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Gap phenomenon in the homogenization of parabolic optimal control problems

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In this paper, we study the asymptotic behaviour of a parabolic optimal control problem in a domain $\Omega_\varepsilon \subset \mathbb{R}^n$, whose boundary $\partial\Omega_\varepsilon$ contains a highly oscillating part. We consider this problem with two different classes of Dirichlet boundary controls, and, as a result, we provide its asymptotic analysis with respect to the different topologies of homogenization. It is shown that the mathematical descriptions of the homogenized optimal control problems have different forms and these differences appear not only in the state equation and boundary conditions but also in the control constraints and the limit cost functional.

Keywords: optimal control problem; homogenization; thick multi-structure; variational convergence; set convergence; gap phenomenon.

1. Introduction

The aim of this paper is to study the asymptotic behaviour of the following class of the parabolic optimal control problems:

$$I_\varepsilon(u_\varepsilon, y_\varepsilon) = \int_0^T \int_{\Omega^+} (y_\varepsilon - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon^2 \,\mathrm{d}x'\,\mathrm{d}t \to \inf, \tag{1.1}$$

$$\left.\begin{aligned}
&y_\varepsilon' - \Delta_x y_\varepsilon + y_\varepsilon = f_\varepsilon, && \text{in } (0,T)\times\Omega_\varepsilon,\\
&\partial_\nu y_\varepsilon = -\varepsilon k_0 y_\varepsilon, && \text{on } (0,T)\times S_\varepsilon,\\
&y_\varepsilon = u_\varepsilon, && \text{on } (0,T)\times\Gamma_\varepsilon,\\
&\partial_\nu y_\varepsilon = 0, && \text{on } (0,T)\times\partial\Omega_\varepsilon\setminus(\Gamma_\varepsilon\cup S_\varepsilon),\\
&y_\varepsilon(0,x) = y_\varepsilon^0 && \text{a.e. } x\in\Omega_\varepsilon,
\end{aligned}\right\} \tag{1.2}$$

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as a small parameter ε tends to zero. Here, $\Omega_\varepsilon \subset \mathbb{R}^n$ denotes a thick multi-structure for which the following representation holds true (see Fig. 1 for 3D example):

$$\Omega_\varepsilon = (B \times (0, c)) \bigcup \left(\bigcup_{\mathbf{k} \in \theta_\varepsilon} (\varepsilon C + \varepsilon \mathbf{k}) \times (-d, 0] \right) = \Omega^+ \bigcup \left(\bigcup_{\mathbf{k} \in \theta_\varepsilon} G_\varepsilon^{\mathbf{k}} \right),$$

where $B = (0, a)^{n-1}$ and C are bounded open smooth domains in $\mathbb{R}^{n-1}$ $(n \geqslant 2)$, $C \subset\subset (0, 1)^{n-1}$, $\theta_\varepsilon = \{\mathbf{k} = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}: \varepsilon C + \varepsilon \mathbf{k} \subset\subset B\}$,

$$\begin{gathered} \Omega = B \times (-d, c), \quad G_\varepsilon^{\mathbf{k}} = \{(x', x_n): x' \in \varepsilon C + \varepsilon \mathbf{k}, -d < x_n \leqslant 0\}, \\ \Sigma = B \times \{0\}, \quad \Omega^+ = B \times (0, c), \quad \Omega^- = B \times (-d, 0), \\ \Gamma_0 = B \times \{-d\}, \quad \Omega_\varepsilon^- = \Omega_\varepsilon \cap \Omega^-, \end{gathered} \tag{1.3}$$

Γ_ε is the union of the lower bases $\Gamma_\varepsilon^{\mathbf{k}} = \{(x', x_n): x' \in \varepsilon C + \varepsilon \mathbf{k}, x_n = -d\}$ of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ when $\mathbf{k} \in \theta_\varepsilon$ (i.e. $\Gamma_\varepsilon = \Gamma_0 \cap \partial\Omega_\varepsilon$), S_ε is the union of their boundaries along the axis Ox_n: $S_\varepsilon^{\mathbf{k}} = \{(x', x_n): x' \in \varepsilon\partial C + \varepsilon \mathbf{k}, -d < x_n < 0\}$, k_0 is a positive constant, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative and $q_0 \in L^2(0, T; L^2(\Omega^+))$, $y_\varepsilon^0 \in L^2(\Omega_\varepsilon)$ and $f_\varepsilon \in L^2(0, T; L^2(\Omega))$ are given functions. In the sequel, we shall always assume that $\varepsilon = a/N$, where N is a large positive integer. For this kind of domains and boundary-value problems in Ω_ε, we refer to Brizzi & Chalot (1997), Mel'nyk & Nazarov (1994) and Mel'nyk (2001).

We consider the optimal control problem (1.1), (1.2) assuming that there are two different classes of admissible boundary controls U_ε^a (so-called regular controls) and U_ε^b (so-called contrast controls) which are realized via the Dirichlet boundary conditions posed on the lower bases Γ_ε of the thin cylinders $G_\varepsilon^{\mathbf{k}}$, where

$$u_\varepsilon \in U_\varepsilon^a = \left\{ u|_{\Gamma_\varepsilon}: u \in L^2(0, T; H^1(\Gamma_0)), \|u\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C_0} \right\}, \tag{1.4}$$

$$u_\varepsilon \in U_\varepsilon^b = \left\{ u \in L^2(0, T; H^1(\Gamma_\varepsilon)), \|u\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} \leqslant \mathbf{C_0} \right\}. \tag{1.5}$$

We denote the problems (1.1), (1.2), (1.4) and (1.1), (1.2), (1.5) by $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$, respectively. It is well known that the computational calculation of the optimal solutions of these problems is very complicated through the singularities of the thick junctions Ω_ε. Therefore, the study of their asymptotic behaviour as

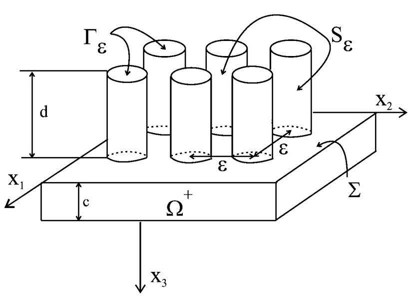

FIG. 1. Thick multi-structure Ω_ε.

the parameter ε tends to zero is one of the main approach to the analysis of optimization problems in such domains. As a rule it gives the possibility to replace the original problems by the corresponding limit problems defined in a more 'simpler' domain with the preservation of the main variational property: both optimal solution and minimal value of the cost functional for the original problem converge to the corresponding characteristics of a limit optimal control problem as ε tends to zero. So, our goal is to obtain an appropriate asymptotical limit for the optimal control problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ as the parameter ε tends to zero.

For comparison we note that the asymptotic analysis of an optimal control problem for linear elliptic equations in the thick multi-structures with U_ε^a-admissible boundary controls was given in the recent work Kogut & Mel'nyk (2004). The optimal control problems for linear parabolic equations in the same domains with unconstrained distributed L^2-controls were studied by De Maio *et al.* (2004). The characteristic feature of considered optimal control problems is the fact that each of these problems has a unique solution for every ε.

The optimal control problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ we consider are such that the existence of an optimal solution for the control problem $\mathbb{P}_\varepsilon^b$ must be taken as probable, but not certainly proved. This circumstance is atypical for the overwhelming majority investigations in this field. We show that the result of homogenization for these problems as $\varepsilon \to 0$, i.e. when the number of attached thin cylinders infinitely increases and their thickness vanishes, essentially depends on the classes of admissible controls. Namely, let $|C|$ be the $(n-1)$-dimensional Lebesgue measure of the set C, $\widetilde{v}_\varepsilon$ be the zero-extension to Ω of a function v defined on Ω_ε and χ_{Ω^+} and χ_{Ω^-} be the characteristic functions of the sets Ω^+ and Ω^-, respectively. Having assumed

$$\widetilde{y}_\varepsilon^0 \to (|C|\chi_{\Omega^-} + \chi_{\Omega^+})y^0 \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0, \tag{1.6}$$

$$\widetilde{f}_\varepsilon \to (|C|\chi_{\Omega^-} + \chi_{\Omega^+})\, f_0 \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0, \tag{1.7}$$

we prove that for the $\mathbb{P}_\varepsilon^a$-problem there exists a unique homogenized one $(\mathbb{P}_{\text{hom}}^a)$ as $\varepsilon \to 0$ that can be represented in the form

$$\left.\begin{aligned}
(y^+)' - \Delta_x y^+ + y^+ &= f_0, && \text{in } (0,T)\times\Omega^+,\\
(y^-)' - \partial_{x_n}^2 y^- + \frac{|C| + k_0|\partial C|_H}{|C|} y^- &= f_0, && \text{in } (0,T)\times\Omega^-,\\
\partial_\nu y^+ &= 0, && \text{in } (0,T)\times\partial\Omega^+\setminus\Sigma,\\
y^- &= u, && \text{on } (0,T)\times\Gamma_0,\\
y^+ = y^-, \quad \partial_{x_n} y^+ &= |C|\partial_{x_n} y^-, && \text{on } (0,T)\times\Sigma,\\
y(0,x) &= y^0(x), && \text{a.e. } x\in\Omega,
\end{aligned}\right\} \tag{1.8}$$

$$u \in U_a = \left\{u \in L^2(0,T;H^1(\Gamma_0))\colon \|u\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C_0}\right\}, \tag{1.9}$$

$$I_a(u,y^+,y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2\, \mathrm{d}x\, \mathrm{d}t + |C| \int_0^T \int_{\Gamma_0} u^2\, \mathrm{d}x'\, \mathrm{d}t \to \inf, \tag{1.10}$$

whereas for the optimal control problem $\mathbb{P}_\varepsilon^b$ the homogenized one ($\mathbb{P}_{\mathrm{hom}}^b$) has another analytical representation

$$\left.\begin{aligned} (y^+)' - \Delta_x y^+ + y^+ &= f_0, && \text{in } (0,T)\times\Omega^+,\\ (y^-)' - \partial_{x_n}^2 y^- + \frac{|C| + k_0|\partial C|_H}{|C|} y^- &= f_0, && \text{in } (0,T)\times\Omega^-,\\ \partial_\nu y^+ &= 0, && \text{in } (0,T)\times\partial\Omega^+\setminus\Sigma,\\ \upsilon_0^- &= |C|^{-1}u, && \text{on } (0,T)\times\Gamma_0,\\ y^+ = y^-, \quad \partial_{x_n} y^+ &= |C|\partial_{x_n} y^-, && \text{on } (0,T)\times\Sigma,\\ y(0,x) &= y^0(x), && \text{a.e. } x\in\Omega, \end{aligned}\right\} \tag{1.11}$$

$$u \in U_b = \left\{ u \in L^2((0,T)\times\Gamma_0)\colon \|u\|_{L^2((0,T)\times\Gamma_0)} \leqslant \sqrt{|C|}\mathbf{C_0} \right\}, \tag{1.12}$$

$$I_b(u, y^+, y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|C|}\int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t \to \inf. \tag{1.13}$$

Here by υ^+ and υ^- we denote the restrictions of a function $\upsilon\colon (0,T)\times\Omega \to \mathbb{R}$ to the sets $(0,T)\times\Omega^+$ and $(0,T)\times\Omega^-$, respectively.

The plan of this paper is as follows. In Section 2, following the approach of Zhikov (2000), we give the description of the Robin boundary conditions for the boundary-value problem (1.2) in terms of the so-called singular measures and reformulate the original optimal control problems. We give *a priori* norm estimate for their solutions and study also the solvability of such problems at a fixed value of ε.

In Section 3, we deal with the question of definition of an appropriate topology for the homogenization of the original optimal control problems (see Definitions 3.3 and 3.4). We prove that any sequence of admissible pairs for the corresponding problem is relatively compact with respect to the so-called w^a- and w^b-convergence, respectively. Section 4 is devoted to the definition of the homogenized problems and their main variational properties. In Section 5, we establish the analytical representation for the limit sets of admissible solutions Ξ_a and Ξ_b. We show that each of these sets can be represented in an explicit form (see Theorems 5.1 and 5.2).

In Section 6, we give the result of identification for the limit cost functionals I_a and I_b. We show that these functionals have different analytical representation and prove the main results of homogenization for problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ as $\varepsilon \to 0$.

2. On solvability of the original optimal control problems

We begin this section with the description of the geometry of the set S_ε in terms of a singular measure in $\mathbb{R}^n$ (see Bouchitte & Fragala, 2001; Zhikov, 2000). Let μ_0 be a periodic finite positive Borel measure in $\mathbb{R}^{n-1}$ with the torus of periodicity $\square = [0,1)^{n-1}$. We assume that the Borel measure μ_0 is the probability measure, concentrated and uniformly distributed on the set ∂C, so $\int_\square \mathrm{d}\mu_0 = 1$.

REMARK 2.1 By definition we have $\mu_0(\square\setminus\partial C) = 0$. Therefore, any functions, taking the same values on the manifold ∂C, coincide as elements of $L^2(\square, \mathrm{d}\mu_0)$. Here, the Lebesgue space $L^2(\square, \mathrm{d}\mu_0)$ is defined in a usual way with the corresponding norm $\|f\|^2_{L^2(\square,\mathrm{d}\mu_0)} = \int_\square |f(x)|^2 \,\mathrm{d}\mu_0$ (we adopt the standard notation $L^2(\square)$ when μ_0 is the Lebesgue measure).

Now, we set $\square_n = \square \times [0,1) = [0,1)^n$ and consider the measure $\mathrm{d}\mu = \mathrm{d}\mu_0 \times \mathrm{d}x_n$ in $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1)$, and for any smooth function g we have

$$\int_{\square_n} g\,\mathrm{d}\mu = \int_0^1 \int_{\square} g\,\mathrm{d}x_n\,\mathrm{d}\mu_0 = [\mathcal{H}^{n-1}(\partial C \times [0,1))]^{-1} \int_{\partial C \times [0,1)} g\,\mathrm{d}\mathcal{H}^{n-1}$$

(see Evans & Gariepy, 1992). However, $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$. Then, using in the sequel the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$, the previous relation can be rewritten in the form

$$\int_{\square_n} g\,\mathrm{d}\mu = \int_0^1 \int_{\square} g\,\mathrm{d}x_n\,\mathrm{d}\mu_0 = |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g\,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.1}$$

For instance, let us consider the plane thick multi-structure $\Omega_\varepsilon \subset \mathbb{R}^2$. Then, $n = 2$ and the set C is some part of the segment $(0,1)$, e.g. $C = \{x_1 \in (0,1) \colon |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed number. So, in this case $|\partial C|_H = 2$ and the 1-periodic measure μ_0 in $\mathbb{R}^1$ can be defined by the rule

$$\mu_0 = \frac{1}{|\partial C|_H}\left(\delta_{M_1} + \delta_{M_2}\right) = \frac{1}{2}\left(\delta_{M_1} + \delta_{M_2}\right), \quad \text{where } M_i = \frac{1}{2} + \left(i - \frac{3}{2}\right)h, \;\; i = 1, 2.$$

Here by δ_{M_i} we denote the Dirac measures located at the points M_i. Thus, the multiplier $|\partial C|_H^{-1}$ in (2.1) is equal to $1/2$.

