_n\big)_{\mathbb{R}^N}\ dx\\ &\xrightarrow{\text{by (3.8), (4.15), and (4.19)}} \int_\Omega \big(A^{-1}\xi, Av\big)_{\mathbb{R}^N}\ dx\\ &= \int_\Omega (\xi, v)_{\mathbb{R}^N}\ dx \quad \forall\,\xi \in C_0^\infty(\Omega)^N.\end{aligned}$$

Thus, in view of the weak compactness property of $\{\nabla u_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^N)$, we conclude

$$\nabla u_n \rightharpoonup v \text{ in } \ L^1(\Omega;\mathbb{R}^N) \quad \text{as} \quad n\to\infty. \tag{4.20}$$

Since $u_n \in W^{1,1}(\Omega;\Gamma_D)$ for all $n\in\mathbb{N}$ and the Sobolev space $W^{1,1}(\Omega;\Gamma_D)$ is complete, (4.14) and (4.20) imply $\nabla u = v$, and consequently $u \in W^{1,1}(\Omega;\Gamma_D)$. To end the proof, it remains to observe that (4.14)–(4.15) guarantee the finiteness of the norm $\|u\|_A$ (see (2.8)). Hence, $u \in W_A(\Omega;\Gamma_D)$ and this concludes the proof. □

5. Setting of the Optimal Control Problem. Let $M \in \mathbb{S}^N$ be a given constant matrix satisfying the condition

$$(M\xi,\xi)_{\mathbb{R}^N} \ge m\|\xi\|^2_{\mathbb{R}^N} \ \text{ for some } \ m > 0.$$

Let Q be a closed nonempty subdomain of Ω for which $\operatorname{dist}(\partial\Omega, \partial Q) \ge \delta > 0$, where δ is a prescribed value. Let $B \in L^\infty(Q;\mathbb{S}^N)$ be a given matrix-valued function such that

$$\sigma_1\|\xi\|^2_{\mathbb{R}^N} \le (B(x)\xi,\xi)_{\mathbb{R}^N} \le \sigma_2\|\xi\|^2_{\mathbb{R}^N} \text{ a. e. in } \ Q \quad \forall\,\xi\in\mathbb{R}^N$$

with some $\sigma_2 > \sigma_1 > 0$.

Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ be given functions. We consider the following boundary value problem

$$-\mathrm{div}\,\big(A(x)\nabla y\big) = f \quad \text{in} \ \ \Omega, \tag{5.1}$$

$$y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial\nu_A = g \text{ on } \Gamma_N. \tag{5.2}$$

Here

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial y}{\partial x_j} \cos(n, x_i),$$

$\cos(n, x_i)$ is i-th directional cosine of n, and n is the outward unit normal at Γ_N to Ω.

To introduce the class of admissible controls in coefficients, we adopt the following concept:

Definition 5.1. *We say that a matrix-valued function $A = A(x) \in \mathbb{S}^N$ is an admissible control for the boundary value problem* (5.1)–(5.2) *(it is written as $A \in \mathfrak{A}_{ad}$) if*

$$A \in BV(\Omega \setminus Q; \mathbb{S}^N), \quad \int_{\Omega \setminus Q} A(x)\, dx = M, \tag{5.3}$$

$$A \in \mathfrak{M}_\alpha^\beta(\Omega \setminus Q), \quad A(x) = B(x) \textit{ a.e. in } Q. \tag{5.4}$$

Hereafter we assume that the set $\mathfrak{A}_{ad}$ is nonempty. Moreover, it is easy to see that for a given $B \in L^\infty(Q; \mathbb{S}^N)$, we can always guarantee the fulfilment of condition $\mathfrak{A}_{ad} \neq \emptyset$ by an appropriate choice of the matrix $M \in \mathbb{S}^N$ and functions $\zeta_{ad} \in L^1(\Omega)$ and $\beta \in L^1(\Omega)$.

