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Homogenization of Dirichlet optimal control problems with exact partial controllability constraints *

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Abstract. We consider an elliptic distributed quadratic optimal control problem with exact controllability constraints on a part of the domain which, in turn, is parametrized by a small parameter ε. The quadratic tracking type functional is defined on the remaining part of the domain. We thus consider a family of optimal control problems with state equality constraints. The purpose of this paper is to study the asymptotic limit of the optimal control problems as the parameter ε tends to zero. The analysis presented is in the spirit of the direct approach of the calculus of variations. This is achieved in the framework of relaxed problems. We finally apply the procedure to an optimal control problem on a perforated domain with holes of critical size. It is shown that a *strange term* in the terminology of Cioranescu and Murat (Prog. Nonlinear Diff. Eq. Appl., Vol. 31, Birkhäuser-Verlag, Boston, 1997, pp. 49–93) appears in the limiting homogenized problem.

Keywords: controllability constraints, relaxed problems, γ^{Δ} and Γ-convergence, perforated domains

0. Introduction

The aim of this paper is to study the homogenization of optimal control problems with partial exact controllability constraints. More precisely, for the sake of simplicity, the main control object is a Dirichlet boundary value problem for the linear Laplacian with bounded controls in the space of Radon measures $\mathcal{M}(\Omega)$. We suppose that for every ε, where ε takes its values in a sequence of positive numbers tending to zero, there are some closed "holes" T_i^{ε}, $1 \leqslant i \leqslant n(\varepsilon)$, such that on the sub-domain $S_\varepsilon = \bigcup T_i^{\varepsilon}$ the state is supposed to exactly match a certain profile Ψ_ε^i on each T_i^ε. We do not make any assumption on the holes other than meas $T_i^\varepsilon > 0\ \forall \varepsilon > 0$. The problem is to describe the asymptotic behavior of the parametrized optimal control problem as ε tends to zero.

We would like to emphasize that in contrast to the approach of Kesavan and Saint Jean Paulin [15] and [16], Zoubairi [26], Saint Jean Paulin and Zoubairi [22] and Conca, Osses and Saint Jean Paulin [8] we do not just look for a limit of optimal control functions and for a limit of minimal values of the cost functionals. Rather, we stay with the optimal control problem in the original sense and look for a homogenized problem as some variational limit of the original one. This limiting problem should be unique (as a result of some passage to the limit), and should preserve the well-known variational properties such as the convergence of both optimal solutions and minimal values of a cost function and, of course, should finally have a clearly defined structure including the limit form of a state equation, control and state constraints and a limit cost functional. Our approach, the so-called "direct approach"

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of calculus of variations, is based on the concept of variational convergence of constrained minimization problems [17,18] and its variational properties. See also Rajesh [21] for a simpler problem.

The plan of this paper is as follows. In Section 1 we present the state constrained optimal control problem with generalized controls and show that, in general, this problem has no solutions. In Section 2, we give an appropriate modification of the original state constrained optimal control problem which always has a solution and good stability properties. In Section 3 we prove the existence of a homogenized optimal control problem and recover its structure and variational properties. The last section is devoted to the homogenization of the Dirichlet optimal control problem on perforated domains. We show that this problem can be viewed as a particular example covered by the theory developed in this paper. In particular, we consider holes having a critical size depending on their number and distribution. We show that the state equation and the cost functional in the homogenized optimal control problem include additional terms, so-called "strange terms", that are associated with the holes.

The analysis is very much in the spirit of Dal Maso and Murat [11,12], where, however, optimal control problems have not been considered. It is to be noted that the homogenization method used in this paper does not provide estimates of the convergence of solutions with respect to some distance measure between the domains. Convergence estimates have been investigated by Savaré and Schimperna [23], again for problems without control.

1. The statement of the optimal control problem and preliminary results

Let Ω be a bounded connected and open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $E = (0; \varepsilon^0]$ be an index set and let $\{S_\varepsilon\}_{\varepsilon \in E}$ be a family of closed subsets such that $S_\varepsilon \subset \Omega$ and S_ε has a non-empty interior and a Lipschitz boundary for every $\varepsilon \in E$.

Let $\{\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)\}_{\varepsilon \in E}$ and $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon \in E}$ be given functions such that

$$\Psi_\varepsilon \to \Psi_0 \quad \text{weakly in } H_0^1(\Omega), \qquad f_\varepsilon \to f_0 \quad \text{weakly in } L^2(\Omega).$$

As usual by $H_0^1(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$-functions in the Sobolev space $H^1(\Omega)$ and the space $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. On $H_0^1(\Omega)$ we consider the norm

$$\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla y|^2 \, \mathrm{d}x \right)^{1/2}$$

and $H^{-1}(\Omega)$ is endowed with the corresponding dual norm.

By $\mathrm{cap}(B, \Omega)$ we will denote the capacity of a set B with respect to Ω which is defined as the infimum of $\int_\Omega |\nabla y|^2 \, \mathrm{d}x$ over the set of all functions $y \in H_0^1(\Omega)$ such that $y \geqslant 1$ almost everywhere (a.e.) in a neighborhood of B. Following [25] we say that some property $\mathcal{P}(\alpha)$ holds quasi-everywhere (q.e.) in the set Ω if it holds for all $x \in \Omega$ except for a subset B of Ω with $\mathrm{cap}(B, \Omega) = 0$. It is well known that every function $y \in H_0^1(\Omega)$ has a quasi-continuous representative which is uniquely defined up to a set of capacity zero. Here a function $y : \Omega \to \mathbb{R}$ is said to be quasi-continuous if for every $\delta > 0$ there exists a subset $B \subseteq \Omega$ with $\mathrm{cap}(B, \Omega) < \delta$ such that the restriction of y to $\Omega \setminus B$ is continuous. Thus we will always identify elements of $H_0^1(\Omega)$ with their quasi-continuous representatives. Moreover, in order to obtain a strong setting of the control problem we describe the class of bounded controls in the space of Radon measures on Ω. For this we say that λ is a Radon measure on Ω if it is a continuous linear functional on the space $C_0(\Omega)$ of all continuous functions with compact support in Ω. It is well known that for every Radon measure λ there exists a countable additive set function ν defined on the family of all relatively compact Borel subsets of Ω such that

$$\lambda(y) = \int_\Omega y \, \mathrm{d}\nu \quad \text{for every } y \in C_0(\Omega).$$

We will always identify the functional λ with the set function ν.

Definition 1.1. We say that the sequence of Radon measures $\{\lambda_\varepsilon\}_{\varepsilon \in E}$ is bounded if for every compact set $K \subseteq \Omega$ there exists a constant $c_K > 0$ such that

$$|\lambda_\varepsilon|(K) \leqslant c_K \quad \text{for every } \varepsilon \in E.$$

We will frequently use the notion of a filter on the set E.

Definition 1.2. The Fréchet filter on E is the family $\mathcal{H}$ of all subsets of E which contain an interval of the form $(0, \delta)$ for some $\delta > 0$.

We denote the family of all subsets of E that meet all sets H in $\mathcal{H}$ by $\mathcal{H}^\#$.