Let Λ be any Borel set of $\mathbb{R}^n$. We introduce the so-called 'scaling' measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon^n \mu(\varepsilon^{-1}\Lambda)$. This measure has the period ε. Since $\mu(\varepsilon\square_n) = \varepsilon \cdot \mu_0(\varepsilon\square)$ by definition of μ, it follows that

$$\mu_\varepsilon(\varepsilon\square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon\square} \mathrm{d}\mu_0(x'/\varepsilon)\mathrm{d}(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_{\square} \mathrm{d}\mu_0\,\mathrm{d}x_n = \varepsilon^n.$$

It means that the measure μ_ε weakly converges to the Lebesgue measure in $\mathbb{R}^n$ as $\varepsilon \to 0$ (in symbols $\mathrm{d}\mu_\varepsilon \rightharpoonup \mathrm{d}x$), i.e. $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi\,\mathrm{d}x$ for all functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ (see Zhikov, 2000).

Since the Sobolev space $H^1(\Omega_\varepsilon)$ can be viewed as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\left(\int_{\Omega_\varepsilon}(y^2 + |\nabla y|^2)\mathrm{d}x\right)^2$, it follows that $y_\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon))$ is the weak solution of the above-mentioned problem whenever (see Lions, 1971)

$$\begin{aligned}\int_0^T \int_{\Omega_\varepsilon} (-y_\varepsilon\varphi\psi' + \nabla y_\varepsilon \cdot \nabla\varphi\,\psi + y_\varepsilon\varphi\psi)\mathrm{d}x\,\mathrm{d}t + k_0\varepsilon \int_0^T \int_{S_\varepsilon} y_\varepsilon\varphi\psi\,\mathrm{d}x'\,\mathrm{d}t \\ = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon\varphi\psi\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon), \;\; \forall\,\psi \in C_0^\infty(0,T),\end{aligned} \tag{2.2}$$

where by $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$ we denote the set of all functions of $C_0^\infty(\mathbb{R}^n)$ such that $\varphi|_{\Gamma_\varepsilon} = 0$.

Let us consider the last term in the left part of identity (2.2). Using the notations introduced above, we may write down

$$\begin{aligned}k_0\varepsilon \int_0^T \int_{S_\varepsilon} y_\varepsilon\varphi\psi\,\mathrm{d}x'\,\mathrm{d}t &= k_0\varepsilon \int_0^T \left(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C + \mathbf{k}_j)} \int_{-d}^0 y_\varepsilon\varphi\,\mathrm{d}\mathcal{H}^{n-2}\,\mathrm{d}x_n\right)\psi\,\mathrm{d}t \\ &= k_0\varepsilon|\partial C|_H \int_0^T \left(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square + \mathbf{k}_j)} \int_{-d}^0 \widehat{y}_\varepsilon\varphi\varepsilon^{n-2}\,\mathrm{d}\mu_0(x'/\varepsilon)\mathrm{d}x_n\right)\psi\,\mathrm{d}t\end{aligned}$$

$$= k_0|\partial C|_H \int_0^T \left(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square+\mathbf{k}_j)} \int_{-d}^{0} \widehat{y}_\varepsilon \varphi \varepsilon^n \, \mathrm{d}\mu_0(x'/\varepsilon)\mathrm{d}(x_n/\varepsilon) \right) \psi \, \mathrm{d}t$$

$$= k_0|\partial C|_H \int_0^T \left(\sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square+\mathbf{k}_j)} \int_{-d}^{0} \widehat{y}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon \right) \psi \, \mathrm{d}t$$

$$= k_0|\partial C|_H \int_0^T \int_{\Omega^-} \widehat{y}_\varepsilon \varphi \psi \, \mathrm{d}\mu_\varepsilon \, \mathrm{d}t.$$

Here by $\widehat{y}_\varepsilon$ we denote a function of $L^2(0, T; L^2(\Omega^-, \mathrm{d}\mu_\varepsilon))$ taking the same values with y_ε on the set S_ε. Note that the integral $\int_{\Omega^-} \widehat{y}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$. Indeed, since the set Ω^- is bounded and $\widehat{y}_\varepsilon \, \mathrm{d}\mu_\varepsilon$ is a Radon measure, it follows that $\int_{\Omega^-} \widehat{y}_\varepsilon \varphi \, \mathrm{d}\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$.

Let $\mathbf{X}_{\mu_\varepsilon}$ be the vector space of functions $y_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon))$ such that $y_\varepsilon \in L^2(0, T; L^2(\Omega^-, \mathrm{d}\mu_\varepsilon))$, i.e. for any function $y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}$ the integral $\int_0^T \int_{\Omega^-} y_\varepsilon^2 \, \mathrm{d}\mu_\varepsilon$ is well defined. It is easy to see that $\mathbf{X}_{\mu_\varepsilon}$ is the Hilbert space with respect to the following scalar product:

$$(y_\varepsilon, v_\varepsilon)_{\mathbf{X}_{\mu_\varepsilon}} = \int_0^T \int_{\Omega_\varepsilon} (\nabla y_\varepsilon \cdot \nabla v_\varepsilon + y_\varepsilon v_\varepsilon) \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Omega^-} y_\varepsilon v_\varepsilon \, \mathrm{d}\mu_\varepsilon \, \mathrm{d}t.$$

As a result of this motivation, we give the following variational formulation of the initial boundary-value problem (1.2).

DEFINITION 2.1 We say that a function $y_\varepsilon = y_\varepsilon(u_\varepsilon)$ is a weak solution of the parabolic problem (1.2) for a given function $u_\varepsilon \in L^2(0, T; H^1(\Gamma_\varepsilon))$ if

$$\int_0^T \int_{\Omega_\varepsilon} (-y_\varepsilon \varphi \psi' + \nabla y_\varepsilon \cdot \nabla \varphi \psi + y_\varepsilon \varphi \psi) \mathrm{d}x \, \mathrm{d}t + k_0|\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon \varphi \psi \, \mathrm{d}\mu_\varepsilon \, \mathrm{d}t$$
$$= \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon \varphi \psi \, \mathrm{d}x \, \mathrm{d}t, \tag{2.3}$$

$$y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}, \quad y_\varepsilon(0, x) = y_\varepsilon^0 \text{ a.e. } x \in \Omega_\varepsilon, \quad y_\varepsilon|_{\Gamma_\varepsilon} = u_\varepsilon \text{ a.e. } t \in (0, T), \tag{2.4}$$

holds for every $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$ and $\psi \in C_0^\infty(0, T)$.

Then using the standard Hilbert space method, we have the following result.

PROPOSITION 2.1 For any given function $u_\varepsilon \in L^2(0, T; H^1(\Gamma_\varepsilon))$, problem (1.2) admits a unique weak solution in the sense of Definition 2.1 such that

$$y_\varepsilon' \in L^2(0, T; (H^1(\Omega_\varepsilon))'),$$
$$\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} \leqslant c \left(\| f_\varepsilon \|_{L^2((0,T)\times\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))} \right), \quad \forall \varepsilon > 0, \tag{2.5}$$

where a constant $c > 0$ is independent of ε (see Lions, 1971).

Now, we can return to the question on solvability of the optimal control problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$. For this, we rewrite the original problems as follows:

$$(\mathbb{P}_\varepsilon^a)\colon \left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon,y_\varepsilon)\right\rangle,\quad (\mathbb{P}_\varepsilon^b)\colon \left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon)\right\rangle, \tag{2.6}$$

where by Ξ_ε^a and Ξ_ε^b we denote the sets of admissible pairs for the corresponding control problems, i.e.

$$\Xi_\varepsilon^i = \left\{ (u_\varepsilon,y_\varepsilon) \;\middle|\; \begin{array}{r} u_\varepsilon\in U_\varepsilon^i,\quad y_\varepsilon\in \mathbf{X}_{\mu_\varepsilon},\quad y_\varepsilon|_{\Gamma_\varepsilon}=u_\varepsilon,\\ y_\varepsilon(0,x)=y_\varepsilon^0 \quad \text{a.e. } x\in\Omega_\varepsilon,\\ \displaystyle\int_0^T\int_{\Omega_\varepsilon}(-y_\varepsilon\varphi\psi'+\nabla y_\varepsilon\cdot\nabla\varphi\psi+y_\varepsilon\varphi\psi)\mathrm{d}x\,\mathrm{d}t \\ \displaystyle +k_0|\partial C|_H\int_0^T\int_{\Omega^-} y_\varepsilon\varphi\psi\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t=\int_0^T\int_{\Omega_\varepsilon} f_\varepsilon\varphi\psi\,\mathrm{d}x\,\mathrm{d}t,\\ \forall\varphi\in C_0^\infty(\mathbb{R}^n;\Gamma_\varepsilon),\ \forall\psi\in C_0^\infty(0,T)\end{array}\right\},\quad i=a,b. \tag{2.7}$$

Using the direct method of the calculus of variations, it can be easily shown that $\mathbb{P}_\varepsilon^a$-problem has the unique solution for every value $\varepsilon>0$: $I_\varepsilon(u_\varepsilon^a,y_\varepsilon^a)=\inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon,y_\varepsilon)$. As for the $\mathbb{P}_\varepsilon^b$-problem, we observe that its set of admissible pairs Ξ_ε^b is convex and closed in $L^2(0,T;H^1(\Omega_\varepsilon))\times\mathbf{X}_{\mu_\varepsilon}$, and the cost functional (1.1) is strictly convex and lower semicontinuous with respect to the weak topology of $L^2(0,T;H^1(\Omega_\varepsilon))\times L^2(0,T;H^1(\Omega_\varepsilon))$. Hence, we cannot assert the solvability of this problem in general, i.e. the existence of an optimal solution $(u_\varepsilon^b,y_\varepsilon^b)\in\Xi_\varepsilon^b$ for the $\mathbb{P}_\varepsilon^b$-problem must be taken as probable, but not certainly proved. However, if the $\mathbb{P}_\varepsilon^b$-problem is solvable, then its solution is unique. Let τ_ε^a be the product of the weak topologies of $L^2(0,T;H^1(\Gamma_\varepsilon))$ and $L^2(0,T;H^1(\Omega_\varepsilon))$, and τ_ε^b be the product of the weak topologies of $L^2(0,T;L^2(\Gamma_\varepsilon))$ and $L^2(0,T;H^1(\Omega_\varepsilon))$. Let us denote by $\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b$ the closure of the set Ξ_ε^b with respect to the τ_ε^b-topology and consider the following constrained minimization problem (so-called τ_ε^b-relaxed problem for the optimal control problem $\mathbb{P}_\varepsilon^b$): $\left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in \mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon)\right\rangle$. It is clear that this problem is solvable for every ε. Indeed, $\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b$ is a convex, closed and bounded subset of $L^2(0,T;L^2(\Gamma_\varepsilon))\times\mathbf{X}_{\mu_\varepsilon}$, and $I_\varepsilon\colon L^2(0,T;L^2(\Gamma_\varepsilon))\times\mathbf{X}_{\mu_\varepsilon}\to\mathbb{R}$ is the strictly convex τ_ε^b-lower semicontinuous functional. This means that this problem has a unique solution $(u_\varepsilon^*,y_\varepsilon^*)\in\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b$.

THEOREM 2.1 If $(u_\varepsilon^b,y_\varepsilon^b)$ is an optimal pair for $\mathbb{P}_\varepsilon^b$-problem, then $(u_\varepsilon^b,y_\varepsilon^b)$ is the unique solution of τ_ε^b-relaxed problem.

Proof. Let ε be any fixed value (we recall that $\varepsilon=a/N$). Since $\Xi_\varepsilon^b\subset\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b$, we have

$$\inf_{(u_\varepsilon,y_\varepsilon)\in\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon)\leqslant \inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon).$$

Let $(u_\varepsilon^*,y_\varepsilon^*)$ be a solution of τ_ε^b-relaxed problem. Assume that

$$I_\varepsilon(u_\varepsilon^*,y_\varepsilon^*)<\inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon,y_\varepsilon)=I_\varepsilon(u_\varepsilon^b,y_\varepsilon^b)=:\alpha. \tag{2.8}$$

It means that $(u_\varepsilon^*, y_\varepsilon^*) \in \mathrm{cl}_{\tau_\varepsilon^b} \Xi_\varepsilon^b \setminus \Xi_\varepsilon^b$. At the same time there exists a sequence of admissible pairs $\{(u_{\varepsilon,n}, y_{\varepsilon,n}) \subset \Xi_\varepsilon^b \colon n \in \mathbb{N}\}$ such that $(u_{\varepsilon,n}, y_{\varepsilon,n}) \overset{\tau_\varepsilon^b}{\to} (u_\varepsilon^*, y_\varepsilon^*)$. Obviously,

$$I_\varepsilon(u_{\varepsilon,n}, y_{\varepsilon,n}) \geqslant \inf_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon) = \alpha, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

By virtue of the τ_ε^b-lower semicontinuity property of the cost functional I_ε, we just have $\liminf_{n\to\infty} I_\varepsilon(u_{\varepsilon,n}, y_{\varepsilon,n}) \geqslant I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*)$. Then taking into account relation (2.9), we conclude that $I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*) \geqslant \alpha$. However, this contradicts inequality (2.8). As a result,

$$\alpha = I_\varepsilon(u_\varepsilon^*, y_\varepsilon^*) = \inf_{(u_\varepsilon, y_\varepsilon) \in \mathrm{cl}_{\tau_\varepsilon^b} \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon),$$

i.e. $(u_\varepsilon^*, y_\varepsilon^*) \equiv (u_\varepsilon^b, y_\varepsilon^b)$. □

3. Formalism of convergence in variable Banach spaces

It is clear that $\Xi_\varepsilon^a \subset \Xi_\varepsilon^b$, and these inclusions are strict for every fixed ε. So, the problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$ are drastically different from the control theory point of view. It means that the following inequality can be held for every $\varepsilon > 0$:

$$I_\varepsilon(u_\varepsilon^a, y_\varepsilon^a) = \min_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon, y_\varepsilon) > \min_{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

Hence, in the 'limit' as ε tends to zero we can obtain one homogenized problem for the (a)-case and another one for the (b)-case.