Remark 5.1. *As immediately follows from definition of the set $\mathfrak{M}_\alpha^\beta(\Omega \setminus Q)$ and the properties of the matrix B,* (5.4) *implies*

$$A \in \mathfrak{M}_\alpha^{\widetilde{\beta}}(\Omega) \cap L^1(\Omega; \mathbb{S}^N) \quad \textit{with} \quad \widetilde{\beta} = \max\{\beta, \sigma_2\}.$$

Remark 5.2. *In view of* $(5.4)_1$ *and* (2.5) *(see also Remark 2.2), we deal with a boundary value problem for the degenerate elliptic equation. It means that for some admissible controls $A \in \mathfrak{A}_{ad}$ the boundary value problem* (5.1)–(5.2) *can exhibit the Lavrentieff phenomenon and nonuniqueness of the weak solutions.*

Definition 5.2. *We say that a function $y = y(A, f, g)$ is a weak solution to the boundary value problem* (5.1)–(5.2) *for a fixed control $A \in \mathfrak{A}_{ad}$ and given functions $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ if*

$$y \in W_A(\Omega; \Gamma_D)$$

and the integral identity

$$\int_\Omega \left(\nabla\varphi, A(x)\nabla y\right)_{\mathbb{R}^N} dx = \int_\Omega f\varphi\, dx + \int_{\Gamma_N} g\varphi\, d\mathcal{H}^{N-1} \tag{5.5}$$

holds for any $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

Remark 5.3. *It is worth to notice that the original boundary value problem* (5.1)–(5.2) *is ill-possed and in general the existence of a weak solution to* (5.1)–(5.2) *for fixed $A \in \mathfrak{A}_{ad}$, $f \in L^2(\Omega)$, and $g \in L^2(\Gamma_N)$ seems to be an open question. This means that there are no reasons to expect that for every admissible given data $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, and $A \in \mathfrak{A}_{ad}$, this problem admits at least one weak solution $y \in W_A(\Omega; \Gamma_D)$ in the sense of Definition 5.2. So, it is not possible to write in this case $y = y(A, f, g)$. Even if a weak solution to the above problem exists, the question about its uniqueness leads us to the problem of density of the subspace of smooth functions $C_0^\infty(\Omega; \Gamma_D)$ in*

$W_A(\Omega;\Gamma_D)$. *However, as was indicated in [29], there exists a diagonal matrix-valued function* $A(x)=\rho(x)I$ *with* $\rho\in\Psi_*$ *such that the subspace* $C_0^\infty(\Omega;\Gamma_D)$ *is not dense in* $W_A(\Omega;\Gamma_D)$*, and, hence, there is no uniqueness of weak solutions (for more details and other types of solutions we refer to [3, 27, 29]). Thus, the mapping* $A\mapsto y(A,f,g)$ *can be multivalued, in general.*

To avoid this situation in our analysis, we introduce the set of admissible solutions to the original optimal control problem as follows:

$$\Xi_w=\big\{(A,y)\ \mid A\in\mathfrak{A}_{ad},\ y\in W_A(\Omega;\Gamma_D),\ (A,y)\text{ are related by (5.5)}\big\}. \tag{5.6}$$

In what follows, we make use the following result:

PROPOSITION 5.3. *Let* $A\in\mathfrak{A}_{ad}$ *be a given matrix-valued function. Then there exist bounded linear operators*

$$\gamma_A^0: W_A(\Omega;\Gamma_D)\to H^{1/2}(\partial\Omega)\quad\textit{and}\quad \gamma_A^1: W_A(\Omega;\Gamma_D)\to H^{-1/2}(\partial\Omega) \tag{5.7}$$

such that

(i) $\gamma_A^0(y)=y|_{\partial\Omega}$ *and* $\gamma_A^1(y)=\frac{\partial y}{\partial\nu_A}\Big|_{\partial\Omega}$ *provided* $y\in W_A(\Omega;\Gamma_D)\cap C(\overline{\Omega})$*;*

(ii) for any function $y\in W_A(\Omega;\Gamma_D)$

$$\|\gamma_A^0(y)\|_{H^{1/2}(\partial\Omega)}\le C\|y\|_{W_A(\Omega;\Gamma_D)},\quad \|\gamma_A^1(y)\|_{H^{-1/2}(\partial\Omega)}\le C_1\|y\|_{W_A(\Omega;\Gamma_D)}$$

with the positive constants C *and* C_1 *independent of* y.