Now for every fixed $\varepsilon \in E$ we define the control problem in the class of bounded Radon measures as follows

$$-\Delta y = f_\varepsilon + \nu, \quad \text{in } \mathcal{D}'(\Omega) \tag{1.1}$$

$$y \in H_0^1(\Omega), \qquad \nu \in \mathcal{M}(\Omega), \tag{1.2}$$

$$y = \Psi_\varepsilon \qquad \text{quasi-everywhere in } S_\varepsilon, \tag{1.3}$$

$$I(\nu, y) = \int_{\Omega \setminus S_\varepsilon} |y - z_d|^2 \, \mathrm{d}x + \|\nu\|_{\mathcal{M}(\Omega)} \to \inf, \tag{1.4}$$

where z_d is some fixed function of $L^2(\Omega)$ and by $\|\nu\|_{\mathcal{M}(\Omega)}$ we denote the norm of ν in the space of Radon measures $\mathcal{M}(\Omega)$, i.e.

$$\|\nu\|_{\mathcal{M}(\Omega)} = \sup_{\|\varphi\|_{C_0(\Omega)} \leqslant 1} |\nu(\varphi)|.$$

It should be stressed here that in general this problem has no optimal solutions. The reason for this is as follows. First, for a Radon measure $\nu \in \mathcal{M}(\Omega)$, in general, the Dirichlet problem for the equation $-\Delta y = f_\varepsilon + \nu$ has no solution in $H_0^1(\Omega)$. Secondly, even if the set of admissible pairs

$$\Xi_\varepsilon = \{(\nu, y) \in \mathcal{M}(\Omega) \times H_0^1(\Omega)\colon -\Delta y = f_\varepsilon + \nu, \text{ in } \mathcal{D}'(\Omega),\ y = \Psi_\varepsilon \text{ q.e. on } S_\varepsilon\}$$

is non-empty for a fixed value ε, it is not clear that this set is closed with respect to the product of the weak-$*$ topology for $\mathcal{M}(\Omega)$ and the topology of the weak convergence in $H_0^1(\Omega)$ (see [14,19]). So, for the solvability of the original problem we have to use some regularization approaches. The simplest way for this is penalizing the controllability constraints on S_ε and choosing a narrower class of admissible controls compared with the class of all bounded Radon measures $\mathcal{M}(\Omega)$.

2. On modification of the optimal control problem with controllability constraints

We begin with some notions of general measure theory. By a non-negative Borel measure on Ω we mean a countably additive set function defined in the Borel σ-field of Ω with values in $[0,+\infty]$. If μ is a non-negative Borel measure on Ω we will use $L^2_\mu(\Omega)$ to denote the Lebesgue space with respect to the measure μ. It is well known that every non-negative Borel measure which is finite on all compact subsets of Ω is a non-negative Radon measure. Following [9] we denote by $\mathcal{M}^2_0(\Omega)$ the set of all non-negative Borel measures μ on Ω such that

(i) $\mu(B)=0$ for every Borel set $B\subseteq\Omega$ with $\operatorname{cap}(B,\Omega)=0$;
(ii) $\mu(B)=\inf\{\mu(U)\colon U \text{ quasi-open}, B\subseteq U\}$ for every Borel set $B\subseteq\Omega$.

We recall that a subset U of Ω is said to be quasi-open if for every $\delta>0$ there exists an open subset V of Ω with $\operatorname{cap}(V,\Omega)<\delta$ such that $U\cup V$ is open.

By analogy with [4] we introduce the space $V_\mu=H^1_0(\Omega)\cap L^2_\mu(\Omega)$ which is well defined for every $\mu\in\mathcal{M}^2_0(\Omega)$ since all functions in $H^1_0(\Omega)$ are defined μ-almost everywhere in Ω. In general this space is not dense in $L^2(\Omega)$. Moreover this space is a Hilbert space with respect to the scalar product

$$(y,g)_\mu=\int_\Omega\nabla y\cdot\nabla g\,\mathrm{d}x+\int_\Omega yg\,\mathrm{d}\mu\quad\text{(see [4]).}$$

Let V'_μ denote the dual space of V_μ and $\langle\cdot,\cdot\rangle_\mu$ be the duality pairing. We note that even if the transposed mappings to the embeddings of V_μ into $H^1_0(\Omega)$ and into $L^2(\Omega)$ are not injective, the spaces $H^{-1}(\Omega)$ and $L^2(\Omega)$ can be considered as the linear subspaces of V'_μ and we have:

$$\langle f,y\rangle_\mu=\langle f,y\rangle_{H^1_0(\Omega)},\quad\text{for } f\in H^{-1}(\Omega), y\in V_\mu$$

and, in particular,

$$\langle f,y\rangle_\mu=\int_\Omega fy\,\mathrm{d}x,\quad\text{for } f\in L^2(\Omega), y\in V_\mu$$

where we denote by $\langle\cdot,\cdot\rangle_{H^1_0(\Omega)}$ the duality pairing between $H^1_0(\Omega)$ and $H^{-1}_0(\Omega)$.

Let $f\in H^{-1}(\Omega)$, $\Psi\in H^1_0(\Omega)$ and $\mu\in\mathcal{M}^2_0$ be fixed elements. We introduce the following Dirichlet problem: find $y\in H^1_0(\Omega)$ such that $y-\Psi\in V_\mu$ and

$$\int_\Omega(\nabla y,\nabla\phi)\,\mathrm{d}x+\int_\Omega(y-\Psi)\varphi\,\mathrm{d}\mu=\langle f,\varphi\rangle\quad\forall\varphi\in V_\mu.\tag{2.5}$$

It is easy to see that this problem has a unique solution since the operator $B\colon V_\mu\to V'_\mu$ which is defined by

$$\langle Bz,\varphi\rangle=\int_\Omega(\nabla z,\nabla\varphi)\,\mathrm{d}x+\int_\Omega z\varphi\,\mathrm{d}\mu\quad\forall\varphi\in V_\mu$$

is monotone, continuous and coercive. Moreover, if we take $\varphi=y-\Psi$ as test function in (2.5) we obtain

$$\int_\Omega\left|\nabla(y-\Psi)\right|^2\mathrm{d}x+\int_\Omega(y-\Psi)^2\,\mathrm{d}\mu=\langle f+\Delta\Psi,y-\Psi\rangle.$$

Therefore by Young's inequality we have

$$\|y - \Psi\|_{V_\mu} \leqslant \|f + \Delta\Psi\|_{H^{-1}(\Omega)} \leqslant \|f\|_{H^{-1}(\Omega)} + \|\Delta\Psi\|_{H^{-1}(\Omega)}. \tag{2.6}$$

Now we note that, whenever $f \in L^2(\Omega)$ and $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$, the solution of (2.5) is actually the solution of a new equation involving a Radon measure λ.

Indeed, let y be the corresponding solution of the problem (2.5). Since $(f + \Delta\Psi) \in L^2(\Omega)$ it follows that the positive part of $f + \Delta\Psi$ is well defined. So, we define the element λ_1 of $H^{-1}(\Omega)$ such that

$$-\Delta(y - \Psi)^+ + \lambda_1 = (f + \Delta\Psi)^+, \tag{2.7}$$

where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, $f = f^+ - f^-$.

Let $v \in H_0^1(\Omega)$ be any function so that $v \geqslant 0$ q.e. in Ω. Following the lines of proof of Proposition 2.8 from [11] we put

$$v_n = \frac{1}{n} v \wedge (y - \Psi)^+,$$

where by $y \wedge v$ we denote the minimum of $\{y, v\}$. Then it is easy to see that

$$v_n \geqslant 0 \quad \text{q.e. in } \Omega, \text{ and } v_n \in V_\mu \ \forall n \in N.$$

Since

$$\begin{aligned} &(y - \Psi) \cdot v_n \geqslant 0 && \text{q.e. in } \Omega, \text{ and} \\ &(f + \Delta\Psi) v_n \leqslant (f + \Delta\Psi)^+ v_n && \text{a.e. in } \Omega, \end{aligned}$$

it follows that by taking v_n as test function in (2.5) we obtain

$$\int_\Omega \big(\nabla(y - \Psi), \nabla v_n\big)\,\mathrm{d}x \leqslant \int_\Omega (f + \Delta\Psi)^+ v_n \,\mathrm{d}x \leqslant \frac{1}{n}\int_\Omega (f + \Delta\Psi)^+ v \,\mathrm{d}x. \tag{2.8}$$