To study the asymptotic behaviour of the problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$, we adopt the concept of the variational convergence of constrained minimization problems (see Attouch, 1984; Buttazzo, 1993; Kogut & Leugering, 2001). Then the homogenization procedure can be reduced to the limit analysis of the following sequences:

$$\left\{\left\langle \inf_{(u,y) \in \Xi_\varepsilon^a} I_\varepsilon(u, y)\right\rangle : \varepsilon \to 0\right\}, \quad \left\{\left\langle \inf_{(u,y) \in \Xi_\varepsilon^b} I_\varepsilon(u, y)\right\rangle : \varepsilon \to 0\right\}, \tag{3.1}$$

where the cost functionals $I_\varepsilon \colon \Xi_\varepsilon^i \to \mathbb{R}$, $i = a, b$, and the corresponding sets of admissible pairs are defined in (1.1) and (2.7), respectively.

Note that because of the specific construction of the domains Ω_ε, we have rather delicate situation with the limit passage in (2.6) as $\varepsilon \to 0$. Indeed, each of the admissible pairs $(u_\varepsilon, y_\varepsilon)$ belongs to the corresponding space

$$\mathbb{Y}_\varepsilon := L^2(0, T; H^1(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon} \tag{3.2}$$

and this fact is common as for $\mathbb{P}_\varepsilon^a$-problem so for $\mathbb{P}_\varepsilon^b$-one. Therefore, we focus our attention in this section on working up of the convergence formalism in such spaces.

3.1 *The convergence concept for $\mathbb{P}_\varepsilon^a$-problems*

Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be a sequence of pairs such that $u_\varepsilon \in U_\varepsilon^a$, $y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}$, $\forall \varepsilon > 0$, and

$$\limsup_{\varepsilon>0} \|y_\varepsilon\|^2_{\mathbf{X}_{\mu_\varepsilon}} = \limsup_{\varepsilon>0} \left[\int_0^T \int_{\Omega_\varepsilon} (|\nabla y_\varepsilon|^2 \,\mathrm{d}x + y_\varepsilon^2)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega^-} y_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon\right] < +\infty.$$

It is clear that any sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a\}_{\varepsilon>0}$ satisfies these assumptions.

As the definition of the sets U_ε^a indicates (see (1.4)), for every $\varepsilon > 0$ and for every control function $u_\varepsilon \in U_\varepsilon^a$ there exists an extension operator $P_\varepsilon\colon L^2(0,T;H^1(\Gamma_\varepsilon)) \to L^2(0,T;H^1(\Gamma_0))$ such that $\|P_\varepsilon(u_\varepsilon)\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C_0}$. However, the weak limits of any two weakly convergent sequences $\{P_\varepsilon^{(1)}(u_\varepsilon)\}_{\varepsilon>0}$ and $\{P_\varepsilon^{(2)}(u_\varepsilon)\}_{\varepsilon>0}$ are the same. Indeed, let us assume that

$$P_\varepsilon^{(1)}(u_\varepsilon) \to u_1^* \text{ and } P_\varepsilon^{(2)}(u_\varepsilon) \to u_2 \text{ weakly in } L^2(0,T;H^1(\Gamma_0)).$$

Let $\chi_{\Gamma_\varepsilon}$ be the characteristic function of the set Γ_ε. Since $\chi_{\Gamma_\varepsilon}$ is the $\varepsilon\,\square$-periodic function, it follows that $\chi_{\Gamma_\varepsilon} \to |C|$ weakly-$*$ in $L^2(B)$ as $\varepsilon \to 0$. Then passing to the limit in the integral identity

$$\int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} P_\varepsilon^{(1)}(u_\varepsilon)\varphi(x)\psi(t)\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} P_\varepsilon^{(2)}(u_\varepsilon)\varphi(x)\psi(t)\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\psi \in C_0^\infty(0,T), \;\; \forall\,\varphi \in H^1(\Gamma_0),$$

as ε tends to zero, we just conclude that

$$|C|\int_0^T \int_{\Gamma_0} u_1^*\varphi(x)\psi(t)\mathrm{d}x\,\mathrm{d}t = |C|\int_0^T \int_{\Gamma_0} u_2^*\varphi(x)\psi(t)\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\psi \in C_0^\infty(0,T), \;\; \forall\,\varphi \in H^1(\Gamma_0).$$

Hence, $u_1^* = u_2^*$ and we have obtained the required.

In view of this, we give the following definition.

DEFINITION 3.1 We say that a sequence of controls $\{u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))\}_{\varepsilon>0}$ is weakly convergent to a function u^* with respect to the space $L^2(0,T;H^1(\Gamma_0))$ if some sequence of its images $\{P_\varepsilon(u_\varepsilon)\}_{\varepsilon>0} \subset L^2(0,T;H^1(\Gamma_0))$ converges to u^* weakly in $L^2(0,T;H^1(\Gamma_0))$.

As a consequence, we have the following result.

LEMMA 3.1 Any sequence of admissible controls $\{u_\varepsilon \in U_\varepsilon^a\}_{\varepsilon>0}$ is relatively compact with respect to the weak convergence introduced above. Moreover, its weak limit u^* belongs to the set $U_a = \left\{u \in L^2(0,T;H^1(\Gamma_0)) \,\middle|\, \|u\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C_0}\right\}$.

Now, we give the convergence formalism for the sequences of the type $\left\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\right\}$. By analogy with Brizzi & Chalot (1997), we extend each of the function y_ε by zero into the whole of domain Ω, namely,

$$\widetilde{y}_\varepsilon(x) := \begin{cases} y_\varepsilon(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \Omega \setminus \Omega_\varepsilon, \end{cases} \tag{3.3}$$

and introduce the following functions: $y_\varepsilon^+(x) = y_\varepsilon(x)$ if $x \in \Omega^+$ and $\widetilde{y}_\varepsilon^-(x) = \widetilde{y}_\varepsilon(x)$ if $x \in \Omega^-$. Thanks to the rectilinear boundaries of S_ε with respect to x_n, we have

$$\partial_{x_n}(\widetilde{y}_\varepsilon^-) = \widetilde{\partial_{x_n}(y_\varepsilon^-)} \quad \text{in } \Omega^-. \tag{3.4}$$

This means that $\widetilde{y}_\varepsilon^- \in L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$, where $W_2^{(\mathbf{0},1)}(\Omega^-)$ is the anisotropic Sobolev space $\left\{v \in L^2(\Omega^-)\colon \partial_{x_n} v \in L^2(\Omega^-)\right\}$.

Let χ_C be the $\square$-periodic characteristic function of the set C. It is easy to see that $\chi_C(\cdot/\varepsilon) \to |C|$ weakly-$*$ in $L^\infty(B)$ as $\varepsilon \to 0$, where $B = (0,a)^{n-1}$ and $|C|$ is the $(n-1)$-dimensional Lebesgue measure of C. We recall also that (see Brizzi & Chalot, 1997)

$$\chi_{\Omega_\varepsilon^-} \to |C| \quad \text{weakly-}* \text{ in } L^\infty(\Omega^-) \text{ as } \varepsilon \to 0, \tag{3.5}$$

$$\chi_{\Omega_\varepsilon \cap \Sigma} \to |C| \quad \text{weakly-}* \text{ in } L^\infty(\Sigma) \text{ as } \varepsilon \to 0, \tag{3.6}$$

$$\chi_{\Gamma_\varepsilon} \to |C| \quad \text{weakly-}* \text{ in } L^\infty(\Gamma_0) \text{ as } \varepsilon \to 0. \tag{3.7}$$

DEFINITION 3.2 We say that a sequence $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}_{\varepsilon>0}$ is weakly convergent to a function $y_* = (y_*^+, y_*^-)$ (with respect to the space $L^2(0,T;H^1(\Omega^+) \times W_2^{(\mathbf{0},1)}(\Omega^-)))$ as ε tends to zero (in symbols, $y_\varepsilon \rightsquigarrow y_* = (y_*^+, y_*^-)$) if:

(a) $y_\varepsilon^+ \to y_*^+$ weakly in $L^2(0,T;H^1(\Omega^+))$;

(b) $\widetilde{y}_\varepsilon^- \to |C| y_*^-$ weakly in $L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$.

To show the correctness of this definition, we prove the following compactness property.

PROPOSITION 3.1 Let $\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\}_{\varepsilon>0}$ be a bounded sequence. Then there exist a subsequence $\{y_{\varepsilon'}\}_{\varepsilon'>0}$ and a function

$$y_0 = (y_0^+, y_0^-) \in L^2(0,T;H^1(\Omega^+)) \times L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))$$

such that $y_{\varepsilon'} \rightsquigarrow y_0 = (y_0^+, y_0^-)$.

Proof. In accordance with the initial assumptions, there exists a constant $\mathbf{C} > 0$ independent of ε such that $\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} \leqslant \mathbf{C}$. Hence,

$$\|y_\varepsilon^+\|_{L^2(0,T;H^1(\Omega^+))} + \|\widetilde{y}_\varepsilon^-\|_{L^2(0,T;W_2^{(\mathbf{0},1)}(\Omega^-))} + \|y_\varepsilon\|_{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))} \leqslant \mathbf{C}.$$

Therefore, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and elements $y_0^+ \in L^2(0,T;H^1(\Omega^+))$, $y_0^- \in L^2(0,T;L^2(\Omega^-))$ and $y^* \in L^2(0,T;L^2(\Omega^-))$ such that

$$\left.\begin{aligned} y_\varepsilon^+ &\to y_0^+ && \text{weakly in } L^2(0,T;H^1(\Omega^+)),\\ \widetilde{y}_\varepsilon^- &\to v =: |C| y_0^- && \text{weakly in } L^2(0,T;L^2(\Omega^-)),\\ y_\varepsilon &\to y^* && \text{weakly in the scale } L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon)),\\ \partial_{x_n}\widetilde{y}_\varepsilon^- &\to |C|\partial_{x_n} y_0^- && \text{weakly in } L^2(0,T;L^2(\Omega^-)). \end{aligned}\right\} \tag{3.8}$$

REMARK 3.1 Here, we have used the fact that the bounded sequence $\{y_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the weak convergence in $\{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\}$. Indeed, since $\limsup_{\varepsilon\to 0}\int_0^T\int_{\Omega^-}(y_\varepsilon)^2 \mathrm{d}\mu_\varepsilon \mathrm{d}t < +\infty$, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by ε) and an element $y^* \in L^2(0,T;L^2(\Omega^-))$ such that (see Zhikov, 2000) $\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^-}\varphi\psi y_\varepsilon \,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t = \int_0^T\int_{\Omega^-}\varphi\psi y^*\,\mathrm{d}x\,\mathrm{d}t$ for any functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R})$.

Note also that the last limit in (3.8) is the consequence of (3.4). Moreover, by analogy with Brizzi & Chalot (1997) one can easily prove the following relation:

$$y_0^+ = y_0^- \qquad \text{a.e. on } (0,T)\times\Sigma, \tag{3.9}$$

i.e. in this case the traces of the limit functions (y_0^+, y_0^-) coincide on Σ.

Obviously, to establish the fact $y_\varepsilon \rightsquigarrow (y_0^+, y_0^-)$, it suffices to prove that

$$y^* = y_0^- \quad \text{a.e. in } (0,T)\times\Omega^-. \tag{3.10}$$

For this, we introduce some periodic finite Borel measure ν on $\mathbb{R}^n$. Let $\square_n = [0,1)^n$ be the cell of periodicity for ν. Assume that ν is the probability measure, concentrated and uniformly distributed on the set $C \times [0,1)$, so $\int_{\square_n} \mathrm{d}\nu = 1$. It is easy to see that for any smooth function, the equality

$$\int_{\square_n} f\,\mathrm{d}\nu = [\mathcal{L}^n(C\times[0,1))]^{-1}\int_{C\times[0,1)} f\,\mathrm{d}x = |C|^{-1}\int_{C\times[0,1)} f\,\mathrm{d}x \tag{3.11}$$

is valid. Now, define a scaling measure ν_ε by the relation $\nu_\varepsilon(A) = \varepsilon^n\nu(\varepsilon^{-1}A)$, where A is an arbitrary Borel set in $\mathbb{R}^n$ and $\varepsilon^{-1}A = \{\varepsilon^{-1}x, x \in A\}$. Then the measure ν_ε is ε-periodic, and $\nu_\varepsilon(\varepsilon\square) = \varepsilon^n\int_{\square_n}\mathrm{d}\nu = \varepsilon^n$. Therefore, this measure weakly converges to the Lebesgue measure as $\varepsilon \to 0$, i.e. $\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\varphi\,\mathrm{d}\nu_\varepsilon = \int_{\mathbb{R}^n}\varphi\,\mathrm{d}x$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. It means that the weak limits of both the sequences $\{y_\varepsilon^- \in L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\}$ and $\{y_\varepsilon^- \in L^2(0,T;L^2(\Omega^-,\mathrm{d}\nu_\varepsilon))\}$ in the sense of Remark 4.1 have to be the same, namely,

$$\begin{aligned}\lim_{\varepsilon\to0}\int_0^T\int_{\Omega^-}\varphi\psi y_\varepsilon^-\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t &= \int_0^T\int_{\Omega^-}\varphi\psi y^*\,\mathrm{d}x\,\mathrm{d}t\\ &= \lim_{\varepsilon\to0}\int_0^T\int_{\Omega^-}\varphi\psi y_\varepsilon^-\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t.\end{aligned} \tag{3.12}$$

At the same time for every function $y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}$ and every fixed ε, the set Ω^- can be covered by a system of cubes with edges ε. We denote these cubes by the symbols $\varepsilon(\square + \mathbf{k}_j)$. Then in accordance with the definition of measure ν_ε, we may write down

$$\begin{aligned}\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t &= |C|\int_0^T\sum_j\int_{\varepsilon(\square+\mathbf{k}_j)} y_\varepsilon^-\varphi\psi\varepsilon^n\,\mathrm{d}\nu(x/\varepsilon)\mathrm{d}t\\ &= |C|\int_0^T\int_{\Omega^-} y_\varepsilon^-\varphi\psi\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t,\end{aligned} \tag{3.13}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R})$. Now, using (3.13) and (3.12) and taking into account the fact that $\widetilde{y}_\varepsilon^- \to |C|y_0^-$ weakly in $L^2(0,T;L^2(\Omega^-))$ as $\varepsilon \to 0$, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega^-}|C|y_0^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\varepsilon\to0}\int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\\ &= \lim_{\varepsilon\to0}|C|\int_0^T\int_{\Omega^-}y_\varepsilon^-\varphi\psi\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t = |C|\int_0^T\int_{\Omega^-}y^*\varphi\psi\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R})$. Hence, $y^* = y_0^-$ and we obtain the required proposition. □

DEFINITION 3.3 We say that a sequence $\{(u_\varepsilon, y_\varepsilon) \in \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ is w^a-convergent to a triplet (u, y^+, y^-) as ε tends to zero (in symbols, $(u_\varepsilon, y_\varepsilon) \overset{w^a}{\to} (u, y^+, y^-)$) if $u_\varepsilon \to u$ in the sense of Definition 3.1 and $y_\varepsilon \rightsquigarrow (y^+, y^-)$ in the sense of Definition 3.2 (here the space $\mathbb{Y}_\varepsilon$ is defined in (3.2)).