Proof. To begin with, we note that the matrix $A\in\mathfrak{A}_{ad}$ belongs to $\mathfrak{M}_\alpha^\beta(\Omega\setminus Q)$. Hence, it can be associated with an element ζ_* of Ψ such that $A(x)\ge\zeta_* I$ and $\zeta_{ad}<\zeta_*\le\alpha$ almost everywhere in $\Omega\setminus Q$ (see (2.1)–(2.3)). Moreover, in this case $\zeta_*:\Omega\setminus Q\to[0,\alpha]$ is a smooth function along the surface $\partial\Omega$ satisfying the condition $\zeta_*=\alpha$ on $\partial\Omega$. Hence, there exist an open set $\mathcal{O}$ with Lipschitz continuous boundary and a positive constant $\widetilde{\alpha}\le\alpha$ such that

$$\mathcal{O}\subset\Omega\setminus Q,\quad \partial\Omega\subset\partial\mathcal{O},\quad\text{ and }\ \zeta_*\ge\widetilde{\alpha}\ \text{ a.e. in }\ \mathcal{O}.$$

As a result, for any element $y\in W_A(\Omega;\Gamma_D)$, we have $y\in W_A(\mathcal{O};\Gamma_D)$, and, therefore,

$$\begin{aligned}\|y\|^2_{W_A(\Omega;\Gamma_D)}\ge\|y\|^2_{W_A(\mathcal{O};\Gamma_D)}&=\int_{\mathcal{O}}\left(y^2+(\nabla y,A(x)\nabla y)_{\mathbb{R}^N}\right)dx\\ &\ge\min\{1,\widetilde{\alpha}\}\int_{\mathcal{O}}\left(y^2+\|\nabla y\|^2_{\mathbb{R}^N}\right)dx.\end{aligned}$$

Thus, $y\in W^{1,2}(\mathcal{O})$, and, therefore, the existence of the trace operators γ_A^0 and γ_A^1 with (i)–(ii) immediately follows from the Trace Sobolev Theorem (see [18, Section 3.]). □

As an evident consequence of this result, we can give the following observation.

COROLLARY 5.4. *Let* $\big\{(A_n,y_n)\in L^1(\Omega;\mathbb{S}^N)\times W_{A_n}(\Omega;\Gamma_D)\big\}_{n\in\mathbb{N}}$ *be a sequence such that* $\{A_n\}_{n\in\mathbb{N}}\subset\mathfrak{M}_\alpha^\beta(\Omega)$ *and* $(A_n,y_n)\overset{w}{\to}(A,y)$ *in the sense of Definition 4.2, where* $(A,y)\in L^1(\Omega;\mathbb{S}^N)\times W_A(\Omega;\Gamma_D)$*. Then, up to a subsequence, we have*

$$\frac{\partial y_n}{\partial\nu_{A_n}}\rightharpoonup\frac{\partial y}{\partial\nu_A}\quad\textit{in}\ \ H^{-1/2}(\Gamma_D).$$

The optimal control problem we consider here is to minimize the discrepancies (tracking error) between given distributions $y_d\in L^2(\Omega)$, $y^*\in L^2(\Gamma_D)$ and the solution of boundary valued problem (5.1)–(5.2) by choosing an appropriate coefficients matrix $A \in \mathfrak{A}_{ad}$. More precisely, we are concerned with the following optimal control problem

$$\text{Minimize } \Big\{ I(A,y) = \int_\Omega |y(x) - y_d(x)|^2\, dx + \int_\Omega \left(\nabla y(x), A(x)\nabla y(x)\right)_{\mathbb{R}^N}\, dx \\ + \sum_{i,j=1}^N \int_{\Omega\setminus Q} |D\, a_{ij}(x)| dx + \left\| \frac{\partial y}{\partial \nu_A} - y^* \right\|^2_{H^{-1/2}(\Gamma_D)} \Big\} \tag{5.8}$$

subject to the constraints (5.1)–(5.4).

REMARK 5.4. *The second term in* (5.8) *plays a special role in this problem. Its appearance in the cost function* (5.8) *is motivated by the fact that there are no appropriate a priori estimates in the* $W_A(\Omega;\Gamma_D)$*-norm for weak solutions* $y = y(A, f, g)$ *of the degenerate boundary value problem* (5.1)–(5.2) *(in the sense of Definition 5.2). Hence, the term* $\int_\Omega (\nabla y(x), A(x)\nabla y(x))_{\mathbb{R}^N}\, dx$ *together with the first one in* (5.8) *ensures the coercivity of the cost function on the space of weak solutions* $W_A(\Omega;\Gamma_D)$.