We note that

$$\begin{aligned} &\nabla v_n = \frac{\nabla v}{n} && \text{a.e. in the domain } \left\{\frac{1}{n} v < (y - \Psi)^+\right\}, \\ &\nabla v_n = \nabla (y - \Psi)^+ && \text{a.e. in } \left\{\frac{1}{n} v \geqslant (y - \Psi)^+\right\}. \end{aligned}$$

Therefore, from (2.8) we have

$$\begin{aligned} &\frac{1}{n}\int_{\{v < n(y - \Psi)^+\}} \big(\nabla(y - \Psi), \nabla v\big)\,\mathrm{d}x + \int_{\{v \geqslant n(y - \Psi)^+\}} \big|\nabla(y - \Psi)^+\big|^2 \,\mathrm{d}x \\ &\quad \leqslant \frac{1}{n}\int_\Omega (f + \Delta\Psi)^+ v \,\mathrm{d}x. \end{aligned} \tag{2.9}$$

As the second term in (2.9) is non-negative one gets

$$\int_{\{\frac{1}{n}v<(y-\Psi)^+\}} \big(\nabla(y-\Psi),\nabla v\big)\,\mathrm{d}x \leqslant \int_\Omega (f+\Delta\Psi)^+ v\,\mathrm{d}x.$$

Then taking the limits as $n\to\infty$ we have

$$\int_{\{(y-\Psi)^+>0\}} \big(\nabla(y-\Psi),\nabla v\big)\,\mathrm{d}x \leqslant \int_\Omega (f+\Delta\Psi)^+ v\,\mathrm{d}x.$$

Since $\nabla(y-\Psi)=\nabla(y-\Psi)^+$ a.e. in $\{(y-\Psi)^+>0\}$ and $\nabla(y-\Psi)=0$ a.e. in $\{(y-\Psi)^+=0\}$ it follows that

$$\int_\Omega \big(\nabla(y-\Psi)^+,\nabla v\big)\,\mathrm{d}x \leqslant \int_\Omega (f+\Delta\Psi)^+ v\,\mathrm{d}x \quad \forall v\in H_0^1(\Omega) \text{ with } v\geqslant 0 \text{ q.e. in } \Omega.$$

Hence this implies that the element λ_1, which was defined in (2.7), is non-negative, i.e. λ_1 is a Radon measure.

By analogy we deduce that if $\lambda_2\in H^{-1}(\Omega)$ is defined as

$$-\Delta\big[-(y-\Psi)^-\big]-\lambda_2=-(f+\Delta\Psi)^-, \tag{2.10}$$

then λ_2 is a non-negative Radon measure as well.

Since $-\Delta y+\Delta\Psi=-\Delta(y-\Psi)^+-\Delta[-(y-\Psi)^-]$ it follows that the element $\lambda=\lambda_1-\lambda_2$ of $H^{-1}(\Omega)$ satisfies the equality

$$-\Delta(y-\Psi)+\lambda=f+\Delta\Psi, \tag{2.11}$$

and it is a Radon measure with $|\lambda|\leqslant\lambda_1+\lambda_2$. Thus we have obtained the following result.

Lemma 2.1. *Let $\mu\in\mathcal{M}_0^2(\Omega)$, $\Psi\in H_0^1(\Omega)\cap H^2(\Omega)$, $f\in L^2(\Omega)$, and let $y\in H_0^1(\Omega)$ be the corresponding solution of the problem* (2.5)*. Let λ_1, λ_2, λ be the elements of $H^{-1}(\Omega)$ defined by* (2.7), (2.10) *and* (2.11)*, respectively. Then λ_1, λ_2, λ are Radon measures such that $\lambda=\lambda_1-\lambda_2$, $|\lambda|\leqslant\lambda_1+\lambda_2$, $\lambda_1\geqslant 0$, $\lambda_2\geqslant 0$ and for every compact set $K\subseteq\Omega$ the following inequality holds*

$$|\lambda|(K)\leqslant 2\sqrt{\mathrm{cap}(K,\Omega)}\big(\|\nabla y-\nabla\Psi\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}+\|\Delta\Psi\|_{L^2(\Omega)}\big), \tag{2.12}$$

where

$$\mathrm{cap}(K,\Omega)=\inf\left\{\int_\Omega |\nabla z|^2\,\mathrm{d}x,\ z\geqslant 1 \text{ a.e. in the neighborhood of } K\right\}.$$

In order to prove the estimate (2.12) we note that from (2.7) we have the following upper bound

$$\|\lambda_1\|_{H^{-1}(\Omega)}\leqslant\|\nabla y-\nabla\Psi\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}+\|\Delta\Psi\|_{L^2(\Omega)}.$$

By analogy from (2.10) we have

$$\|\lambda_2\|_{H^{-1}(\Omega)} \leqslant \|\nabla y - \nabla\Psi\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\Delta\Psi\|_{L^2(\Omega)}.$$

For every $\delta > 0$ we fix a function $z \in H_0^1(\Omega)$ such that $\|z\|^2_{H_0^1(\Omega)} \leqslant \mathrm{cap}(K,\Omega) + \delta$ and $z \geqslant 0$ q.e. in Ω, $z \geqslant 1$ q.e. in a neighborhood of K, where $K \subseteq \Omega$ is any compact set. Then

$$\begin{aligned}|\lambda|(K) &\leqslant \int_\Omega z\,\mathrm{d}\lambda_1 + \int_\Omega z\,\mathrm{d}\lambda_2 \leqslant \|z\|_{H_0^1(\Omega)}\big(\|\lambda_1\|_{H^{-1}(\Omega)} + \|\lambda_2\|_{H^{-1}(\Omega)}\big)\\ &\leqslant 2\big(\mathrm{cap}(K,\Omega)+\delta\big)^{1/2}\big(\|\nabla y - \nabla\Psi\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\Delta\Psi\|_{L^2(\Omega)}\big).\end{aligned}$$

Passing to the limit as $\delta \to 0$ we obtain (2.12).

Let us consider the following modification of the state constrained optimal control problem (1.1)–(1.4)

$$-\Delta y + \mu_\varepsilon \cdot (y - \Psi_\varepsilon) = f_\varepsilon + u, \quad \text{in } V'_{\mu_\varepsilon}, \tag{2.13}$$

$$y - \Psi_\varepsilon \in V_{\mu_\varepsilon}, \qquad u \in L^2(\Omega), \tag{2.14}$$

$$\begin{aligned}I_\varepsilon(u,y) = &\int_{\Omega\setminus S_\varepsilon} |y - z_d|^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x + \int_\Omega |\nabla(y-\Psi_\varepsilon)|^2\,\mathrm{d}x\\ &+ \int_\Omega (y-\Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon \to \inf,\end{aligned} \tag{2.15}$$

where $f_\varepsilon \in L^2(\Omega)$, $\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$, $z_d \in L^2(\Omega)$, and a Borel measure μ_ε is defined in such form that $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$ and for any $u \in L^2(\Omega)$ the corresponding solution of the problem (2.13)–(2.14) satisfies the condition (1.3) (see [11]). For instance this can be achieved by the rule

$$\mu_\varepsilon(B) = \begin{cases} 0, & \text{if } \mathrm{cap}(B \cap S_\varepsilon, \Omega) = 0,\\ +\infty, & \text{otherwise.}\end{cases} \tag{2.16}$$

As we will see later this problem always has a solution and good stability properties. In particular, in view of the results mentioned above and well-known Lax–Milgram lemma we may conclude by the following proposition.