As follows from the result obtained above and estimate (2.5) the following statement holds.

PROPOSITION 3.2 Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^a\}_{\varepsilon>0}$ be any sequence of admissible pairs for the $\widehat{\mathbb{P}}_\varepsilon^a$-problem. Then there exist a subsequence $\{(u_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ and a triplet

$$(u, y^+, y^-) \in \mathbb{Y}_0^a := L^2(0,T; H^1(\Gamma_0)) \times L^2(0,T; H^1(\Omega^+)) \times L^2(0,T; W_2^{(\mathbf{0},1)}(\Omega^-)) \tag{3.14}$$

such that $u \in U_a$ and $(u_{\varepsilon'}, y_{\varepsilon'}) \overset{w^a}{\to} (u, y^+, y^-)$, where

$$U_a = \left\{ u \in L^2(0,T; H^1(\Gamma_0)) \middle| \|u\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C_0} \right\}. \tag{3.15}$$

3.2 *The convergence concept for $\mathbb{P}_\varepsilon^b$-problems*

Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be any sequence of admissible pairs for $\mathbb{P}_\varepsilon^b$-problems. Since we cannot assert in this case the existence of extension operators $P_\varepsilon : U_\varepsilon^b \to L^2(0,T; H^1(\Gamma_0))$ that would be uniformly bounded with respect to ε, it follows that we have to give another convergence concept in the variable space (3.2). Let us denote by $\widetilde{u}_\varepsilon$ the extension by zero of a function $u_\varepsilon \in L^2(0,T; H^1(\Gamma_\varepsilon))$ into Γ_0. Then $\widetilde{u}_\varepsilon \in L^2(0,T; L^2(\Gamma_0))$.

DEFINITION 3.4 A sequence $\left\{(u_\varepsilon, y_\varepsilon) \in L^2(0,T; H^1(\Gamma_\varepsilon)) \times \mathbf{X}_{\mu_\varepsilon}\right\}_{\varepsilon>0}$ is said to be w^b-convergent to a triplet (u, y^+, y^-) $\left(\text{in symbols, } (u_\varepsilon, y_\varepsilon) \overset{w^b}{\to} (u, y^+, y^-)\right)$ as ε tends to zero if $\widetilde{u}_\varepsilon \to u$ weakly in $L^2(0,T; L^2(\Gamma_0))$ and $y_\varepsilon \rightsquigarrow (y^+, y^-)$ in the sense of Definition 3.3.

Then, taking the definition of the sets U_ε^b, estimate (2.5) and Proposition 3.1 into account, we have the following obvious result.

PROPOSITION 3.3 Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon^b\}_{\varepsilon>0}$ be a sequence of admissible pairs for the $\mathbb{P}_\varepsilon^b$-problems such that $\sup_{\varepsilon>0} \|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. Then there exist a subsequence $\{(u_{\varepsilon'}, y_{\varepsilon'})\}_{\varepsilon'>0}$ and a triplet

$$(u, y^+, y^-) \in \mathbb{Y}_0^b := L^2(0,T; L^2(\Gamma_0)) \times L^2(0,T; H^1(\Omega^+)) \times L^2(0,T; W_2^{(\mathbf{0},1)}(\Omega^-)) \tag{3.16}$$

for which $(u_{\varepsilon'}, y_{\varepsilon'}) \overset{w^b}{\to} (u, y^+, y^-)$ as $\varepsilon' \to 0$.

Let us denote by τ^a the topology associated with w^a-convergence in $\mathbb{Y}_0^a$ and by τ^b the topology associated with w^b-convergence in $\mathbb{Y}_0^b$. Then as follows from Propositions 3.2 and 3.3, these topologies can be taken as the most natural ones for the homogenization of the optimal control problems $\mathbb{P}_\varepsilon^a$ and $\mathbb{P}_\varepsilon^b$, respectively.

4. Definition of the homogenized problems and their properties

As follows from the previous sections, each of the sets of admissible solutions Ξ_ε^i $(i = a, b)$ belongs to the corresponding Banach space ($\Xi_\varepsilon^i \subset \mathbb{Y}_\varepsilon$). We introduce the convergence concept of such sets using the w^i-sequential version of the set convergence in Kuratowski's sense (see Pankov, 1997; hereinafter $i = a, b$).

DEFINITION 4.1 We say that a set $\Xi_i \subset \mathbb{Y}_0^i$ is the sequential w^i-limit in the Kuratowski's sense (or $K(w^i)$-limit) of the sequence $\{\Xi_\varepsilon^i \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ if the following conditions are satisfied:

1. for every triplet $(u, y^+, y^-) \in \Xi_i$, there exist a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ w^i-converging to (u, y^+, y^-) and a positive value $\varepsilon_0 > 0$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi^i_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$;
2. for every sequence of admissible pairs $\{(u_k, y_k) \in \Xi^i_{\varepsilon_k}\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ and $(u_k, y_k) \stackrel{w^i}{\to} (u, y^+, y^-)$ as $k \to \infty$, the triplet (u, y^+, y^-) belongs to Ξ_i.

Let us show that the sets of admissible pairs for the $\mathbb{P}^a_\varepsilon$-problems possess the compactness property with respect to the $K(w^a)$-convergence.

THEOREM 4.1 For the sequence of sets $\{\Xi^a_\varepsilon\}_{\varepsilon>0}$, there exist a subsequence $\{\Xi^a_{\varepsilon'}\}_{\varepsilon'>0}$ and a set $\Xi_a \subset \mathbb{Y}^a_0$ such that $K(w^a) - \lim_{\varepsilon'\to 0} \Xi^a_{\varepsilon'} = \Xi_a$.

Proof. We begin with the obvious fact that the w^a-convergence of any sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon\}_{\varepsilon>0}$ is equivalent to the weak convergence of its image $\{(P_\varepsilon(u_\varepsilon), y^+_\varepsilon, \widetilde{y}^-_\varepsilon)\}_{\varepsilon>0}$ in the space $\mathbb{Y}^a_0 = L^2(0, T; H^1(\Gamma_0)) \times L^2(0, T; H^1(\Omega^+)) \times L^2(0, T; W^{(\mathbf{0},1)}_2(\Omega^-))$. Since the space $\mathbb{Y}^a_0$ is separable and reflexive, there exists a metric d such that for any sequence $\{p_k = (w_k, y_k, v_k)\}_{k\in\mathbb{N}}$ in $\mathbb{Y}^a_0$ the following conditions are equivalent (see, e.g. Dunford & Schwartz, 1957):
(j) $\{p_k\} \to p = (w, y, v)$ weakly in $\mathbb{Y}^a_0$; (jj) $\{p_k\}$ is bounded in $\mathbb{Y}^a_0$ and $d(p_k, p) \to 0$ as $k \to \infty$.
Let η be the topology associated to the metric d on $\mathbb{Y}^a_0$ and $\{\widetilde{\Xi}^a_\varepsilon\}_{\varepsilon>0}$ be the image sequence of the sets Ξ^a_ε in $\mathbb{Y}^a_0$, i.e. $\widetilde{\Xi}^a_\varepsilon = \{(P_\varepsilon(u_\varepsilon), y^+_\varepsilon, \widetilde{y}^-_\varepsilon) : (u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon\}$.

Since the η-topology has a countable base, by the Kuratowski compactness theorem (see Dal Maso, 1993) there exists a subsequence of $\{\widetilde{\Xi}^a_\varepsilon\}_{\varepsilon>0}$ still denoted by $\{\widetilde{\Xi}^a_\varepsilon\}_{\varepsilon>0}$ that $K(\eta)$-converges to a set $A \subset \mathbb{Y}^a_0$. Now, we prove that the set A coincides with $K(w^a)$-limit of the family $\{\Xi^a_\varepsilon\}_{\varepsilon>0}$. With this aim, it is enough to show that

$$\Xi_a \subseteq A, \tag{4.1}$$

$$A \subseteq \Xi_a, \tag{4.2}$$

where by Ξ_a we denoted the $K(w^a)$-limit of the sequence $\{\Xi^a_\varepsilon\}_{\varepsilon>0}$ in the sense of Definition 4.1.

First, let us verify inclusion (4.1). Let (u, y^+, y^-) be any triplet in $\mathbb{Y}^a_0$ for which one can found a sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}}$, w^a-converging to (u, y^+, y^-), and a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $(u_k, y_k) \in \Xi^a_{\varepsilon_k}$ for every $k \in \mathbb{N}$. Then $(u, y^+, y^-) \in \Xi^a_0$ by Definition 4.1. Let $\{P_k\}$ be any sequence of the extension operators $P_k: L^2(0, T; H^1(\Gamma_\varepsilon)) \to L^2(0, T; H^1(\Gamma_0))$. Then $\{(P_k u_k, y^+_k, \widetilde{y}^-_k)\} \to (u, y^+, y^-)$ weakly in $\mathbb{Y}^a_0$, and $(P_k u_k, y^+_k, \widetilde{y}^-_k) \in \widetilde{\Xi}^a_{\varepsilon_k}$ for every $k \in \mathbb{N}$. Therefore, the equivalence between conditions (j) and (jj) yields η-convergence of $\{(P_k u_k, y^+_k, \widetilde{y}^-_k)\}$ to (u, y^+, y^-). Hence, $(u, y^+, y^-) \in A$ by definition of Kuratowski's limit. So, inclusion (4.1) is proved.

Now we verify (4.2). Let (u, y^+, y^-) be any triplet of A. Then there exists a sequence $\{(v_\varepsilon, p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ η-converging to (u, y^+, y^-) such that $(v_\varepsilon, p_\varepsilon, q_\varepsilon) \in \widetilde{\Xi}_\varepsilon$ for ε small enough. It follows that each pair $(p_\varepsilon, q_\varepsilon)$ can be represented as $p_\varepsilon = y^+_\varepsilon$, $q_\varepsilon = \widetilde{y}^-_\varepsilon$, where y_ε is a weak solution of the boundary-value problem (1.2) under $u_\varepsilon = v_\varepsilon|_{\Gamma_\varepsilon}$. However, the realization of the condition $(v_\varepsilon, p_\varepsilon, q_\varepsilon) \in \widetilde{\Xi}_\varepsilon$ implies that the pair $(u_\varepsilon, y_\varepsilon)$ is admissible, i.e. $(u_\varepsilon, y_\varepsilon) \in \Xi^a_\varepsilon$. Since the sequence of functions $\{v_\varepsilon\}$ is bounded in $L^2(0, T; H^1(\Gamma_0))$ ($v_\varepsilon \in U_a$ by definition of the sets $\widetilde{\Xi}_\varepsilon$), we get that the sequence of corresponding solutions $\{(p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^2(0, T; H^1(\Omega^+)) \times L^2(0, T; W^{(\mathbf{0},1)}_2(\Omega^-))$ as well. Hence, the equivalence between conditions (j) and (jj) yields the weak convergence of the sequence $\{(v_\varepsilon, p_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ to (u, y^+, y^-). But in view of Definition 3.3, it is equivalent to the w^a-convergence of its prototype $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ to (u, y^+, y^-). Thus, $(u, y^+, y^-) \in \Xi_a$ by Definition 4.1. Thus, the theorem is proved. □

DEFINITION 4.2 We say that $\mathbb{P}_\varepsilon^b$-problem satisfies the property $(\mathcal{N})$ if for any

$$u \in U_b = \left\{ v \in L^2((0,T)\times\Gamma_0) \colon \|v\|_{L^2((0,T)\times\Gamma_0)} \leqslant \sqrt{|C|}\mathbf{C_0} \right\}$$

there exists a sequence $\{u_\varepsilon \in U_\varepsilon^b\}_{\varepsilon>0}$ such that $\widetilde{u}_\varepsilon \to u$ weakly in $L^2((0,T)\times\Gamma_0)$ and $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))} < +\infty$.

Then the following compactness property for the sets Ξ_e^b with respect to the $K(w^b)$-convergence takes place.

THEOREM 4.2 If $\mathbb{P}_\varepsilon^b$-problem possesses the $(\mathcal{N})$-property, then the sequence of the sets $\{\Xi_\varepsilon^b \in \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ has a subsequence (still denoted by ε) for which the exists a non-empty set $\Xi_b \subset \mathbb{Y}_0^b$ which is the $K(w^b)$-limit of $\{\Xi_\varepsilon^b\}_{\varepsilon>0}$ with respect to the space $\mathbb{Y}_0^b$.

REMARK 4.1 It is well known (see Dal Maso, 1993) that the Kuratowski's limit A_0 of a sequence of subsets $\{A_n\}_{n\in\mathbb{N}}$ in a topological space (X,τ) does not change if we replace the sets A_n by their τ-closures, i.e.

$$K(\tau)-\lim_{n\to\infty} A_n = A_0 = K(\tau)-\lim_{n\to\infty} \mathrm{cl}_\tau A_n$$

(τ_ε^b denotes the product of the weak topologies of the spaces $L^2(0,T;L^2(\Gamma_\varepsilon))$ and $L^2(0,T;H^1(\Omega_\varepsilon))$). Thus, the $K(w^b)$-limit of $\{\Xi_\varepsilon^b\}_{\varepsilon>0}$ coincides with the $K(w^b)$-limit of τ_ε^b-closures $\left\{\mathrm{cl}_{\tau_\varepsilon^b}\Xi_\varepsilon^b \subset L^2(0,T;L^2(\Gamma_\varepsilon))\times \mathbf{X}_{\mu_\varepsilon}\right\}$.