REMARK 5.5. *Note that due to* (2.9)–(2.10)*, we have the following obvious inclusion for the set of admissible solutions*

$$\Xi_w \subset L^1(\Omega;\mathbb{S}^N) \times W^{1,1}(\Omega;\Gamma_D).$$

However, the characteristic feature of this set is the fact that for different admissible controls $A \in \mathfrak{A}_{ad}$ *the corresponding admissible solutions* y *of optimal control problem* (5.8) *belong to different weighted spaces. It is a non-typical situation from the point of view of classical optimal control theory.*

It is worth noticing that for any admissible given data $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, verification of $\Xi_w \neq \emptyset$ is a non-trivial matter, in general. In the particular case, when the set of admissible controls $\mathfrak{A}_{ad}$ possesses the property:

$$A \in L^\infty(\Omega;\mathbb{S}^N),\ A(x) \geq \nu I \ \text{ a.e. in } \ \Omega \quad \forall\, A \in \mathfrak{A}_{ad},$$

$\Xi_w \neq \emptyset$ is obvious since the corresponding boundary value problem (5.1)–(5.2) has a unique weak solution $y = y(A)$. Therefore, we adopt the following hypothesis, which is mainly motivated by Remark 5.3.

Hypothesis A. *The set of admissible solutions* Ξ_w *is nonempty.*

We say that a pair $(A^0, y^0) \in L^1(\Omega;\mathbb{S}^N) \times W_A(\Omega;\Gamma_D)$ is optimal for problem (5.8) on the class Ξ_w (or shortly, weakly optimal), if

$$(A^0, y^0) \in \Xi_w \ \text{ and } \ I(A^0, y^0) = \inf_{(A,y)\in\Xi_w} I(A,y). \tag{5.9}$$

6. Existence of Weak Optimal Solutions. Since our prime interest is the solvability of optimal control problem (5.8), we begin with the study of the topological properties of the set of admissible solutions Ξ_w. To do so, we give a some auxiliary results.

DEFINITION 6.1. *We say that a sequence* $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ *is bounded if*

$$\sup_{n\in\mathbb{N}} \left[\|A_n\|_{BV(\Omega\setminus Q;\mathbb{S}^N)} + \|y_n\|_{A_n} \right] < +\infty.$$

Lemma 6.2. *Let $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ be a bounded sequence in the sense of Definition 6.1. Then there exists a pair $(A, y) \in L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega; \Gamma_D)$ such that, up to a subsequence,*

$$(A_n, y_n) \xrightarrow{w} (A, y) \text{ see Definition 4.2}, \; A \in \mathfrak{A}_{ad}, \quad \text{and} \;\; y \in W_A(\Omega; \Gamma_D). \tag{6.1}$$

Proof. By the compactness result for BV-functions (see Proposition 2.5), there exists a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, still denoted by the same indices, and a matrix $A \in BV(\Omega \setminus Q; \mathbb{S}^N)$ such that $A_n \to A$ in $L^1(\Omega \setminus Q; \mathbb{S}^N)$. Since $A_n(x) = B(x)$ a.e. in Q and $B \in L^\infty(Q; \mathbb{S}^N)$, it follows that the strong L^1-convergence $A_n \to A$ can be extended to the entire domain Ω. Thus,

$$A \in BV(\Omega \setminus Q; \mathbb{S}^N), \quad A(x) = B(x) \text{ a.e. in } Q, \tag{6.2}$$

$$\int_{\Omega\setminus Q} A(x)\, dx = \lim_{n\to\infty} \int_{\Omega\setminus Q} A_n(x)\, dx = M, \tag{6.3}$$

and the condition (4.6) of Definition 4.2 holds true. In order to check the remaining conditions (4.7)–(4.8) of this definition and to show that $A \in \mathfrak{A}_{ad}$, we make use of the following observation.

We have $(A_n, y_n) \in \Xi_w$ for all $n \in \mathbb{N}$. Hence, there is a sequence $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ in Ψ_* such that (see Lemma 4.1 for the details) $\zeta_{*,n} \to \zeta_*$ and $\zeta_{*,n}^{-1} \to \zeta_*^{-1}$ in $L^1(\Omega)$ as $n \to \infty$. Moreover, by properties of Ψ_*, the L^1-limit element ζ_* satisfies (2.1)–(2.3). Then, in view of L^1-convergence $A_n \to A$, we may assume that $A_n^{-1} \to A^{-1}$ almost everywhere in Ω. Since $A_n(x) \geq \zeta_{*,n} I$ a. e. in Ω, it follows that