Proposition 2.1. *Assume that for a fixed $\varepsilon \in E$, $f_\varepsilon \in L^2(\Omega)$, $\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$, and the Borel measure $\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)$ is defined in* (2.16). *Then for every $u \in L^2(\Omega)$ there exists a unique solution of the problem*

$$\begin{aligned}&y - \Psi_\varepsilon \in V_{\mu_\varepsilon},\\ &\int_\Omega (\nabla y, \nabla\varphi)\,\mathrm{d}x + \int_\Omega (y-\Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u)\cdot\varphi\,\mathrm{d}x \quad \forall\varphi \in V_{\mu_\varepsilon}\end{aligned}$$

such that $y_\varepsilon = \Psi_\varepsilon$ quasi-everywhere in S_ε, and $\lambda = \Delta y_\varepsilon + f_\varepsilon$ is a Radon measure satisfying the inequality (2.12) *for every compact set $K \subseteq \Omega$.*

3. Homogenization of the modified optimal control problem

The aim of this section is to study the limiting behavior of the modified problem (2.13)–(2.15) as $\varepsilon \to 0$. We will look for the homogenized limit of this problem using so-called "direct approach" which is based on the concept of variational convergence of constrained minimization problems. In order to introduce the formal concept for the homogenization process we represent the optimal control problem (2.13)–(2.15) in the following form

$$\Big\{ \Big\langle \inf_{(u,y)\in \Xi_\varepsilon} I_\varepsilon(u,y) \Big\rangle,\ \varepsilon \in E \Big\}, \tag{3.17}$$

where the cost functional $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ and their domains Ξ_ε are

$$\Xi_\varepsilon = \left\{ (u,y) \;\middle|\; \begin{array}{ll} -\Delta y + \mu_\varepsilon(y - \Psi_\varepsilon) = f_\varepsilon + u & \text{in } V'_{\mu_\varepsilon}, \\ y - \Psi_\varepsilon \in V_{\mu_\varepsilon}, & u \in L^2(\Omega). \end{array} \right\}, \tag{3.18}$$

$$I_\varepsilon(u,y) = \int_{\Omega\setminus S_\varepsilon} (y - z_d)^2 \,\mathrm{d}x + \int_\Omega u^2 \,\mathrm{d}x + \int_\Omega (y - \Psi_\varepsilon)\,\mathrm{d}\mu_\varepsilon + \int_\Omega \big|\nabla(y - \Psi_\varepsilon)\big|^2 \,\mathrm{d}x. \tag{3.19}$$

First of all we emphasize the fact that for every $\varepsilon \in E$ each of the sets of admissible pairs Ξ_ε is non-empty convex and closed in the following sense: we say that a sequence$\{(u_n, y_n) \in \Xi_\varepsilon\}_{n\in\mathbb{N}}$ τ_ε-converges to some pair (u,y) if

$$u_n \to u \quad \text{weakly in } L^2(\Omega),$$
$$\int_\Omega (\nabla y_n, \nabla\varphi)\,\mathrm{d}x + \int_\Omega (y_n - \Psi_\varepsilon)\varphi \,\mathrm{d}\mu_\varepsilon \to \int_\Omega (\nabla y, \nabla\varphi)\,\mathrm{d}x + \int_\Omega (y - \Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon \quad \forall\varphi \in V_{\mu_\varepsilon}.$$

It should be remarked that for different values $\varepsilon \in E$ the sets Ξ_ε belong to different functional spaces. We proceed to define the concept of convergence in "varying functional spaces". For this we recall the notion of γ^Δ-convergence in $\mathcal{M}_0^2(\Omega)$ that was introduced in [10].

Definition 3.1. Let $\{\mu_\varepsilon\}_{\varepsilon\in E}$ be a sequence of Borel measures of $\mathcal{M}_0^2(\Omega)$ and let $\mu \in \mathcal{M}_0^2(\Omega)$. We say that $\{\mu_\varepsilon\}_{\varepsilon\in E}$ γ^Δ-converges to μ in Ω if for every $f \in H^{-1}(\Omega)$ the solution y_ε of the problem

$$y_\varepsilon \in V_{\mu_\varepsilon}, \quad \langle -\Delta y_\varepsilon, \varphi\rangle + \int_\Omega y_\varepsilon \varphi \,\mathrm{d}\mu_\varepsilon = \langle f, \varphi\rangle \quad \forall\varphi \in V_{\mu_\varepsilon} \tag{3.20}$$

converges weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$ to the solution $y \in V_\mu$ of the problem

$$\langle -\Delta y, \varphi\rangle + \int_\Omega y\varphi\,\mathrm{d}\mu = \langle f, \varphi\rangle \quad \forall\varphi \in V_\mu. \tag{3.21}$$

We quote the following compactness result by Dal Maso and Murat [11,12].

Theorem 3.1. *Every sequence of measures of $\mathcal{M}_0^2(\Omega)$ contains a γ^Δ-convergent subsequence, the γ^Δ-limit of which is unique.*

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Remark 3.1. Note also that since the solutions of (3.20) depend continuously on f uniformly with respect to the measure μ_ε (see [11]) it follows that if $\mu \in \mathcal{M}_0^2(\Omega)$ is the γ^Δ-limit of $\{\mu_\varepsilon\}_{\varepsilon\in E}$ and $\{f_\varepsilon \in H^{-1}(\Omega)\}_{\varepsilon\in E}$ is strongly convergent to a $f \in H^{-1}(\Omega)$ then the functions $y_\varepsilon \in V_{\mu_\varepsilon}$ defined as the solutions of the problem

$$\langle -\Delta y_\varepsilon, \varphi\rangle + \int_\Omega y_\varepsilon \varphi \,\mathrm{d}\mu_\varepsilon = \langle f_\varepsilon, \varphi\rangle \quad \text{for every } \varphi \in V_{\mu_\varepsilon}$$

converge weakly in $H_0^1(\Omega)$ to the solution $y \in H_0^1(\Omega)$ of the limit problem (3.21).

Due to this remark we immediately have the following result.

Lemma 3.1. *Let $\{\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)\}_{\varepsilon\in E}, \{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}, \{u_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}, \{\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)\}_{\varepsilon\in E}$ be sequences such that*

$$\begin{cases} \mu_\varepsilon \to \mu_0 & \text{in the sense of } \gamma^\Delta\text{-convergence,} \\ f_\varepsilon \to f_0 & \text{weakly in } L^2(\Omega), \\ u_\varepsilon \to u_0 & \text{weakly in } L^2(\Omega), \\ \Psi_\varepsilon \to \Psi_0 & \text{weakly in } H_0^1(\Omega), \\ \Delta\Psi_\varepsilon \to \Delta\Psi_0 & \text{strongly in } H^{-1}(\Omega). \end{cases} \tag{3.22}$$

Then the solution y_ε of the problem

$$\begin{cases} y_\varepsilon - \Psi_\varepsilon \in V_{\mu_\varepsilon}, \\ \langle -\Delta y_\varepsilon, \varphi\rangle + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)\varphi \quad \forall \varphi \in V_{\mu_\varepsilon} \end{cases} \tag{3.23}$$

converges weakly in $H_0^1(\Omega)$ to the solution y_0 of the problem

$$\begin{cases} y_0 - \Psi_0 \in V_{\mu_0}, \\ \langle -\Delta y_0, \varphi\rangle + \int_\Omega (y_0 - \Psi_0)\varphi \,\mathrm{d}\mu_0 = \int_\Omega (f_0 + u_0)\varphi \,\mathrm{d}x \quad \forall \varphi \in V_{\mu_0}. \end{cases} \tag{3.24}$$

In view of this lemma we give the definition of the limiting object for the sequence of constrained minimization problems (3.17).