Let us turn back to the main object of this section, namely, to the sequences of constrained minimization problems (3.1). Using the concept of variational convergence (see Kogut & Leugering, 2001) we give the definition of the 'appropriate limits' for these sequences.

DEFINITION 4.3 We say that the minimization problem

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_i} I_i(u,y^+,y^-) \right\rangle \quad (i=a,b), \tag{4.3}$$

where $\Xi_i \subset \mathbb{Y}_0^i$, is the variational w^i-limit of the sequence (3.1) with respect to the w^i-convergence if: (i) Ξ_i is the $K(w^i)$-limit of the sets $\{\Xi_\varepsilon^i\}$; (ii) for any triplet $(u,y^+,y^-)\in\Xi_i$ and for any sequence $\left\{(u_k,y_k)\in\Xi_{\varepsilon_k}^i\right\}_{k\in\mathbb{N}}$ such that $\varepsilon_k\to 0$ and $(u_k,y_k)\overset{w^i}{\Longrightarrow}(u,y^+,y^-)$ as $k\to\infty$, we have

$$I_i(u,y^+,y^-) \leqslant \liminf_{k\to\infty} I_{\varepsilon_k}(u_k,y_k); \tag{4.4}$$

(iii) for every triplet $(u,y^+,y^-)\in\Xi_i$, there exist a positive constant ε_0 and a sequence $\{(u_\varepsilon,y_\varepsilon)\}_{\varepsilon>0}$ such that

$$(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon^i \quad \text{for every}\varepsilon\leqslant\varepsilon_0;\quad (u_\varepsilon,y_\varepsilon)\overset{w^i}{\Longrightarrow}(u,y^+,y^-);\ I_i(u,y^+,y^-)\geqslant \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon,y_\varepsilon). \tag{4.5}$$

REMARK 4.2 In fact, Definition 4.3 is the natural extension of the well-known notion of Γ-convergence. We will prove that the variational w^i-convergence of the corresponding sequence (3.1) to problem (4.3) implies the convergence of the minimum values of I_ε on Ξ_ε^i to the minimum one of I_i on Ξ_i; in addition every w^i-cluster 'point' of the sequence of the minimizers for I_ε is the minimizer for I_i.

THEOREM 4.3 Assume that the constrained minimization problem

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-) \right\rangle \tag{4.6}$$

is the variational w^a-limit of the corresponding sequence (3.1) and this problem has a unique solution $(u^a,(y^a)^+,(y^a)^-)$ in Ξ_a. Let $\{(u^a_\varepsilon,y^a_\varepsilon)\in\Xi^a_\varepsilon\}_{\varepsilon>0}$ be a sequence of the optimal pairs for $\mathbb{P}^a_\varepsilon$-problem. Then

$$(u^a_\varepsilon,y^a_\varepsilon)\overset{w^a}{\to}(u^a,(y^a)^+,(y^a)^-) \quad \text{as } \varepsilon\to 0, \tag{4.7}$$

and furthermore

$$\inf_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-) = I_a(u^a,(y^a)^+,(y^a)^-) = \lim_{\varepsilon\to 0} I_\varepsilon(u^a_\varepsilon,y^a_\varepsilon). \tag{4.8}$$

Proof. Let $\left\{\left(u^a_{\varepsilon_k},y^a_{\varepsilon_k}\right)\in\Xi^a_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ be any w^a-convergent subsequence of the sequence of minimizers $\{(u^a_\varepsilon,y^a_\varepsilon)\}_{\varepsilon>0}$. Note that in view of Proposition 3.2 such choice is always possible. Let $(u^*,(y^*)^+,(y^*)^-)$ be its w^a-limit. Then, by Definition 4.1, we have $(u^*,(y^*)^+,(y^*)^-)\in\Xi_a$. Moreover, due to part (ii) of Definition 4.3,

$$\begin{aligned}\liminf_{k\to\infty}\min_{(u,y)\in\Xi^a_{\varepsilon_k}} I_{\varepsilon_k}(u,y) &= \liminf_{k\to\infty} I_{\varepsilon_k}\left(u^a_{\varepsilon_k},y^a_{\varepsilon_k}\right)\geqslant I_a(u^*,(y^*)^+,(y^*)^-)\\ &\geqslant \min_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-) = I_a(u^a,(y^a)^+,(y^a)^-),\end{aligned} \tag{4.9}$$

where $(u^a,(y^a)^+,(y^a)^-)\in\Xi_a$ is the unique solution of the limit problem (4.6).

As follows from Definition 4.3 (see (iii)) there exist a constant $\varepsilon^0>0$ and a sequence $\{(u^a_\varepsilon,y^a_\varepsilon)\}$ such that $(u^a_\varepsilon,y^a_\varepsilon)\in\Xi^a_\varepsilon$ for all values $\varepsilon\in(0,\varepsilon^0)$, $(u^a_\varepsilon,y^a_\varepsilon)\overset{w^a}{\to}(u^a,(y^a)^+,(y^a)^-)$ as $\varepsilon\to 0$ and $I_a(u^a,(y^a)^+,(y^a)^-)\geqslant\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon,y_\varepsilon)$. Using this fact, we get

$$\begin{aligned}\min_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-) &= I_a(u^a,(y^a)^+,(y^a)^-)\geqslant\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon,y_\varepsilon)\\ &\geqslant\limsup_{\varepsilon\to 0}\min_{(u,y)\in\Xi^a_\varepsilon} I_\varepsilon(u,y)\geqslant\limsup_{k\to\infty}\min_{(u,y)\in\Xi^a_{\varepsilon_k}} I_{\varepsilon_k}(u,y)\\ &=\limsup_{k\to\infty} I_{\varepsilon_k}(u^a_{\varepsilon_k},y^a_{\varepsilon_k}).\end{aligned} \tag{4.10}$$

From (4.9) it follows that $\liminf_{k\to\infty} I_{\varepsilon_k}\left(u^a_{\varepsilon_k},y^a_{\varepsilon_k}\right)\geqslant\limsup_{k\to\infty} I_{\varepsilon_k}\left(u^a_{\varepsilon_k},y^a_{\varepsilon_k}\right)$. Combining (4.9) and (4.10), we conclude that $I_a(u^a,(y^a)^+,(y^a)^-)=\lim_{k\to\infty}\min_{(u,y)\in\Xi^a_{\varepsilon_k}} I_{\varepsilon_k}(u,y)$ and

$$I_a(u^*,(y^*)^+,(y^*)^-) = I_a(u^a,(y^a)^+,(y^a)^-) = \min_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-).$$

Taking into account these relations and the uniqueness of the solution to problem (4.3), we obtain $(u^*,(y^*)^+,(y^*)^-)=(u^a,(y^a)^+,(y^a)^-)$. Since this equality holds for the limits of any converging subsequences of $\{(u^a_\varepsilon,y^a_\varepsilon)\}_{\varepsilon>0}$, it yields that $(u^a,(y^a)^+,(y^a)^-)$ is the w^a-limit of the sequence $\{(u^a_\varepsilon,y^a_\varepsilon)\}_{\varepsilon>0}$. Making for the sequence of minimizers what we did before with the subsequence

$\{(u^a_{\varepsilon_k}, y^a_{\varepsilon_k})\}_{k\in\mathbb{N}}$, we obtain

$$\liminf_{\varepsilon\to 0} \min_{(u,y)\in\Xi^a_\varepsilon} I_\varepsilon(u,y) = \liminf_{\varepsilon\to 0} I_\varepsilon(u^a_\varepsilon, y^a_\varepsilon) \geqslant I_a(u^a, (y^a)^+, (y^a)^-)$$

$$= \min_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-) \geqslant \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \geqslant \limsup_{\varepsilon\to 0} \min_{(u,y)\in\Xi^a_\varepsilon} I_\varepsilon(u,y)$$

$$= \limsup_{\varepsilon\to 0} I_\varepsilon(u^a_\varepsilon, y^a_\varepsilon).$$

Thus, the relations (4.8) hold. □

Using the same arguments and taking into account the $(\mathcal{N})$-property, one can prove the analogous result for the variational w^b-limits.

THEOREM 4.4 Assume that the constrained minimization problem

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_b} I_b(u,y^+,y^-)\right\rangle \tag{4.11}$$

is the variational w^b-limit of the corresponding sequence (3.1) and this problem has a unique solution $(u^b, (y^b)^+, (y^b)^-) \in \Xi_b \subset \mathbb{Y}^b_0$. Let $\{(u^b_\varepsilon, y^b_\varepsilon) \in \Xi^b_\varepsilon\}_{\varepsilon>0}$ be a sequence of optimal pairs of $\mathbb{P}^b_\varepsilon$-problems such that $\sup_{\varepsilon>0} \|y^b_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. Then $(u^b_\varepsilon, y^b_\varepsilon) \overset{w^b}{\to} (u^b, (y^b)^+, (y^b)^-)$ as $\varepsilon \to 0$, and furthermore

$$\inf_{(u,y^+,y^-)\in\Xi_b} I_b(u,y^+,y^-) = I_b(u^b, (y^b)^+, (y^b)^-) = \lim_{\varepsilon\to 0} I_\varepsilon(u^b_\varepsilon, y^b_\varepsilon).$$

DEFINITION 4.4 We say that the family of optimal control problems $\{\mathbb{P}^i_\varepsilon\}_{\varepsilon>0}$ $(i = a, b)$ admits the homogenization as $\varepsilon \to 0$ with respect to the w^i-convergence if for the corresponding sequence of constrained minimization problems (3.1), there exists a variational limit which can be represented in the form of some optimal control problem. This problem will be called the homogenized one for $\mathbb{P}^i_\varepsilon$.

5. Analytical representation of the limit sets of admissible solutions

The main objects of our consideration in this section are the sequences of the sets of admissible pairs $\{\Xi^a_e \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ and $\{\Xi^b_e \subset \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ and its Kuratowski's limits with respect to w^a- and w^b -convergence, respectively. In view of Theorems 4.1 and 4.2, we may always suppose that for that sequences there exist sets Ξ_a and Ξ_b such that $\Xi_a = K(w^a) - \lim_{\varepsilon\to 0} \Xi^a_e$ and $\Xi_b = K(w^b) - \lim_{\varepsilon\to 0} \Xi^b_e$. To formulate our next results, we introduce the following space $\mathcal{V}(\Omega) = \left\{y \in L^2(\Omega)\colon \frac{\partial y}{\partial x_n} \in L^2(\Omega^-), y \in H^1(\Omega^+)\right\}$ and endow it with the scalar product

$$(y,v)_{\mathcal{V}(\Omega)} = \int_{\Omega^+} \nabla y \cdot \nabla v \,\mathrm{d}x + \int_{\Omega^+} yv \,\mathrm{d}x + |C| \int_{\Omega^-} \partial_{x_n} y \partial_{x_n} v \,\mathrm{d}x + (|C| + k_0|\partial C|_H) \int_{\Omega^-} yv \,\mathrm{d}x.$$

By analogy with Esposito *et al.* (1997), it can easily be shown that $\mathcal{V}(\Omega)$ is a Hilbert space and $H^1(\Omega)$ is dense in $\mathcal{V}(\Omega)$. Moreover, as follows from (3.8–3.9), for any function $y_* = (y^+_*, y^-_*)$, which is a weak limit in the sense of Definition 3.2, we have (1) $y_* = (y^+_*, y^-_*) \in L^2(0,T;\mathcal{V}(\Omega))$ and (2) $y^+_*(\cdot,\cdot) = y^-_*(\cdot,\cdot)$ almost everywhere on $(0,T) \times \Sigma$. Moreover, it should be stressed here that any function of $\mathcal{V}(\Omega)$ has a trace on any hyperplane L in Ω^- such that $L = \{(x', x_n) \in \Omega^-\colon x_n = \text{constant}\}$.

5.1 *Recovery of the set Ξ_a*

The crucial point in the study of the $K(w^a)$-limit properties for the sequence of admissible pairs is the following result.

LEMMA 5.1 Let $\{u_\varepsilon\}_{\varepsilon>0}$ be any sequence of admissible controls for $\mathbb{P}^a_\varepsilon$-problems which is weakly convergent to a function u_0 in $L^2(0,T;H^1(\Gamma_0))$. Let $\left\{y_\varepsilon \in \mathbf{X}_{\mu_\varepsilon}\right\}$ be the corresponding solutions of problem (1.2). Then $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u_0, v_0^+, v_0^-)$ as $\varepsilon \to 0$, where

$$v_0(x) = \begin{cases} v_0^+(x), & x \in \Omega^+, \\ v_0^-(x), & x \in \Omega^-, \end{cases} \tag{5.1}$$

is a unique weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the following limit problem:

$$\left.\begin{aligned} (v_0^+)' - \Delta_x v_0^+ + v_0^+ &= f_0, && \text{in } (0,T)\times\Omega^+, \\ (v_0^-)' - \partial^2_{x_n} v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|} v_0^- &= f_0, && \text{in } (0,T)\times\Omega^-, \\ \partial_\nu v_0^+ &= 0, && \text{in } (0,T)\times\partial\Omega^+\setminus\Sigma, \\ v_0^- &= u_0, && \text{on } (0,T)\times\Gamma_0, \\ v_0^+ = v_0^-, \quad \partial_{x_n} v_0^+ &= |C|\partial_{x_n} v_0^-, && \text{on } (0,T)\times\Sigma, \\ v_0(0,x) &= y^0(x), && \text{a.e. } x\in\Omega. \end{aligned}\right\} \tag{5.2}$$

REMARK 5.1 Here, the weak formulation of problem (5.2) means that

$$\left.\begin{aligned} & v_0 \in L^2(0,T;\mathcal{V}(\Omega)), \\ & -\int_0^T \int_\Omega (\chi_{\Omega^+} + |C|\chi_{\Omega^-}) v_0 \varphi \psi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T (v_0,\varphi)_{\mathcal{V}(\Omega)} \psi \,\mathrm{d}t \\ & = \int_0^T \int_\Omega (\chi_{\Omega^+} + |C|\chi_{\Omega^-}) f_0 \varphi\psi \,\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\varphi\in\mathcal{V}(\Omega;\Gamma_0), \\ & \qquad \forall\,\psi \in C_0^\infty(0,T), \\ & v_0^- = u_0 \text{ on } (0,T)\times\Gamma_0, \quad v_0(0,x) = y^0(x) \text{ a.e. } x\in\Omega, \end{aligned}\right\} \tag{5.3}$$

where

$$\mathcal{V}(\Omega;\Gamma_0) = \left\{ v \in L^2(\Omega) \colon \partial_{x_n} v \in L^2(\Omega^-), v \in H^1(\Omega^+), v = 0 \text{ a.e. on } \Gamma_0 \right\}.$$

Moreover, in this case we have $v_0' \in L^2(0,T;(\mathcal{V}(\Omega))')$ (see Lions, 1971, p. 107).