$$\int_K \left(\xi, A_n^{-1}\xi\right)_{\mathbb{R}^N}\, dx \leq \int_K \zeta_{*,n}^{-1}\, dx \|\xi\|^2_{\mathbb{R}^N} \quad \forall\, n \in \mathbb{N}$$

for any subset $K \subset \Omega$. Hence, due to the strong L^1-convergence $\zeta_{*,n}^{-1} \to \zeta_*^{-1}$, the sequence $\{A_n^{-1}\}_{n\in\mathbb{N}}$ is equi-integrable. Then, by Lebesgue's Theorem (see Theorem 2.1) we obtain $A_n^{-1} \to A^{-1}$ in $L^1(\Omega; \mathbb{S}^N)$ as $n \to \infty$. As a result,

$$A \in \mathfrak{M}_\alpha^\beta(\Omega \setminus Q) \cap L^1(\Omega; \mathbb{S}^N)$$

by Lemma 4.3. Combining this fact with properties (6.2)–(6.3), we conclude $A \in \mathfrak{A}_{ad}$.

To end of this proof, it remains to observe that the remaining conditions (4.7)–(4.8) of Definition 4.2 and $y \in W_A(\Omega; \Gamma_D)$ for the w-limiting component (A, y) of the sequence $\{(A_n, y_n)\}_{n\in\mathbb{N}}$, are ensured by Lemma 4.3. This concludes the proof. ∎

Our next step deals with the study of topological properties of the set of admissible solutions Ξ_w to the problem (5.8). The following theorem is crucial for our next analysis.

Theorem 6.3. *Assume that the Hypothesis A is valid. Then for any admissible given data $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, the set of admissible solutions Ξ_w is sequentially closed with respect to w-convergence.*

Proof. Let $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ be a bounded w-convergent sequence of admissible solutions to the optimal control problem (5.8). Let $(\widehat{A}, \widehat{y})$ be its w-limit. Our aim is to prove that $(\widehat{A}, \widehat{y}) \in \Xi_w$. By Lemma 6.2, we have: $\widehat{A} \in \mathfrak{A}_{ad}$ and $\widehat{y} \in W_{\widehat{A}}(\Omega; \Gamma_D)$.

It remains to show that the pair $(\widehat{A}, \widehat{y})$ is related to (5.5) for all $\varphi \in C_0^\infty(\Omega; \Gamma_D)$. To do so, we note that for every $n \in \mathbb{N}$, the integral identity (5.5) (with A_n and

y_n instead of A and y, respectively), has to be satisfied for the test functions $\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D)$. Then $\nabla\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D)^N$. However, this class is essentially wider than the space $C_0^\infty(\Omega)^N$ in the definition of the weak convergence in the variable space $L^2(\Omega, A_n\,dx)^N$ (see (3.4)). Therefore, in order to pass to the limit in that integral identity as $n \to \infty$, we make use the following argument (see Buttazzo & Kogut [6]).

For every fixed $n \in \mathbb{N}$ we denote by

$$(\widetilde{A}_n, \widetilde{y}_n) \in BV_{loc}(\mathbb{R}^N \setminus Q) \times W^{1,1}_{loc}(\mathbb{R}^N;\Gamma_D)$$

an extension of the functions (A_n, y_n) to the whole of space $\mathbb{R}^N$ such that the sequence $\{(\widetilde{A}_n, \widetilde{y}_n)\}_{n\in\mathbb{N}}$ satisfies the properties:

$$\widetilde{A} \in \mathfrak{M}_\alpha^\beta(D \setminus Q), \quad \widetilde{A}(x) = B(x) \text{ a.e. in } Q, \tag{6.4}$$

$$\widetilde{A}(x) \le \widetilde{\beta}(x) I \quad \text{a. e. in } \ D \setminus Q, \tag{6.5}$$

$$\widetilde{\zeta}_*\, I \le \widetilde{A}(x) \quad \text{a. e. in } \ D \setminus Q, \tag{6.6}$$

$$\widetilde{\zeta}_{ad} < \widetilde{\zeta}_* \le \alpha \text{ a.e. in } \ D \setminus Q, \quad \widetilde{\zeta}_*^{-1} \in L^1(D \setminus Q), \tag{6.7}$$

$$\sup_{n\in\mathbb{N}} \left[\|\widetilde{A}_n\|_{BV(D\setminus Q;\mathbb{S}^N))} + \|\widetilde{y}_n\|_{L^2(D)} + \|\nabla \widetilde{y}_n)|_{L^2(D,\widetilde{A}_n\,dx)^N} \right] < +\infty \tag{6.8}$$

for any bounded open domain D in $\mathbb{R}^N$ such that $\Omega \subset D$.