Definition 3.2. We say that a constrained minimization problem

$$\Big\langle \inf_{(u,y)\in \Xi_0} I_0(u,y) \Big\rangle \tag{3.25}$$

is the variational limit of the sequence (3.17) with respect to the γ^Δ-convergence if the following conditions are satisfied:

(V1) $$\Xi_0 = \left\{ (u,y) \;\middle|\; \begin{array}{l} u \in L^2(\Omega), \qquad y - \Psi_0 \in V_{\mu_0}, \\ \langle -\Delta y, \varphi\rangle + \int_\Omega (y - \Psi_0)\varphi \,\mathrm{d}\mu_0 = \int_\Omega (f_0 + u)\varphi \,\mathrm{d}x \quad \forall \varphi \in V_{\mu_0} \end{array} \right\}, \tag{3.26}$$

where the elements f_0, Ψ_0 and μ_0 are defined by (3.22);

(V2) For every pair $(u, y) \in \Xi_0$, every index set $H \in \mathcal{H}^\#$, and every sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in H}$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\ \forall\varepsilon \in H$, $u_\varepsilon \to u$ weakly in $L^2(\Omega)$, $y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$, we have $I_0(u, y) \leqslant \liminf_{H\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)$;

(V3) For every pair $(u, y) \in \Xi_0$ one can find an index set $H \in \mathcal{H}$ and a sequence $\{(\bar u_\varepsilon, \bar y_\varepsilon)\}_{\varepsilon\in H}$ such that $(\bar u_\varepsilon, \bar y_\varepsilon) \in \Xi_\varepsilon$ for all $\varepsilon \in H$, $\bar u_\varepsilon \to u$ weakly in $L^2(\Omega)$, $\bar y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$, and $I_0(u, y) \geqslant \limsup_{H\ni\varepsilon\to 0} I_\varepsilon(\bar u_\varepsilon, \bar y_\varepsilon)$.

Remark 3.2. As follows from condition (V1) the set Ξ_0 is the sequential version of the limit for $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ in Kuratowski's sense. Indeed since $\mu_0 \in \mathcal{M}_0^2(\Omega)$ is a γ^Δ-limit of the $\{\mu_\varepsilon\}_{\varepsilon\in E}$ it follows that for any weakly in $L^2(\Omega)$ convergent subsequence of controls $\{u_\varepsilon\}_{\varepsilon\in H}$ to some u^0, $H \in \mathcal{H}^\#$, the corresponding sequence of solutions $\{y_\varepsilon = y(u_\varepsilon, \mu_\varepsilon)\}_{\varepsilon\in H}$ in accordance with Lemma 3.1, converges weakly in $H_0^1(\Omega)$ to the solution $y_0 = y(u_0, \mu_0)$ of the problem (3.24). Hence, the limit pair (u_0, y_0) of $\{(u_\varepsilon, y_\varepsilon)\}$ belongs to Ξ_0. And conversely for any pair $(u_0, y_0) \in \Xi_0$ the sequence $\{u_0, y_\varepsilon = y(u_0, \mu_\varepsilon)\}_{\varepsilon\in H}$ is such that

$$(u_0, y_\varepsilon) \in \Xi_\varepsilon \quad \text{for every } \varepsilon \in H, \qquad y_\varepsilon \to y_0 \quad \text{weakly in } H_0^1(\Omega) \text{ (Lemma 3.1)}.$$

Thus Ξ_0 satisfies all sequential properties of the Kuratowski's limit (see [2]).

Now we are in a position to prove the main result of this section.

Theorem 3.2. *Let* $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$, $\{\mu_\varepsilon \in \mathcal{M}_0^2(\Omega)\}$, $\{\Psi_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)\}_{\varepsilon\in E}$ *be arbitrary sequences such that* $f_\varepsilon \to f_0$ *weakly in* $L^2(\Omega)$, $\Psi_\varepsilon \to \Psi_0$ *weakly in* $H_0^1(\Omega)$, $\Delta\Psi_\varepsilon \to \Delta\Psi_0$ *strongly in* $H^{-1}(\Omega)$, $\mu_\varepsilon(B) = +\infty$ *for every* $B \subseteq \Omega$ *such that* $\operatorname{cap}(B \cap S_\varepsilon, \Omega) > 0$. *Then there exist a non-negative measure* $\mu_0 \in \mathcal{M}_0^2(\Omega)$ *and a subsequence of* (3.17) *for which the variational limit exists, and it can be recovered in the form of the following optimal control problem*

$$-\Delta y + (y - \Psi_0)\mu_0 = f_0 + u \quad \text{in } V'_{\mu_0}, \tag{3.27}$$

$$y - \Psi_0 \in V_{\mu_0}, \qquad u \in L^2(\Omega), \tag{3.28}$$

$$I_0(u, y) = \int_\Omega \chi_*(y - z_d)^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x + \int_\Omega \left|\nabla(y - \Psi_0)\right|^2\mathrm{d}x + \int_\Omega |y - \Psi_0|^2\,\mathrm{d}\mu_0 \to \inf. \tag{3.29}$$

Proof. Let $\{\chi_\varepsilon\}_{\varepsilon\in E}$ be the sequence of the characteristic functions of the sets $\Omega\backslash S_\varepsilon$. We denote by χ_* the weak-$*$ limit point in $L^\infty(\Omega)$ of this sequence. For the sake of simplicity we always assume that $\mu_\varepsilon \to \mu_0$ in the sense of γ^Δ-convergence and $\chi_\varepsilon \to \chi_*$ weakly-$*$ in $L^\infty(\Omega)$. For every $\{y_\varepsilon\}_{\varepsilon\in E}$ such that $y_\varepsilon \to y_0$ weakly in $H_0^1(\Omega)$ we have $y_\varepsilon^2 \to y_0^2$ strongly in $L^1(\Omega)$. Therefore

$$\int_{\Omega\backslash S_\varepsilon} (y_\varepsilon - z_d)^2\,\mathrm{d}x = \int_\Omega \chi_\varepsilon(y_\varepsilon - z_d)^2\,\mathrm{d}x \to \int_\Omega \chi_*(y_0 - z_d)^2\,\mathrm{d}x \quad \text{as } \varepsilon \to 0. \tag{3.30}$$

To prove the representation (3.29) we have to verify the conditions (V2)–(V3). Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in H}$ be any sequence satisfying the condition (V2), i.e.

$$(u_\varepsilon, y_\varepsilon) \to (u, y) \in \Xi_0 \quad \text{weakly in } L^2(\Omega) \times H_0^1(\Omega).$$

Then by (3.30) and the lower semi-continuity property of the norm in $L^2(\Omega)$ with respect to the weak convergence we have

$$\begin{aligned}
&\liminf_{H\ni\varepsilon\to 0}\left[\int_\Omega \chi_\varepsilon (y_\varepsilon - z_d)^2\,\mathrm{d}x + \int_\Omega u_\varepsilon^2\,\mathrm{d}x + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon + \int_\Omega \left|\nabla(y_\varepsilon - \Psi_\varepsilon)\right|^2\mathrm{d}x\right] \\
&\quad\geqslant \int_\Omega \chi_* (y - z_d)^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x + \liminf_{H\ni\varepsilon\to 0}\left[\int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon + \int_\Omega \left|\nabla(y_\varepsilon - \Psi_\varepsilon)\right|^2\mathrm{d}x\right]. \qquad (3.31)
\end{aligned}$$

Since for every $\varepsilon \in H$ each of the pairs $(u_\varepsilon, y_\varepsilon)$ satisfies the equality

$$\langle -\Delta y_\varepsilon, \varphi\rangle + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)\varphi\,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)\varphi\,\mathrm{d}x \quad \forall \varphi \in V_{\mu_\varepsilon} \qquad (3.32)$$

one obtains, upon putting $\varphi = y_\varepsilon - \Psi_\varepsilon$ into (3.32) and integrating by parts:

$$\int_\Omega \left|\nabla(y_\varepsilon - \Psi_\varepsilon)\right|^2\mathrm{d}x + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon)\,\mathrm{d}x + \langle \Delta\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon\rangle. \qquad (3.33)$$