Proof. From Proposition 3.1, it follows that there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a triplet $(u_0, v_0^+, v_0^-) \in \mathbb{Y}^a_0$ such that $(u_\varepsilon, y_\varepsilon) \xrightarrow{w^a} (u_0, v_0^+, v_0^-)$ as $\varepsilon \to 0$. Similar to the proof of

Proposition 3.1 (see relations (3.8) and (3.9)), we can show that

$$\left.\begin{array}{ll} y_\varepsilon^+ \to v_0^+ & \text{weakly in } L^2(0,T;H^1(\Omega^+)),\\ \widetilde{y}_\varepsilon^- \to |C|v_0^- & \text{weakly in } L^2(0,T;L^2(\Omega^-)),\\ v_0^+ = v_0^-, \quad \partial_{x_n}v_0^+ = |C|\partial_{x_n}v_0^-, & \text{a.e. on } (0,T)\times\Sigma,\\ y_\varepsilon^- \to v_0^- & \text{weakly in } \{L^2(0,T;L^2(\Omega^-,\mathrm{d}\mu_\varepsilon))\}. \end{array}\right\} \tag{5.4}$$

Moreover, there exist functions $\gamma_i \in L^2(0,T;L^2(\Omega^-))$ $(i=1,\dots,n-1)$ such that

$$\widetilde{\nabla_x y_\varepsilon^-} \to (\gamma_1,\dots,\gamma_{n-1},|C|\partial v_0^-/\partial x_n) \ \text{ weakly in } [L^2(0,T;L^2(\Omega^-))]^n. \tag{5.5}$$

Let us prove that the function v_0^- satisfies the following boundary condition:

$$v_0^- = u_0 \ \text{ almost everywhere on } \ (0,T)\times\Gamma_0. \tag{5.6}$$

We note that any function $f \in \mathcal{V}(\Omega)$ has a trace $f|_{\Gamma_0} \in L^2(\Gamma_0)$ (see Brizzi & Chalot, 1997), so the equality (5.6) has a sense. It is easy to see that the following statements hold:

$$\widetilde{y}_\varepsilon^- = \chi_{\Gamma_\varepsilon} P_\varepsilon(u_\varepsilon), \qquad \text{a.e. on } (0,T)\times\Gamma_0, \ \ \forall\,\varepsilon>0, \tag{5.7}$$

$$\chi_{\Gamma_\varepsilon} P_\varepsilon(u_\varepsilon) \to |C|u_0 \quad \text{weakly in } L^2((0,T)\times\Gamma_0) \tag{5.8}$$

(as product of strongly and weakly convergent sequences). Then from the integral identity

$$\begin{aligned} \int_0^T\int_{\Gamma_0} \widetilde{y}_\varepsilon^- \varphi\psi\,\mathrm{d}x'\,\mathrm{d}t &= -\int_0^T\int_{\Omega^-} \partial\widetilde{y}_\varepsilon^-/\partial x_n \varphi\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad -\int_0^T\int_{\Omega^-} \widetilde{y}_\varepsilon^- \partial\psi/\partial x_n\varphi\,\mathrm{d}x\,\mathrm{d}t,\\ &\quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^n;\partial\Omega^-\setminus\Gamma_0),\ \ \forall\,\psi\in C_0^\infty(\mathbb{R}), \end{aligned} \tag{5.9}$$

where $C_0^\infty(\mathbb{R}^n;\partial\Omega^-\setminus\Gamma_0) = \{\varphi \in C_0^\infty(\mathbb{R}^n)\colon \varphi = 0 \text{ on } \partial\Omega^-\setminus\Gamma_0\}$, we immediately get $(\widetilde{y}_\varepsilon^-)|_{\Gamma_0} \to |C|(v_0^-)|_{\Gamma_0}$ weakly in $L^2((0,T)\times\Gamma_0)$. Thus, passing to the limit in (5.7) as $\varepsilon\to 0$, we obtain the required relation (5.6).

Now, let us show that the function v_0 is the unique weak solution of problem (5.2). With this aim, we rewrite the integral identity (2.3) as follows:

$$\begin{aligned} &-\int_0^T\int_{\Omega^+} y_\varepsilon^+\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^+}\nabla y_\varepsilon^+\cdot\nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\int_{\Omega^-}\widetilde{\nabla y_\varepsilon^-}\cdot\nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^+} y_\varepsilon^+\varphi\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\int_{\Omega^-}\widetilde{y}_\varepsilon^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t + k_0|\partial C|_H\int_0^T\int_{\Omega^-} y_\varepsilon^-\varphi\psi\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t\\ &= \int_0^T\int_{\Omega^+} f_\varepsilon\varphi\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^-}\chi_{\Omega_\varepsilon^-} f_\varepsilon\varphi\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.10}$$

Passing to the limit in (5.10) as $\varepsilon \to 0$ and using the properties (3.5), (5.4) and (5.5), we get

$$\begin{aligned}
&-\int_0^T \int_{\Omega^+} v_0^+ \varphi \psi' \,\mathrm{d}x\,\mathrm{d}t - |C| \int_0^T \int_{\Omega^-} v_0^- \varphi \psi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega^+} \nabla v_0^+ \cdot \nabla \varphi \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^-} \sum_{i=1}^{n-1} \gamma_i (\partial \varphi / \partial x_i) \psi \,\mathrm{d}x\,\mathrm{d}t + |C| \int_0^T \int_{\Omega^-} (\partial v_0^- / \partial x_n)(\partial \varphi / \partial x_n) \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^+} v_0^+ \varphi \psi \,\mathrm{d}x\,\mathrm{d}t + |C| \int_0^T \int_{\Omega^-} v_0^- \varphi \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + k_0 |\partial C|_H \int_0^T \int_{\Omega^-} v_0^- \varphi \psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega^+} f_0 \varphi \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + |C| \int_0^T \int_{\Omega^-} f_0 \varphi \psi \,\mathrm{d}x\,\mathrm{d}t, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_0), \ \forall \psi \in C_0^\infty(0, T).
\end{aligned} \tag{5.11}$$

Let us fix $i \in \{1, \ldots, n-1\}$ and let w_ε^i be a sequence in $W^{1,\infty}(\Omega^-)$ satisfying the following conditions:

$$w_\varepsilon^i \to x_i \ \text{ strongly in } L^\infty(\Omega^-), \tag{5.12}$$

$$D w_\varepsilon^i = 0 \ \text{ a.e. in } \Omega_\varepsilon^- \tag{5.13}$$

for every $\varepsilon > 0$. The existence of such sequence is proved in Brizzi & Chalot (1997) and Esposito *et al.* (1997). Let us prove that $\gamma_i = 0$ a.e. in $(0, T) \times \Omega^-$. Take the following test functions $\varphi = w_\varepsilon^i \phi$ and $\varphi = x_i \phi$ with $\phi \in C_0^\infty(\Omega^-)$ in (5.10). Then, by virtue of (5.13), we have

$$\begin{aligned}
&-\int_0^T \int_{\Omega^-} y_\varepsilon^- \phi w_\varepsilon^i \psi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla \phi w_\varepsilon^i \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^-} \widetilde{y}_\varepsilon^- \phi w_\varepsilon^i \psi \,\mathrm{d}x\,\mathrm{d}t + k_0 |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon^- \phi w_\varepsilon^i \psi \,\mathrm{d}\mu_\varepsilon \,\mathrm{d}t \\
&= \int_0^T \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \phi w_\varepsilon^i \psi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&-\int_0^T \int_{\Omega^-} y_\varepsilon^- \phi x_i \psi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega^-} \widetilde{\nabla y_\varepsilon^-} \cdot \nabla (\phi x_i) \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega^-} \widetilde{y}_\varepsilon^- \phi x_i \psi \,\mathrm{d}x\,\mathrm{d}t + k_0 |\partial C|_H \int_0^T \int_{\Omega^-} y_\varepsilon^- \phi x_i \psi \,\mathrm{d}\mu_\varepsilon \,\mathrm{d}t \\
&= \int_0^T \int_{\Omega^-} \chi_{\Omega_\varepsilon^-} f_\varepsilon \phi x_i \psi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{5.15}$$

for every $\varepsilon > 0$, $\phi \in C_0^\infty(\Omega^-)$ and $\psi \in C_0^\infty(0,T)$. Hence, passing to the limit in (5.14) and (5.15) as $\varepsilon \to 0$, and using the properties (3.5), (5.4), (5.5), (5.12) and Proposition 3.1, we obtain

$$-|C|\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^-}\sum_{k=1}^{n-1}\gamma_k(\partial\phi/\partial x_k)x_i\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+|C|\int_0^T\int_{\Omega^-}(\partial v_0^-/\partial x_n)(\partial\phi/\partial x_n)x_i\psi\,\mathrm{d}x\,\mathrm{d}t + |C|\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+k_0|\partial C|_H\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t = |C|\int_0^T\int_{\Omega^-} f_0\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t, \tag{5.16}$$

$$-|C|\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega^-}\sum_{k=1}^{n-1}\gamma_k(\partial(\phi x_i)/\partial x_k)\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+|C|\int_0^T\int_{\Omega^-}(\partial v_0^-/\partial x_n)(\partial\phi/\partial x_n)x_i\psi\,\mathrm{d}x\,\mathrm{d}t + |C|\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+k_0|\partial C|_H\int_0^T\int_{\Omega^-} v_0^-\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t = |C|\int_0^T\int_{\Omega^-} f_0\phi x_i\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{5.17}$$

Comparing (5.16) with (5.17) we conclude that $\int_0^T\int_{\Omega^-}\gamma_k\phi\psi\,\mathrm{d}x\,\mathrm{d}t = 0$, $\forall k \in \{1,\dots,n-1\}$, $\forall\phi \in C_0^\infty(\Omega^-)$ and $\forall\psi \in C_0^\infty(0,T)$. Thus, $\gamma_i = 0$ a.e. in $(0,T)\times\Omega^-$ and we obtain the required. As for the function v_0 we have the following identity:

$$-\int_0^T\int_{\Omega^+} v_0^+\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t - |C|\int_0^T\int_{\Omega^-} v_0^-\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega^+}\nabla v_0^+\cdot\nabla\varphi\,\psi\,\mathrm{d}x\,\mathrm{d}t + |C|\int_0^T\int_{\Omega^-}(\partial v_0^-/\partial x_n)(\partial\varphi/\partial x_n)\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega^+} v_0^+\varphi\psi\,\mathrm{d}x\,\mathrm{d}t + |C|\int_0^T\int_{\Omega^-} v_0^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+k_0|\partial C|_H\int_0^T\int_{\Omega^-} v_0^-\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega^+} f_0\varphi\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$+|C|\int_0^T\int_{\Omega^-} f_0\varphi\psi\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n;\Gamma_0),\ \ \forall\psi \in C_0^\infty(0,T). \tag{5.18}$$

However, using the facts that $C_0^\infty(\mathbb{R}^n;\Gamma_0)$ is dense in $H^1(\Omega;\Gamma_0) = \{\varphi \in H^1(\Omega)\colon \varphi = 0 \text{ on } \Gamma_0\}$ and $H^1(\Omega;\Gamma_0)$ is dense in

$$\mathcal{V}(\Omega;\Gamma_0) = \Big\{v \in L^2(\Omega)\colon \partial_{x_n}v \in L^2(\Omega^-), v \in H^1(\Omega^+), \quad v = 0 \text{ a.e. on } \Gamma_0\Big\}$$

with the continuous injection $H^1(\Omega;\Gamma_0)\hookrightarrow \mathcal{V}(\Omega;\Gamma_0)$ (see Esposito *et al.*, 1997), we observe that the integral identity (5.18) is valid with $\varphi\in\mathcal{V}(\Omega;\Gamma_0)$. Hence, it can be rewritten in the form

$$-\int_0^T\int_\Omega(\chi_{\Omega^+}+|C|\chi_{\Omega^-})v_0\varphi\psi'\,\mathrm{d}x\,\mathrm{d}t+\int_0^T(v_0,\varphi)_{\mathcal{V}(\Omega)}\psi\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega(\chi_{\Omega^+}+|C|\chi_{\Omega^-})f_0\varphi\psi\,\mathrm{d}x\,\mathrm{d}t,\quad\forall\varphi\in\mathcal{V}(\Omega;\Gamma_0),\ \forall\psi\in C_0^\infty(0,T).\tag{5.19}$$

Besides, taking the initial supposition (1.6) into account and using the approach of De Maio *et al.* (2004), one can prove the relation $v_0(0,x)=y^0(x)$ a.e. in Ω. Following the standard Hilbert space method and the arguments in De Maio *et al.* (2004), we can state that the function v_0 is a unique weak solution of problem (5.2) in the sense of Remark 5.1. However, due to the uniqueness of the solution to problem (5.2), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Thus, the lemma is proved. □

We are now in a position to state the first important result which deals with the recovery problem of the Kuratowski's $K(w^a)$-limit set Ξ_a in the analytical form.

THEOREM 5.1 For the sequence of the sets of admissible pairs for $\mathbb{P}_\varepsilon^a$-problems $\{\Xi_\varepsilon^a\}$, there exists a non-empty $K(w^a)$-limit set $\Xi_a\subset\mathbb{Y}_0^a$ with the following structure:

$$\Xi_a=\left\{(u_0,v_0^+,v_0^-)\;\middle|\;\begin{array}{r}u_0\in U_a,\\(v_0^+)'-\Delta_x v_0^++v_0^+=f_0,\ \text{in}\ (0,T)\times\Omega^+,\\(v_0^-)'-\partial_{x_n}^2v_0^-+\dfrac{|C|+k_0|\partial C|_H}{|C|}v_0^-\\=f_0,\ \text{in}\ (0,T)\times\Omega^-,\\\partial_\nu v_0^+=0,\ \text{in}\ (0,T)\times\partial\Omega^+\setminus\Sigma,\\v_0^-=u_0,\ \text{on}\ (0,T)\times\Gamma_0,\\v_0^+=v_0^-,\quad\partial_{x_n}v_0^+=|C|\partial_{x_n}v_0^-,\ \text{on}\ (0,T)\times\Sigma,\\v_0(0,x)=y^0(x),\ \text{a.e.}\ x\in\Omega.\end{array}\right\}\tag{5.20}$$

Here, $U_a=\left\{u\in L^2(0,T;H^1(\Gamma_0)):\|u\|_{L^2(0,T;H^1(\Gamma_0))}\leqslant \mathbf{C_0}\right\}$.