Here $\widetilde{\beta} \in L^1_{loc}(\mathbb{R}^N)$, $\widetilde{\zeta}_{ad} \in L^1_{loc}(\mathbb{R}^N)$, and $\widetilde{\zeta}_* \in L^1_{loc}(\mathbb{R}^N)$ are non-negative functions such that $\widetilde{\beta}\Big|_\Omega = \beta$, $\widetilde{\zeta}_{ad}\Big|_\Omega = \zeta_{ad}$, and $\widetilde{\zeta}_*\Big|_\Omega \in \Psi_*$, respectively.

Then by analogy with Lemmas 4.3 and 6.2, it can be proved that for every bounded domain $D \subset \mathbb{R}^N$ there exists a matrix $\widetilde{A} \in BV(D \setminus Q;\mathbb{S}^N)$ and a function $\widetilde{y} \in W_{\widetilde{A}}(D;\Gamma_D)$ such that

$$\widetilde{A}_n \to \widetilde{A} \text{ in } L^1(D)), \qquad \widetilde{y}_n \rightharpoonup \widetilde{y} \text{ in } L^2(D), \tag{6.9}$$

$$\nabla\widetilde{y}_n \rightharpoonup \nabla\widetilde{y} \ \text{ in the variable space } \ L^2(D, \widetilde{A}_n\,dx)^N. \tag{6.10}$$

It is important to note that in this case we have

$$\widetilde{y} = \widehat{y} \ \text{ and } \ \widetilde{A} = \widehat{A} \quad \text{a.e. in } \Omega. \tag{6.11}$$

Taking this fact into account, we can rewrite (5.5) in the equivalent form

$$\int_{\mathbb{R}^N} \left(\nabla\varphi, \widetilde{A}_n(x)\nabla\widetilde{y}_n\right)_{\mathbb{R}^N} \chi_\Omega\,dx = \int_{\mathbb{R}^N} f\varphi\chi_\Omega\,dx + \int_{\Gamma_N} g\varphi\,d\mathcal{H}^{N-1} \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D). \tag{6.12}$$

In what follows, we note that due to (6.9), we have $\widetilde{A}_n \to \widetilde{A}$ strongly in $L^1_{loc}(\mathbb{R}^N;\mathbb{S}^N)$. Hence,

$$\int_{\mathbb{R}^N} \Big(\chi_\Omega\,\xi, \widetilde{A}_n\,\chi_\Omega\,\xi\Big)_{\mathbb{R}^N}\,dx = \int_{\mathbb{R}^N} \Big(\xi, \widetilde{A}_n\,\xi\Big)_{\mathbb{R}^N} \chi_\Omega\,dx \longrightarrow \int_{\mathbb{R}^N} \Big(\xi, \widetilde{A}\,\xi\Big)_{\mathbb{R}^N} \chi_\Omega\,dx = \int_{\mathbb{R}^N} \Big(\chi_\Omega\,\xi, \widetilde{A}\,\chi_\Omega\,\xi\Big)_{\mathbb{R}^N}\,dx \tag{6.13}$$

for any vector $\xi \in \mathbb{R}^N$. As follows from convergence property (3.9), (6.13) implies strong convergence $\chi_\Omega \xi \to \chi_\Omega \xi$ in the variable space $L^2(\mathbb{R}^N, \widetilde{A}_n\, dx)^N$. Taking (6.9)–(6.10) into account, we can pass to the limit in (6.12) as $n \to \infty$ and obtain

$$\int_{\mathbb{R}^N} \left(\nabla\varphi, \widetilde{A}(x)\nabla\widetilde{y}\right)_{\mathbb{R}^N} \chi_\Omega\, dx = \int_{\mathbb{R}^N} f\varphi\chi_\Omega\, dx + \int_{\Gamma_N} g\varphi\, d\mathcal{H}^{N-1} \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$$

which, due to (6.11), is equivalent to

$$\int_\Omega \left(\nabla\varphi, \widehat{A}(x)\nabla\widehat{y}\right)_{\mathbb{R}^N} dx = \int_\Omega f\varphi\, dx + \int_{\Gamma_N} g\varphi\, d\mathcal{H}^{N-1} \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D).$$

Hence, $\widehat{y} \in W_{\widehat{A}}(\Omega; \Gamma_D)$ is a weak solution to (5.1)–(5.2) under $A = \widehat{A}$ in the sense of Definition 5.2. Thus, the w-limit pair $(\widehat{A}, \widehat{y})$ belongs to set Ξ_w, and this concludes the proof. ▯

We are now in a position to state the existence of weak optimal solution to the problem (5.8).