As a result, taking into account that

$$\begin{array}{llll}
f_\varepsilon \to f_0 & \text{weakly in } L^2(\Omega), & u_\varepsilon \to u & \text{weakly in } L^2(\Omega), \\
\Psi_\varepsilon \to \Psi_0 & \text{weakly in } H_0^1(\Omega), & \Delta\Psi_\varepsilon \to \Delta\Psi_0 & \text{strongly in } H^{-1}(\Omega),
\end{array}$$

we obtain

$$\begin{cases}
\int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon)\,\mathrm{d}x \to \int_\Omega (f_0 + u)(y - \Psi_0)\,\mathrm{d}x, \\
\langle \Delta\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon\rangle \to \langle \Delta\Psi_0, y - \Psi_0\rangle.
\end{cases} \qquad (3.34)$$

Now we use the fact that $(u, y) \in \Xi_0$. We take as a test function $\varphi = y - \Psi_0$ in (3.26) and integrate by parts. Then

$$\int_\Omega \left|\nabla(y - \Psi_0)\right|^2\mathrm{d}x + \int_\Omega (y - \Psi_0)\,\mathrm{d}\mu_0 = \int_\Omega (f_0 + u)(y - \Psi_0)\,\mathrm{d}x + \langle \Delta\Psi_0, y - \Psi_0\rangle. \qquad (3.35)$$

Using (3.34)–(3.35), from (3.33) we conclude

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left[\int_\Omega \left|\nabla(y_\varepsilon - \Psi_\varepsilon)\right|^2\mathrm{d}x + \int_\Omega (y_\varepsilon - \Psi_\varepsilon)^2\,\mathrm{d}\mu_\varepsilon\right] \\
&\quad= \lim_{\varepsilon\to 0}\left[\int_\Omega (f_\varepsilon + u_\varepsilon)(y_\varepsilon - \Psi_\varepsilon)\,\mathrm{d}x + \langle \Delta\Psi_\varepsilon, y_\varepsilon - \Psi_\varepsilon\rangle\right] \\
&\quad= \int_\Omega (f_0 + u)(y - \Psi_0)\,\mathrm{d}x + \langle \Delta\Psi_0, y - \Psi_0\rangle = \int_\Omega \left|\nabla(y - \Psi_0)\right|^2\mathrm{d}x + \int_\Omega (y - \Psi_0)^2\,\mathrm{d}\mu_0. \qquad (3.36)
\end{aligned}$$

Inserting this equality into (3.31) we immediately obtain

$$\liminf_{H\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) \geqslant I_0(u, y),$$

i.e. the condition (V2) holds true.

In order to prove condition (V3) we take any pair (u, y) of Ξ_0 and construct the corresponding sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon \in E}$ by the rule

$$\bar{u}_\varepsilon = u \quad \text{for every } \varepsilon \in E,$$

and $\bar{y}_\varepsilon = y(u, \mu_\varepsilon)$ are the solutions of the problems (3.23) under $u_\varepsilon = \bar{u}_\varepsilon = u$.

It is easy to see that $(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon \in E$. Moreover, due to Theorem 6.8 in [11] and the fact that $\mu_\varepsilon \to \mu_0$ in the sense of γ^Δ-convergence we have

$$\bar{y}_\varepsilon \to y \quad \text{weakly in } H_0^1(\Omega).$$

Using (3.30) and (3.36) we obtain

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} I_\varepsilon(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \\
&\quad = \limsup_{\varepsilon \to 0} \left[\int_\Omega \chi_\varepsilon (\bar{y}_\varepsilon - z_d)^2 \, \mathrm{d}x + \int_\Omega u^2 \, \mathrm{d}x + \int_\Omega \left| \nabla (\bar{y}_\varepsilon - \Psi_\varepsilon) \right|^2 \mathrm{d}x + \int_\Omega (\bar{y}_\varepsilon - \Psi_\varepsilon)^2 \, \mathrm{d}\mu_\varepsilon \right] \\
&\quad = I_0(u, y),
\end{aligned}$$

i.e. condition (V3) holds true as well. This concludes the proof. □

In view of this theorem we will always assume that the sequence of optimal control problems (3.17) is convergent in the sense of Definition 3.2.

Remark 3.3. The conditions (V2) and (V3) coincide with the definition of the Γ-limit for the following sequence of uniformly coercive in $L^2(\Omega) \times H_0^1(\Omega)$ functionals (see [2,4])

$$\{ P_{\Xi_\varepsilon} I_\varepsilon : L^2(\Omega) \times H_0^1(\Omega) \to \bar{\mathbb{R}} \}_{\varepsilon \in E},$$

where the extension operators P_{Ξ_ε} are defined as

$$P_{\Xi_\varepsilon} I(u, y) = \begin{cases} I(u, y), & \text{if } (u, y) \in \Xi_\varepsilon, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore, the variational limit (3.25) inherits all variational properties of Γ-limits. Namely, if $(u^*, y^*) \in \Xi_0$ is a weak limit in $L^2(\Omega) \times H_0^1(\Omega)$ of the sequence of optimal pairs for the original problem (3.17) then:

(A1) (u^*, y^*) is an optimal solution for the limit problem (3.25);
(A2) $I_0(u^*, y^*) = \lim_{\varepsilon \to 0} \inf_{(u,y) \in \Xi_\varepsilon} I_\varepsilon(u, y)$.

Indeed, the following result holds.

Corollary 3.1. *The unique optimal pair* (u^0, y^0) *for the limit problem* (3.27)–(3.29) *is the weak limit in* $L^2(\Omega) \times H_0^1(\Omega)$ *of the sequence of optimal solutions* $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon \in E}$ *for the original problem* (3.17) *and* *moreover*

$$\lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) = I_0(u^0, y^0). \tag{3.37}$$

Proof. Using the direct method of calculus of variations, it is easy to show that each of the optimal control problems (3.17) and (3.27)–(3.29) admits a unique solution $(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ and $(u^0, y^0) \in \Xi_0$, respectively. Moreover, taking into account the coerciveness property of the cost functional (2.15) the sequence of optimal pairs $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in E}$ is uniformly bounded in $L^2(\Omega) \times H^1_0(\Omega)$ with respect to the small parameter ε. So, we may always extract from $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in E}$ a weakly convergent subsequence $\{(v^0_\sigma, p^0_\sigma)\}_{\sigma\in H}$ $(H \in \mathcal{H}^\#)$. Let $(u^*, y^*) \in L^2(\Omega) \times H^1_0(\Omega)$ be its limit. Then, by Lemma 3.1, we have $(u^*, y^*) \in \Xi_0$. Moreover, due to the (V2)-part of Definition 3.2,

$$\liminf_{H\ni\sigma\to 0} \min_{(u,y)\in\Xi_\sigma} I_\sigma(u, y) = \liminf_{H\ni\sigma\to 0} I_\sigma(v^0_\sigma, p^0_\sigma) \geqslant I_0(u^*, y^*) \geqslant \min_{(u,y)\in\Xi_0} I_0(u, y) = I_0(u^0, y^0). \tag{3.38}$$

At the same time, as follows from (V3)-part of Definition 3.2, there exist an index set $H \in \mathcal{H}$ and a sequence $\{(u_\varepsilon, y_\varepsilon)\}$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$ for all values $\varepsilon \in H$,

$$(u_\varepsilon, y_\varepsilon) \to (u^0, y^0) \quad \text{as } H \ni \varepsilon \to 0,$$

and $I_0(u^0, y^0) \geqslant \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)$. Using this fact, we get

$$\begin{aligned}\min_{(u,y)\in\Xi_0} I_0(u, y) = I_0(u^0, y^0) &\geqslant \limsup_{H\ni\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)\\ &\geqslant \limsup_{\varepsilon\to 0} \min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) \geqslant \limsup_{H\ni\sigma\to 0} \min_{(u,y)\in\Xi_\sigma} I_\sigma(u, y) = \limsup_{H\ni\sigma\to 0} I_\sigma(v^0_\sigma, p^0_\sigma).\end{aligned} \tag{3.39}$$