Proof. First of all we note that in view of Theorem 4.1, the sequence of sets $\{\Xi_\varepsilon^a\subset\mathbb{Y}_\varepsilon^a\}$ is relatively compact with respect to $K(w^a)$-convergence. We show that the $K(w^a)$-limit set exists for the whole sequence and it can be represented in the form (5.20). For this, in accordance with the definition of $K(w^a)$-limit, we have to verify conditions (1) and (2) of Definition 4.1. From the previous lemma, we see that the set Ξ_a is non-empty. Let (u,y^+,y^-) be any triplet of the set Ξ_a. To construct a sequence $\{(u_\varepsilon,y_\varepsilon)\}_{\varepsilon>0}$ w^a-converging to (u,y^+,y^-), we put: $u_\varepsilon\in L^2(0,T;H^1(\Gamma_\varepsilon))$ is the restriction of the control $u\in U_a$ on Γ_ε given above, and y_ε is the corresponding to u_ε weak solution of the boundary-value problem (1.2). Then, in view of Definition 3.1, we have $u_\varepsilon\to u$ weakly with respect to the space $L^2(0,T;H^1(\Gamma_0))$. Further, using Lemma 5.1 we obtain $(u_\varepsilon,y_\varepsilon)\overset{w^a}{\to}(u,v_0^+,v_0^-)$, where (u,v_0^+,v_0^-) is a solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the limit problem (5.2). Since this problem has a unique solution, we immediately deduce that $(u, v_0^+, v_0^-) = (u, y^+, y^-)$, thereby property (1) of Definition 4.1 is valid.

The second condition of this definition is the evident consequence of Lemma 5.1 and the lower semicontinuity of the norm in $L^2((0,T)\times\Gamma_0)$ with respect to the weak convergence. This concludes the proof. □

5.2 *Recovery of the set* Ξ_b

To establish the structure of the Kuratowski's $K(w^b)$-limit set Ξ_b, we give a result which not only will be useful in the sequel but also seems to be interesting *per se* (for similar one in more complicated case of perforated domains, see Kesavan & Saint Jean Paulin, 1999).

PROPOSITION 5.1 For every bounded sequence $\{u_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon))\}_{\varepsilon>0}$ such that $\widetilde{u}_\varepsilon \to u_*$ weakly in $L^2((0,T)\times\Gamma_0)$, the following inequality holds:

$$\liminf_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon} u_\varepsilon^2\,\mathrm{d}x'\,\mathrm{d}t \geqslant |C|^{-1}\int_0^T\int_{\Gamma_0} u_*^2\,\mathrm{d}x'\,\mathrm{d}t. \tag{5.21}$$

The following assertion can be viewed as an analogue of Lemma 5.1 with respect to w^b-convergence.

LEMMA 5.2 Let $\{u_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))\}_{\varepsilon>0}$ be any sequence of admissible controls for $\mathbb{P}_\varepsilon^b$-problems such that $\widetilde{u}_\varepsilon \to u_*$ weakly in $L^2((0,T)\times\Gamma_0)$. Let $\{y_\varepsilon\}$ be the corresponding solutions of the parabolic problem (1.2) for which $\sup_{\varepsilon>0}\|y_\varepsilon\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. Then $(u_\varepsilon, y_\varepsilon) \overset{w^b}{\to} (u_*, v_0^+, v_0^-)$ as $\varepsilon\to 0$, where

$$v_0(x) = \begin{cases} v_0^+(x), & x\in\Omega^+,\\ v_0^-(x), & x\in\Omega^-,\end{cases} \tag{5.22}$$

is a unique weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the following limit problem:

$$\left.\begin{aligned} (v_0^+)' - \Delta_x v_0^+ + v_0^+ &= f_0, && \text{in } (0,T)\times\Omega^+,\\ (v_0^-)' - \partial_{x_n}^2 v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|}v_0^- &= f_0, && \text{in } (0,T)\times\Omega^-,\\ \partial_\nu v_0^+ &= 0, && \text{in } (0,T)\times\partial\Omega^+\setminus\Sigma,\\ v_0^- &= |C|^{-1}u_*, && \text{on } (0,T)\times\Gamma_0,\\ v_0^+ = v_0^-,\quad \partial_{x_n}v_0^+ &= |C|\partial_{x_n}v_0^-, && \text{on } (0,T)\times\Sigma,\\ v_0(0,x) &= y^0(x), && \text{a.e. } x\in\Omega. \end{aligned}\right\} \tag{5.23}$$

Proof. As follows from Proposition 3.3 and Lemma 5.1, we have to show that the relation

$$v_0^- = |C|^{-1}u_* \text{ on } (0,T)\times\Gamma_0 \tag{5.24}$$

holds true. For this, we note that $(\widetilde{y}_\varepsilon^-)|_{\Gamma_0} \to |C|(v_0^-)|_{\Gamma_0}$ weakly in $L^2((0,T)\times\Gamma_0)$ and the following statements

$$\widetilde{y}_\varepsilon^- = \widetilde{u}_\varepsilon \quad \text{a.e. on } (0,T)\times\Gamma_0,\ \ \forall\,\varepsilon>0,$$

$$\widetilde{u}_\varepsilon \to u_0 \quad \text{weakly in } L^2((0,T)\times\Gamma_0) \tag{5.25}$$

are valid. Then the required relation (5.24) immediately follows from (5.25) after passing to the limit in (5.25) as $\varepsilon \to 0$. In order to conclude our proof, we have to follow the arguments in the proof of Lemma 5.1 closely. □

Now, we are able to prove the theorem concerning the structure of the Kuratowski's $K(w^b)$-limit set Ξ_b.

THEOREM 5.2 Let $\{\Xi_\varepsilon^b\}_{\varepsilon>0}$ be the sets of admissible pairs for $\mathbb{P}_\varepsilon^b$-problems possessing the $(\mathcal{N})$-property. Then for this sequence there exists a non-empty $K(w^b)$-limit set $\Xi_b \subset \mathbb{Y}_0^b$ which can be represented in the form

$$\Xi_b = \left\{ (u_0, v_0^+, v_0^-) \;\middle|\; \begin{aligned} & u_0 \in U_b, \\ & (v_0^+)' - \Delta_x v_0^+ + v_0^+ = f_0 \text{ in } (0,T)\times\Omega^+, \\ & (v_0^-)' - \partial_{x_n}^2 v_0^- + \frac{|C| + k_0|\partial C|_H}{|C|} v_0^- \\ & \qquad = f_0 \text{ in } (0,T)\times\Omega^-, \\ & \partial_\nu v_0^+ = 0 \text{ in } (0,T)\times\partial\Omega^+\setminus\Sigma, \\ & v_0^- = |C|^{-1} u_0 \text{ on } (0,T)\times\Gamma_0, \\ & v_0^+ = v_0^-, \quad \partial_{x_n} v_0^+ = |C|\partial_{x_n} v_0^- \text{ on } (0,T)\times\Sigma, \\ & v_0(0,x) = y^0(x) \text{ a.e. } x\in\Omega. \end{aligned} \right\} \tag{5.26}$$

Here, $U_b = \left\{ u \in L^2((0,T)\times\Gamma_0) : \|u\|_{L^2((0,T)\times\Gamma_0)} \leqslant \sqrt{|C|}\mathbf{C_0} \right\}$.

Proof. To obtain the representation (5.26), we have to verify conditions (1) and (2) of Definition 4.1. Let (u, y^+, y^-) be any triplet of the set Ξ_b. In accordance with $(\mathcal{N})$-property of $\mathbb{P}_\varepsilon^b$-problems, there can be found a w^b-convergent to (u, y^+, y^-) sequence $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon>0}$ such that $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\varepsilon^b$ for every $\varepsilon > 0$. However, due to Lemma 5.2 we have $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \overset{w^b}{\to} (u, v_0^+, v_0^-)$, where $\widehat{v} = (v_0^+, v_0^-)$ is a weak solution in $L^2(0,T;\mathcal{V}(\Omega))$ of the limit problem (5.23). Since this problem has a unique weak solution (see Lions, 1971), we immediately deduce that $(u, v_0^+, v_0^-) = (u, y^+, y^-)$. Thus, property (1) of Definition 4.1 holds for any triplet $(u, y^+, y^-) \in \Xi_0$.

We now verify the second property of Definition 4.1. Let $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ be a w^b-convergent sequence for which there exists a sequence $\{\varepsilon_k \to 0\}$ such that $(u_k, y_k) \in \Xi_{\varepsilon_k}^b$ for all $k \in \mathbb{N}$. Let (u, y^+, y^-) be its w^b-limit. Then by Proposition 5.1, we immediately have

$$\mathbf{C_0} \geqslant \liminf_{k\to\infty} \|u_k\|_{L^2\left((0,T)\times\Gamma_{\varepsilon_k}\right)} \geqslant \sqrt{|C|^{-1}} \cdot \|u\|_{L^2((0,T)\times\Gamma_0)},$$

i.e. $u \in \widehat{U}_b$. In conclusion, it remains only to apply Lemma 5.2. Thus $(u, y^+, y^-) \in \Xi_b$, and we obtained the required. This concludes the proof. □

6. Identification of the cost functionals I_a and I_b

In this section, we show that the cost functionals of the limit constrained minimization problems

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-)\right\rangle \quad \text{and} \quad \left\langle \inf_{(u,y^+,y^-)\in\Xi_b} I_b(u,y^+,y^-)\right\rangle \tag{6.1}$$

can be recovered in an explicit form and their analytical representations are different. We begin with the following results.

THEOREM 6.1 For the sequence of $\mathbb{P}^a_\varepsilon$-problems (3.1) there exists a variational w^a-limit (in the sense of Definition 4.3)

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_a} I_a(u,y^+,y^-)\right\rangle, \tag{6.2}$$

where the set Ξ_a is defined in (5.20) and

$$I_a(u,y^+,y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2 \,\mathrm{d}x\,\mathrm{d}t + |C| \int_0^T \int_{\Gamma_0} u^2 \,\mathrm{d}x'\,\mathrm{d}t. \tag{6.3}$$

Proof. In order to obtain the relation (6.3), we verify conditions (ii) and (iii) of Definition 4.3. Let (u,y^+,y^-) be any triplet of Ξ_a and $\{(u_k,y_k)\}_{k\in\mathbb{N}}$ be a w^a-convergent sequence such that $(u_k,y_k) \stackrel{w^a}{\rightarrow} (u,y^-,y^+)$, $(u_k,y_k)\in\Xi_{\varepsilon_k}$ for every $k\in\mathbb{N}$, where $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ converging to zero. Then using Lemma 5.1, the definition of the class of admissible controls and the properties of w^a-convergence, we get $\int_0^T \int_{\Omega^+} (y^+_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t \rightarrow \int_0^T \int_{\Omega^+} (y^+)^2 \,\mathrm{d}x\,\mathrm{d}t$,

$$\int_0^T \int_{\Gamma_{\varepsilon_k}} u_k^2 \,\mathrm{d}x'\,\mathrm{d}t = \int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(P_{\varepsilon_k} u_k\right)^2 \,\mathrm{d}x'\,\mathrm{d}t, \quad \text{for every } k\in\mathbb{N},$$

$$\int_0^T \int_{\Gamma_0} \chi_{\Gamma_\varepsilon} \left(P_{\varepsilon_k} u_k\right)^2 \,\mathrm{d}x'\,\mathrm{d}t \rightarrow |C| \int_0^T \int_{\Gamma_0} u^2 \mathrm{d}x'\,\mathrm{d}t \quad \text{as } k\rightarrow\infty$$

(as the limit of the product of weakly and strongly convergent sequences) and, therefore,

$$\begin{aligned}\liminf_{k\to\infty} I_{\varepsilon_k}(u_k,y_k) &= \int_0^T \int_{\Omega_0} (y^+ - q_0)^2 \,\mathrm{d}x + |C| \int_0^T \int_{\Gamma_0} u^2 \mathrm{d}x'\,\mathrm{d}t \\ &= I_a(u,y^+,y^-),\end{aligned} \tag{6.4}$$

i.e. property (ii) of Definition 4.3 is valid.

Similarly, we can show the correctness of the 'contrary' inequality (4.5). Indeed, in this case it is enough to consider the 'realized sequence' $\{(u_\varepsilon,y_\varepsilon)\}$ as follows: $u_\varepsilon\in L^2(0,T;H^1(\Gamma_\varepsilon))$ is the restriction of $u\in U_a$ on Γ_ε, and y_ε is the corresponding to u_ε solution of the boundary-value problem (1.2). Then, by Lemma 5.1 we have $(u_\varepsilon,y_\varepsilon) \stackrel{w^a}{\rightarrow} (u,y^+,y^-)$. For the conclusion of this proof, we should repeat the arguments concerning the correctness of the limit passage (6.4). □

THEOREM 6.2

$$\left\langle \inf_{(u,y^+,y^-)\in\Xi_b} I_b(u,y^+,y^-)\right\rangle \tag{6.5}$$

is the variational w^b-limit for the sequence of $\mathbb{P}^b_\varepsilon$-problems (3.1). Here, the set Ξ_b is defined in (5.26) and

$$I_b(u,y^+,y^-) = \int_0^T \int_{\Omega^+} (y^+ - q_0)^2\,\mathrm{d}x + |C|^{-1}\int_0^T\int_{\Gamma_0} u^2\,\mathrm{d}x'\,\mathrm{d}t. \tag{6.6}$$

Proof. To obtain the representation (6.6) it is enough to repeat the same arguments of the proof of Theorem 6.1 and to apply Proposition 5.1 and Lemma 5.2. In this case, for any sequence $\{(u_k, y_k)\in \Xi^b_{\varepsilon_k}\}_{k\in\mathbb{N}}$ w^b-converging to (u,y^+,y^-) we have

$$\liminf_{k\to\infty} I_{\varepsilon_k}(u_k,y_k) \geqslant \int_0^T\int_{\Omega^+} |\nabla y^+|^2\,\mathrm{d}x\,\mathrm{d}t + |C|^{-1}\int_0^T\int_{\Gamma_0} u^2\,\mathrm{d}x'\,\mathrm{d}t = I_b(u,y^+,y^-).$$

To verify the correctness of the inequality (4.5), for arbitrary triplet $(u,y^+,y^-)\in\Xi_b$ we have to construct the special 'realizing sequence' $\{(u_\varepsilon,y_\varepsilon)\}$ satisfying condition (iii) of Definition 4.3. With this aim, we construct the w^b-convergent sequence $\{(u_\varepsilon,y_\varepsilon)\in\Xi^b_\varepsilon\}_{\varepsilon>0}$ to (u,y^+,y^-) as follows. Let $\{\overline{u}_\varepsilon\in L^2((0,T)\times\Gamma_0)\}$ be any sequence such that

$$\overline{u}_\varepsilon \to u \quad \text{weakly in } L^2((0,T)\times\Gamma_0); \quad \|\overline{u}_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_0}$$

for every $\varepsilon>0$. Since the weak topology of $L^2((0,T)\times\Gamma_0)$ is metrizable on the set

$$\widehat{U}_b = \left\{u\in L^2((0,T)\times\Gamma_0)\colon \|u\|_{L^2((0,T)\times\Gamma_0)} \leqslant \sqrt{|C|}\cdot\mathbf{C_0}\right\},$$

one can construct a sequence $\{w_\varepsilon\in L^2((0,T)\times\Gamma_0)\}_{\varepsilon>0}$ satisfying the following condition: each element w_ε is a convex envelope of a finite amount of the elements $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$, and $w_\varepsilon\to u$ strongly in $L^2((0,T)\times\Gamma_0)$. Note that in this case, we have

$$\|w_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_0}, \qquad \text{for every } \varepsilon>0.$$

Thus, a weak convergent sequence $\{u_\varepsilon\}_{\varepsilon>0}$ to u can be taken in the following form: $u_\varepsilon\in L^2(0,T;H^1(\Gamma_\varepsilon))$ are elements such that $\|u_\varepsilon - |C|^{-1}w_\varepsilon\|_{L^2((0,T)\times\Gamma_\varepsilon)} < \varepsilon^2$.