THEOREM 6.4. *Let $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $y_d \in L^2(\Omega)$, and $y^* \in L^2(\Gamma_D)$ be given functions. Assume that the Hypothesis A is valid. Then the optimal control problem* (5.8) *admits at least one solution*

$$(A^0, y^0) \in L^1(\Omega; \mathbb{S}^N) \times W_{A^0}(\Omega; \Gamma_D).$$

Proof. Since the cost functional $I = I(A, y)$ is bounded below and $\Xi_w \neq \emptyset$, it provides the existence of a minimizing sequence $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ to the problem (5.9). Then,

$$\begin{aligned}\inf_{(A,y)\in\Xi_w} I(A, y) = \lim_{n\to\infty} I(A_n, y_n) = \lim_{n\to\infty} \Big[\int_\Omega |y_n(x) - y_d(x)|^2\, dx \\ + \int_\Omega (\nabla y_n(x), A(x)\nabla y_n(x))_{\mathbb{R}^N}\ dx + \sum_{i,j=1}^N \int_{\Omega\setminus Q} |D\, a_{ij}^n(x)| \\ + \left\| \frac{\partial y_n}{\partial \nu_{A_n}} - y^* \right\|^2_{H^{-1/2}(\Gamma_D)} \Big] < +\infty \end{aligned} \tag{6.14}$$

implies the existence of a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|\nabla y_n\|_{L^2(\Omega, A_n\, dx)^N} \le C, \quad \|\frac{\partial y_n}{\partial \nu_{A_n}}\|_{H^{-1/2}(\Gamma_D)} \le C, \tag{6.15}$$

$$\sup_{n\in\mathbb{N}} \|y_n\|_{L^2(\Omega)} \le C, \quad \sup_{n\in\mathbb{N}} \|A_n\|_{BV(\Omega\setminus Q; \mathbb{S}^N)} \le C. \tag{6.16}$$

Hence, the minimizing sequence $\{(A_n, y_n) \in \Xi_w\}_{n\in\mathbb{N}}$ is bounded in the sense of Definition 6.1. Hence, by Lemma 6.2 there exist functions $A^0 \in L^1(\Omega; \mathbb{S}^N)$ and $y^0 \in W_{A^0}(\Omega; \Gamma_D)$ such that, up to a subsequence, $(A_n, y_n) \xrightarrow{w} (A^0, y^0)$. Since the set Ξ_w is sequentially closed with respect to the w-convergence (see Theorem 6.3), it follows that the w-limit pair (A^0, y^0) is an admissible pair of (5.8) (i.e. $(A^0, y^0) \in \Xi_w$). Moreover, by $(6.15)_2$ and Corollary 5.4 of Proposition 5.3, we have:

$$\frac{\partial y_n}{\partial \nu_{A_n}} \rightharpoonup \frac{\partial y^0}{\partial \nu_{A^0}} \quad \text{in} \ \ H^{-1/2}(\Gamma_D). \tag{6.17}$$

To conclude the proof it is enough to observe that by (6.17) and $(A_n, u_n) \stackrel{w}{\to} (A^0, y^0)$, the cost functional I is sequentially lower w-semicontinuous. Hence,

$$I(A^0, y^0) \le \liminf_{n\to\infty} I(A_n, y_n) = \inf_{(A,y)\in\Xi_w} I(A, y),$$

i.e. (A^0, y^0) is an optimal solution. The proof is complete. □

REFERENCES

[1] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Vol. 1694, *Springer-Verlag*, Berlin, 1998.

[2] K. Bredies, *Optimal control of degenerate parabolic equations in image processing*,*Logos Verlag*, Berlin, 2008.

[3] L. Boccardo, T. Gallouët, P. Marcellini, *Anisotropic equations in* L^1, *Diff. Integral Eq.*, **9**(1996), 209–212.

[4] G. Bouchitte, G. Buttazzo, *Characterization of optimal shapes and masses through Monge-Kantorovich equation*, *J. Eur. Math. Soc.*, **3**(2001), 139–168.