From (3.38), it follows that $\liminf_{H\ni\sigma\to 0} I_\sigma(v^0_\sigma, p^0_\sigma) \geqslant \limsup_{H\ni\sigma\to 0} I_\sigma(v^0_\sigma, p^0_\sigma)$. Combining (3.38) and (3.39), we conclude

$$I_0(u^*, y^*) = I_0(u^0, y^0) = \min_{(u,y)\in\Xi_0} I_0(u, y),$$

$$I_0(u^0, y^0) = \lim_{H\ni\sigma\to 0} \min_{(u,y)\in\Xi_\sigma} I_\sigma(u, y).$$

Taking into account these relations and the uniqueness of the solution of the problem (3.27), we obtain $(u^*, y^*) = (u^0, y^0)$. Since this equality holds for the limits of any converging subsequences of $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in E}$, (u^0, y^0) is the weak limit of the sequence $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon\in E}$. Making for the sequence of minimizers what we did before with the subsequence $\{(v^0_\sigma, p^0_\sigma)\}_{H\ni\sigma\to 0}$, we obtain

$$\begin{aligned}\liminf_{E\ni\varepsilon\to 0} \min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) &= \liminf_{E\ni\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \geqslant I_0(u^0, y^0)\\ &= \min_{(u,y)\in\Xi_0} I_0(u, y) \geqslant \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon)\\ &\geqslant \limsup_{E\ni\varepsilon\to 0} \min_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) = \limsup_{E\ni\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon).\end{aligned}$$

Thus, the relations (3.37) hold. □

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4. On the homogenization of Dirichlet optimal control problems in perforated domains

In this section we will study of the homogenization of an optimal control problem involving a canonical linear elliptic equation with Dirichlet boundary conditions in a perforated domain. The problem is to describe the asymptotic behavior of the sequence of such optimal control problems as a small parameter tends to zero. We will show that the study of this problem can be traced by the approach above.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\varepsilon \in E = (0; \varepsilon^0]$ be a small parameter and let $\{\Omega_\varepsilon\}_{\varepsilon \in E}$ be a sequence of open sets contained in Ω. For given $z_d \in L^2(\Omega)$ and $f_\varepsilon \in L^2(\Omega)$ we define the optimal control problem in Ω_ε as follows: find a pair “control–state” $(u_\varepsilon^0, y_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)$ such that

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = \inf_{(u,y) \in \Xi_\varepsilon} I_\varepsilon(u, y), \tag{4.40}$$

where

$$I_\varepsilon(u, y) = \int_{\Omega_\varepsilon} |\nabla y|^2 \, \mathrm{d}x + \int_{\Omega_\varepsilon} (y - z_d)^2 \, \mathrm{d}x + \int_{\Omega_\varepsilon} u^2 \, \mathrm{d}x, \tag{4.41}$$

$$\Xi_\varepsilon = \left\{ (u_\varepsilon, y_\varepsilon) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon) \;\middle|\; \begin{array}{l} -\Delta y_\varepsilon = f_\varepsilon + u_\varepsilon \text{ in } \mathcal{D}'(\Omega_\varepsilon), \\ \tilde{u}_\varepsilon \in \mathcal{U}_\varepsilon \end{array} \right\}. \tag{4.42}$$

Here $\tilde{u}_\varepsilon$ is the trivial extension of $u_\varepsilon \in L^2(\Omega_\varepsilon)$ by zero to the hole of Ω; $\mathcal{U}_\varepsilon$ is a weakly closed convex subset of $L^2(\Omega)$ such that

$$\mathcal{U}_\varepsilon \to \mathcal{U}_0, \quad (\mathcal{U}_0 \neq \emptyset) \text{ in the Kuratowski's sense}$$

with respect to the weak topology of $L^2(\Omega)$; χ_ε is the characteristic function of the set Ω_ε, i.e. $\chi_\varepsilon = 1$ in Ω_ε and $\chi_\varepsilon = 0$ otherwise.

As it is well known for every $\varepsilon \in E$ there exists a unique solution of the problem (4.40)–(4.42) (see [19]). In view of the uniform coerciveness of the cost functionals $I_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$, the sequence of optimal pairs $\{(u_\varepsilon^0, y_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)\}_{\varepsilon \in E}$ is uniformly bounded, i.e.

$$\sup_{\varepsilon \in E} \left[\|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} + \|y_\varepsilon^0\|_{H_0^1(\Omega_\varepsilon)} \right] < \infty.$$

So it is natural to choose as a basic topology for the homogenization procedure the product of the weak topologies for $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Since y_ε^0 is equal to 0 on the boundary $\partial\Omega_\varepsilon$ it is reasonable to extend y_ε^0 by zero to the hole of Ω. We denote this extension by $\tilde{y}_\varepsilon^0$. It belongs to $H_0^1(\Omega)$ and moreover by Poincaré inequality it is easy to show that

$$\|\tilde{y}_\varepsilon^0\|_{H_0^1(\Omega)} \leqslant C \tag{4.43}$$

for all $f_\varepsilon \in L^2(\Omega)$, $z_d \in L^2(\Omega)$ and $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$.

Note the contrast with the case of Neumann boundary conditions on $\partial\Omega_\varepsilon$ (see [15]). The extension by zero of the solution of the Dirichlet problem (4.42) now belongs to the space $H_0^1(\Omega)$ while the same extension for the solution of the Neumann problem is only an element of $L^2(\Omega)$. That is why in that case one needs to construct extension operators.

Further we note that due to the estimate (4.43) one may assume that

$$\tilde{y}^0_\varepsilon \to y \quad \text{weakly in } H^1_0(\Omega).$$

Let us denote by χ_* the weak-$*$ limit in $L^\infty(\Omega)$ of the sequence $\{\chi_\varepsilon \in L^\infty(\Omega)\}_{\varepsilon\in E}$.

Remark 4.1. We shall always assume that supp$(\chi_*) =$ closure(Ω), that is $\chi_* > 0$ almost everywhere on Ω. It is clear that in this case $\Omega \setminus \Omega_\varepsilon$ have to be small enough. Suppose also that $\chi_*^{-1} \in L^\infty(\Omega)$.

Remark 4.2. We also note that if y is a function of $H^1_0(\Omega)$ such that $y = 0$ quasi-everywhere in $\Omega\setminus\Omega_\varepsilon$ then the restriction of y to Ω_ε belongs to $H^1_0(\Omega_\varepsilon)$ (see [3]). Conversely if we extend a function $y \in H^1_0(\Omega_\varepsilon)$ by setting $y = 0$ in $\Omega\setminus\Omega_\varepsilon$, then y is quasi-continuous and belongs to $H^1_0(\Omega)$. Therefore, if we define the non-negative Borel measure $\mu_\varepsilon \in \mathcal{M}^2_0(\Omega)$ as

$$\mu_\varepsilon(B) = \begin{cases} 0, & \text{if } \operatorname{cap}\big(B \cap (\Omega\setminus\Omega_\varepsilon), \Omega\big) = 0, \\ +\infty, & \text{if } \operatorname{cap}\big(B \cap (\Omega\setminus\Omega_\varepsilon), \Omega\big) > 0, \end{cases} \tag{4.44}$$

then a function $y \in H^1_0(\Omega)$ is the solution of the problem

$$\begin{cases} y \in V_{\mu_\varepsilon} = H^1_0(\Omega) \cap L^2_{\mu_\varepsilon}(\Omega), \\ \langle -\Delta y, \varphi\rangle + \displaystyle\int_\Omega y\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + \chi_\varepsilon u)\varphi \,\mathrm{d}x \quad \forall \varphi \in V_{\mu_\varepsilon} \end{cases} \tag{4.45}$$

if and only if the restriction of y to Ω_ε is the solution of the Dirichlet boundary value problem

$$y \in H^1_0(\Omega_\varepsilon) \quad -\Delta y = f_\varepsilon + u \quad \text{in } \mathcal{D}'(\Omega_\varepsilon)$$

and in addition $y = 0$ q.e. in $\Omega\setminus\Omega_\varepsilon$ (see [11,12]).