In view of $(\mathcal{N})$-property, we can suppose that the sequence of norm $\left\{\|u_\varepsilon\|_{L^2(0,T;H^1(\Gamma_\varepsilon))}\right\}$ is uniformly bounded. Since $|C|^{-1}\chi_{\Gamma_\varepsilon}w_\varepsilon\to u$ weakly in $L^2((0,T)\times\Gamma_0)$ (as the limit of the product of weakly and strongly convergent sequences) and

$$\||C|^{-1}w_\varepsilon\|_{L^2((0,T)\times\Gamma_\varepsilon)} = \sqrt{\int_0^T\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}\frac{w_\varepsilon^2}{|C|^{-2}}\,\mathrm{d}x'\,\mathrm{d}t} \to \sqrt{\int_0^T\int_{\Gamma_0}\frac{u^2}{|C|^{-1}}\,\mathrm{d}x'\,\mathrm{d}t} < \mathbf{C_0},$$

it follows that $\widetilde{u}_\varepsilon\to u$ weakly in $L^2((0,T)\times\Gamma_0)$ and $u_\varepsilon\in U^b_\varepsilon$ for ε sufficiently small.

We may always suppose that the elements w_ε have the representation $w_\varepsilon = |C|^{-1}\widehat{w}_\varepsilon$, where the sequence $\{\widehat{w}_\varepsilon\}$ is constructed as follows:

$$L^2((0,T)\times\Gamma_0)\ni\widehat{w}_\varepsilon\to u \quad \text{strongly in } L^2((0,T)\times\Gamma_0),$$

$$\|\widehat{w}_\varepsilon\|_{L^2((0,T)\times\Gamma_0)} < \sqrt{|C|}\cdot\mathbf{C_0}, \quad \forall\,\varepsilon>0.$$

Then

$$\lim_{\varepsilon\to 0}\|w_\varepsilon\|^2_{L^2((0,T)\times\Gamma_\varepsilon)} = \lim_{\varepsilon\to 0}\||C|^{-1}\chi_{\Gamma_\varepsilon}\widehat{w}_\varepsilon\|^2_{L^2((0,T)\times\Gamma_0)}$$

$$= |C|^{-2}\lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_0}\chi_{\Gamma_\varepsilon}\widehat{w}^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t = |C|^{-1}\|u\|^2_{L^2((0,T)\times\Gamma_0)},$$

i.e. for the realizing sequence of the Dirichlet boundary controls $\{u_\varepsilon\}$, we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon}u^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t = |C|^{-1}\int_0^T\int_{\Gamma_0}u^2\,\mathrm{d}x'\,\mathrm{d}t$$

and $u_\varepsilon \in U^b_\varepsilon$ for all $\varepsilon > 0$.

Let y_ε be as the corresponding to u_ε solutions of the boundary-value problem (1.2). Then by $(\mathcal{N})$-property and Lemma 5.2, we have $(u_\varepsilon, y_\varepsilon) \overset{w^b}{\to} (u, y^+, y^-)$ and, therefore

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^+}(y^+_\varepsilon - q_0)^2\,\mathrm{d}x\,\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^T\int_{\Gamma_\varepsilon}u^2_\varepsilon\,\mathrm{d}x'\,\mathrm{d}t$$

$$\geqslant \int_0^T\int_{\Omega^+}(y^+ - q_0)^2\,\mathrm{d}x\,\mathrm{d}t + |C|^{-1}\int_0^T\int_{\Gamma_0}u^2\,\mathrm{d}x'\,\mathrm{d}t.$$

This concludes the proof. □

Thus, in accordance with Definition 4.4 and the results obtained above we may infer: each of the constrained minimization problems (6.1) can be recovered in the form of some optimal control problems, namely, $(\mathbb{P}^a_{\mathrm{hom}})$ (see (1.1), (1.2), (1.4)) and $(\mathbb{P}^b_{\mathrm{hom}})$ (see (1.1), (1.2), (1.5)). Hence, the problems $\mathbb{P}^a_\varepsilon$ and $\mathbb{P}^b_\varepsilon$ admit the homogenization as ε tends to zero. However, the corresponding homogenized problems have the different mathematical descriptions and these differences appear not only in the state equation and boundary conditions but also in the control constraints and limit cost functionals. In fact, the reason of this gap phenomenon is the choice of the different topologies for the homogenization of the original control problem (1.2), (1.4) that were associated with w^a- and w^b-convergence, respectively. It should be stressed that in our case this choice was fated by the characteristic properties of the control constraints.

In conclusion, we give some results concerning the variational properties of the homogenized problems. As was noted before, the problems $(\mathbb{P}^a_{\mathrm{hom}})$ and $(\mathbb{P}^b_{\mathrm{hom}})$ have to preserve the well-known variational property, namely, both optimal pairs and minimal values of the cost functionals for the original problems have to converge to the corresponding characteristics of the limit optimal control problems as ε tends to zero. To establish this result, we begin with the following evident assertion.

PROPOSITION 6.1 Each of the limit optimal control problems (see (1.8–1.10) and (1.11–1.13)) has a unique solution.

Indeed, taking into account the weak lower semicontinuity of the cost functionals $I_i\colon \Xi_i \to \mathbb{R}$ $(i = a, b)$, the topological properties of their domains Ξ_i $(i = a, b)$, and applying the direct method of calculus of variation, we just obtain the required result.

Let us denote by $(u^a, (y^a)^+, (y^a)^-) \in \mathbb{Y}_0^a$ and $(u^b, (y^b)^+, (y^b)^-) \in \mathbb{Y}_0^b$ the optimal triplets for $(\mathbb{P}^a_{\text{hom}})$ and $(\mathbb{P}^b_{\text{hom}})$ problems, respectively.

LEMMA 6.1 If the functions f_ε and y_ε^0 satisfy conditions (1.6) and (1.7), then the sequence of optimal solutions $\{(u_\varepsilon^a, y_\varepsilon^a) \in \Xi_\varepsilon^a\}$ to $\mathbb{P}_\varepsilon^a$-problems and the corresponding minimal values of the cost functional (1.1) possess the following properties:

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^a, y_\varepsilon^a) = \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon^a} I_\varepsilon(u_\varepsilon, y_\varepsilon)$$

$$= \inf_{(u,y^+,y^-)\in \Xi_a} I_a(u, y^+, y^-) = I_a(u^a, (y^a)^+, (y^a)^-), \tag{6.7}$$

$$(u_\varepsilon^a, y_\varepsilon^a) \stackrel{w^a}{\to} (u^a, (y^a)^+, (y^a)^-). \tag{6.8}$$

Proof. As follows from previous results, for every value of ε the optimal control problems (1.1–1.4) has a unique solution $(u_\varepsilon^a, y_\varepsilon^a) \in \Xi_\varepsilon^a$. Since the sequence $\{u_\varepsilon^a\}_{\varepsilon>0} \subset U_\varepsilon^a$ is bounded, there exists a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$, which we again denote by $\{\varepsilon\}$, such that $u_\varepsilon^a \to u^a \in U_a$ weakly with respect to the space $L^2(0, T; H^1(\Gamma_0))$ as $\varepsilon \to 0$. Then, in view of Lemma 5.1, we have $(u_\varepsilon^a, y_\varepsilon^a) \stackrel{w^a}{\to} (u^a, (y^*)^+, (y^*)^-)$ as $\varepsilon \to 0$, where the triplet $(u^a, (y^*)^+, (y^*)^-)$ is the unique solution of problem (1.8) with the Dirichlet condition $y^- = u^a$ on Γ_0. By Theorem 4.3, we immediately conclude that $(u^a, (y^*)^-, (y^*)^+)$ is an optimal solution of homogenized problems (1.8–1.10) and property (6.7) is valid. Hence,

$$(u^a, (y^*)^-, (y^*)^+) = (u^a, (y^a)^+, (y^a)^-).$$

So, we obtained the required result. □

REMARK 6.1 It should be noted that the realization of conditions (6.7) and (6.8) warranty covered by Lemma 6.1 does not imply the strong convergence of optimal states y_ε^a.

LEMMA 6.2 Assume that the functions f_ε and y_ε^0 satisfy conditions (1.6) and (1.7), $\mathbb{P}_\varepsilon^b$-problem is solvable for every value $\varepsilon > 0$ and the sequence of optimal solutions $\{(u_\varepsilon^b, y_\varepsilon^b) \in \Xi_\varepsilon^b\}$ for $\mathbb{P}_\varepsilon^b$-problems is such that $\sup_{\varepsilon>0} \|y_\varepsilon^b\|_{\mathbf{X}_{\mu_\varepsilon}} < +\infty$. Then

$$\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^b, y_\varepsilon^b) = \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon^b} I_\varepsilon(u_\varepsilon, y_\varepsilon)$$

$$= \inf_{(u,y^+,y^-)\in \Xi_b} I_b(u, y^+, y^-) = I_b(u^b, (y^b)^+, (y^b)^-), \tag{6.9}$$

$$(u_\varepsilon^b, y_\varepsilon^b) \stackrel{w^b}{\to} s(u^b, (y^b)^+, (y^b)^-), \tag{6.10}$$

$$\widetilde{u}_\varepsilon^b - |C|^{-1}\chi_{\Gamma_\varepsilon} u^b \to 0 \quad \text{strongly in } L^2((0, T) \times \Gamma_0). \tag{6.11}$$

Proof. Using the arguments of the previous lemma and Theorem 5.2, it can be easily checked that conditions (6.9–6.10) hold. To conclude the proof, it remains to verify the assertion (6.11). However, in view of (3.7) and the fact that $\|u^b_\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_\varepsilon))} \to |C|^{-1}\|u^b\|^2_{L^2((0,T)\times\Gamma_0)}$ (see (6.9)), we get

$$\int_0^T \int_{\Gamma_0} (\widetilde{u}^b_\varepsilon - |C|^{-1}\chi_{\Gamma_e} u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t = \int_0^T \int_{\Gamma_e} (u^b_e)^2 \,\mathrm{d}x'\,\mathrm{d}t - 2|C|^{-1}\int_0^T \int_{\Gamma_0} u^b \widetilde{u}^b_\varepsilon \,\mathrm{d}x'\,\mathrm{d}t$$
$$+|C|^{-2}\int_0^T \int_{\Gamma_0} \chi_{\Gamma_e}(u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t \to |C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t$$
$$-2|C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t + |C|^{-1}\int_0^T \int_{\Gamma_0} (u^b)^2 \,\mathrm{d}x'\,\mathrm{d}t = 0 \quad \text{as } \varepsilon \to 0,$$

which yields (6.11). This completes the proof. □

7. Conclusion

To emphasize the contribution of this paper, we would like to point out one possible application of the above results concerning the approximation of the optimal solutions to the original problem for ε small enough. Since the computational calculation of the solutions of these problems is very complicated, it is particularly relevant to define the so-called suboptimal solutions which have to guarantee the closeness of the corresponding value of the cost functional $I_\varepsilon(u^{\mathrm{sub}}_\varepsilon, y^{\mathrm{sub}}_\varepsilon)$ to its minimum for ε small enough. In view of this we introduce the following concept.

DEFINITION 7.1 We say that a sequence of functions $\{\widetilde{u}^a_\varepsilon\}$ is asymptotically suboptimal for $\mathbb{P}^a_\varepsilon$-problem if for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that

$$\left| \inf_{(u_\varepsilon, y_\varepsilon)\in \Xi^a_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon) - I_\varepsilon(\widetilde{u}^a_\varepsilon, \widetilde{y}_\varepsilon) \right| < \delta, \quad \forall\, \varepsilon > \varepsilon_0,$$

where by $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon(\widetilde{u}^a_\varepsilon)$ we denote the corresponding solution of the parabolic problem (1.2).

As follows from Proposition 6.1 each of the limit optimal control problems (1.8–1.10) and (1.11–1.13) has a unique solution. Then Lemma 6.1 and *a priori* estimate (2.5) immediately lead us to the following result.

THEOREM 7.1 Let $(u^a, (y^a)^+, (y^a)^-) \in \mathbb{Y}^a_0$ be an optimal solution for the homogenized $(\mathbb{P}^a_{\mathrm{hom}})$-problem. Then the sequence $\{u^a|_{\Gamma_\varepsilon}\}_{\varepsilon>0}$ is asymptotically suboptimal for $\mathbb{P}^a_\varepsilon$-problem.

After minor modifications, the similar result can be established for $\mathbb{P}^b_\varepsilon$-problem.

In conclusion, we would like to note that the result of the homogenization of optimal control problems may essentially depend on the differential properties of its solutions. Choosing different topologies on the space of the 'control state', the corresponding limit optimal control problems may have drastically different mathematical descriptions. Thus, the choice of such topologies is a very important and non-trivial matter when dealing with the questions of asymptotic behaviour of the optimal control problems. In the theory of boundary-value problems, this fact is called the Lavrentieff phenomenon (see Zhikov & Lukkassen, 2001).

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