[5] G. Buttazzo, N. Varchon, *On the optimal reinforcement of an elastic membrane*, *Riv. Mat. Univ. Parma*, No. 7 , **4**(2005), 115–125.

[6] G. Buttazzo, P. I. Kogut, *Weak optimal controls in coefficients for linear elliptic problems*, *Revista Matematica Complutense*, **24**(2011), 83–94.

[7] P. Caldiroli, R. Musina, *On a variational degenerate elliptic problem*, *Nonlinear Diff. Equa. Appl.*, **7**(2000), 187–199.

[8] C. Calvo-Jurado, J. Casado-Díaz, *Optimization by the homogenization method for nonlinear elliptic Dirichlet problems*, *Mediterranian J. of Mathematic*, **4**(2007), 53-63.

[9] J. Chabrowski, *Degenerate elliptic equation involving a subcritical Sobolev exponent*, *Portugal. Math.*, **53**(1996), 167–177.

[10] V. Chiadò Piat, F. Serra Cassano, *Some remarks about the density of smooth functions in weighted Sobolev spaces*, *J. Convex Analysis*, No. 2, **1**(1994), 135–142.

[11] R. Dautray, J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, *Physical Origins and Classical Methods*, Vol. 1, *Springer-Verlag*, Berlin, 1985.

[12] K. Deckelnick, M. Hinze, *Identification of matrix parameters in elliptic PDEs*, *Preprint*, SPP1253-115, Hamburg University, 2011.

[13] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, *CBMS Regional Conference Series in Mathematics*, Vol.74, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.

[14] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, *Birkhäuser*, Boston, 1984.

[15] P. I. Kogut, G. Leugering, *Optimal* L^1*-Control in Coefficients for Dirichlet Elliptic Problems: H-Optimal Solutions*, *Zeitschrift für Analysis und ihre Anwendungen*, **31**:1(2011), 31–53.

[16] P. I. Kogut, G. Leugering, *Optimal* L^1*-Control in Coefficients for Dirichlet Elliptic Problems: W-Optimal Solutions*, *Journal of Optimization Theory and Applications*, **150**:2(2011), 205–232.

[17] P. I. Kogut, G. Leugering, *Optimal control problems for partial differential equations on reticulated domains: Approximation and asymptotic analysis*, *Birkhäuser*, Boston, 2011.

[18] J.-L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, *Springer-Verlag*, Berlin, 1972.

[19] F. Murat, *Un contre-exemple pour le prolème de contrôle dans les coefficients*, *C.R.A.S. Paris*, Sér. A **273**(1971), 708–711.

[20] M.K.V. Murthy, V. Stampacchia, *Boundary problems for some degenerate elliptic operators*, *Ann. Mat. Pura e Appl.*, **5**:4(1968), 1–122.

[21] S. E. Pastukhova, *Degenerate equations of monotone type: Lavrentiev phenomenon and attainability problems*, *Sbornik: Mathematics*, **198**:10(2007), 1465–1494.

[22] J. Pendry, *Negative refraction makes a perfect lens*, *Phys. Rev. Lett.* **8** (2000), 3966 - 3969.

[23] U. Leonhardt, *Optical Conformal Mapping*, *Science* **312** (23 June, 2006), 1777-1780.

[24] G. Uhlmann (ed.), *Inside out: Inverse problems and applications*, *Reprint of the 2003 hardback ed. (English) Mathematical Sciences Research Institute Publications* **47**. Cambridge: Cambridge University Press (ISBN 978-0-521-16874-8/pbk), 400 p. (2011).

[25] C. E. Kenig, M. Salo and G. Uhlmann, *Inverse problems for the anisotropic Maxwell equations*, *Duke Math. J.* **157**, No. 2, 369-419 (2011).

[26] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, *Cloaking devices, electromagnetic wormholes, and transformation optics*, *SIAM Rev.* **51**, No. 1, 3-33 (2009).
[27] V. V. Zhikov, *Weighted Sobolev spaces*, *Sbornik: Mathematics*, No. 8, **189**(1998), 27–58.
[28] V. V. Zhikov, *On an extension of the method of two-scale convergence and its applications*, *Sbornik Math.*, **191**(2000), No. 7, 973-1014.
[29] V. V. Zhikov, S. E. Pastukhova, *Homogenization of degenerate elliptic equations*, *Siberian Math. Journal*, No. 1, **49**(2006), 80–101.