Thanks to this result we may rewrite the original optimal control problem (4.40)–(4.42) as (3.17)–(3.19). We thus consider the sequence

$$\Big\{\Big\langle \inf_{(u,y)\in\tilde{\Xi}_\varepsilon} \tilde{I}_\varepsilon(u,y)\Big\rangle, \varepsilon \in E\Big\} \tag{4.46}$$

with

$$\tilde{I}_\varepsilon(u,y) = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega (y - \chi_\varepsilon z_d)^2 \,\mathrm{d}x + \int_\Omega \chi_\varepsilon u^2 \,\mathrm{d}x, \quad \forall (u,y) \in \tilde{\Xi}_\varepsilon, \tag{4.47}$$

$$\tilde{\Xi}_\varepsilon = \left\{ (u,y) \;\middle|\; \begin{array}{c} \langle -\Delta y, \varphi\rangle + \displaystyle\int_\Omega y\varphi \,\mathrm{d}\mu_\varepsilon = \int_\Omega (f_\varepsilon + \chi_\varepsilon u)\varphi \,\mathrm{d}x \quad \forall \varphi \in V_{\mu_\varepsilon}, \\ y \in V_{\mu_\varepsilon},\ u \in L^2(\Omega),\ \chi_\varepsilon u \in \mathcal{U}_\varepsilon \subseteq L^2(\Omega) \end{array} \right\}, \tag{4.48}$$

where the Borel measures $\mu_\varepsilon \in \mathcal{M}^2_0(\Omega)$ are defined in (4.44).

By virtue of compactness of $\mathcal{M}^2_0(\Omega)$ with respect to the γ^Δ-convergence and the fact that for the sequence of constrained sets $\{\mathcal{U}_\varepsilon\}_{\varepsilon\in E}$ there exists a non-empty limit $\mathcal{U}_0$ (in the sense of Kuratowski with respect to the weak topology of $L^2(\Omega)$) we have the following result.

Lemma 4.1. *Assume that $f_\varepsilon \to f_0$ weakly in $L^2(\Omega)$. Then there exists a non-negative Borel measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$ such that the set*

$$\Xi_0 = \left\{ (u,y) \,\middle|\, \begin{array}{l} \langle -\Delta y, \varphi\rangle + \int_\Omega y\varphi \,\mathrm{d}\mu_0 = f_0 + u \quad \forall \varphi \in V_{\mu_0}, \\ y \in V_{\mu_0},\ u \in \mathcal{U}_0 \subseteq L^2(\Omega) \end{array} \right\} \tag{4.49}$$

is the τ-limit of the sequence $\{\tilde{\Xi}_\varepsilon \subset L^2(\Omega) \times V_{\mu_\varepsilon}\}_{\varepsilon\in E}$ in the Kuratowski's sense, where τ is the product of the weak topologies for $L^2(\Omega)$ and $H_0^1(\Omega)$.

Proof. Let us denote by $\tau - \mathrm{Li}\,\tilde{\Xi}_\varepsilon$ and $\tau - \mathrm{Ls}\,\tilde{\Xi}_\varepsilon$ the lower and the upper sequential limit of $\{\tilde{\Xi}_\varepsilon\}$ in the Kuratowski's sense, respectively. Let μ_0 be a γ^Δ-limit of the sequence of measures $\{\mu_\varepsilon\}_{\varepsilon\in E}$. Then due to Remark 3.1 and Lemma 3.1 $\tau - \mathrm{Ls}\,\tilde{\Xi}_\varepsilon \subseteq \Xi_0$. Hence, in order to prove the lemma, it sufficiently to show that $\Xi_0 \subseteq \tau - \mathrm{Li}\,\tilde{\Xi}_\varepsilon$.

Let (u^*, y^*) be any pair of Ξ_0. Then $u^* \in \mathcal{U}_0$, hence there is a sequence $\{u_\varepsilon\}_{\varepsilon\in E}$ such that $u_\varepsilon \to u^*$ weakly in $L^2(\Omega)$, $u_\varepsilon \in \mathcal{U}_\varepsilon$ for every $\varepsilon \in E$, and $u_\varepsilon = \chi_\varepsilon u_\varepsilon$ by definition of $\mathcal{U}_\varepsilon$.

We denote by $y_\varepsilon = y(u_\varepsilon, \mu_\varepsilon)$ the corresponding solutions of problem (4.45) with $u = u_\varepsilon$. Let $y_\varepsilon^* \in V_{\mu_\varepsilon}$ be the solution of (4.45) corresponding to $f_\varepsilon = f_0$ and $u = u^*$. By the definition of γ^Δ-convergence, $\{y_\varepsilon^*\}$ converges to y^* weakly in $H_0^1(\Omega)$. However, as follows immediately from (4.48), we have

$$\|y_\varepsilon^* - y_\varepsilon\|_{H_0^1(\Omega)} + \left(\int_\Omega (y_\varepsilon^* - y_\varepsilon)^2 \,\mathrm{d}\mu_\varepsilon\right)^{1/2} \leqslant \sqrt{2}\big(\|f_\varepsilon - f_0\|_{H^{-1}(\Omega)} + \|u_\varepsilon - u^*\|_{H^{-1}(\Omega)}\big).$$

In view of the standing assumptions we have $y_\varepsilon^* - y_\varepsilon \to 0$ strongly in $H_0^1(\Omega)$. Hence $y_\varepsilon \to y^*$ weakly in $H_0^1(\Omega)$. As a result, the sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon) \in \tilde{\Xi}_\varepsilon\}_{\varepsilon\in E}$ has been constructed such that $(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u^*, y^*)$, i.e. by definition of the Kuratowski limit we have $(u^*, y^*) \in \tau - \mathrm{Li}\,\tilde{\Xi}_\varepsilon$. This implies that

$$\Xi_0 \subseteq \tau - \mathrm{Li}\,\tilde{\Xi}_\varepsilon \subseteq \tau - \mathrm{Ls}\,\tilde{\Xi}_\varepsilon \subseteq \Xi_0.$$

This completes the proof. □

In order to recover the limit functional $I_0 : \Xi_0 \to \mathbb{R}$ with (V2)–(V3) properties, we note that due to the construction (4.44) we have

$$\int_{\Omega_\varepsilon} y^2 \,\mathrm{d}\mu_\varepsilon = 0 \quad \text{for any } (u,y) \in \tilde{\Xi}_\varepsilon.$$

Therefore, for every $\varepsilon \in E$ the cost functionals $\tilde{I}_\varepsilon : \tilde{\Xi}_\varepsilon \to \mathbb{R}$ can be represented in the form

$$\tilde{I}_\varepsilon(u,y) = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega y^2 \,\mathrm{d}\mu_\varepsilon + \int_\Omega (y - \chi_\varepsilon z_d)^2 \,\mathrm{d}x + \int_\Omega \chi_\varepsilon u^2 \,\mathrm{d}x \quad \forall (u,y) \in \tilde{\Xi}_\varepsilon. \tag{4.50}$$

Taking formula (3.36) into account we have

$$\lim_{H \ni \varepsilon \to 0} \left[\int_\Omega |\nabla y_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega y_\varepsilon^2 \,\mathrm{d}\mu_\varepsilon\right] = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \int_\Omega y^2 \,\mathrm{d}\mu_0, \tag{4.51}$$

$$\lim_{h\ni\varepsilon\to 0}\int_\Omega (y_\varepsilon - \chi_\varepsilon z_d)^2\,\mathrm{d}x = \int_\Omega (y-\chi_* z_d)^2\,\mathrm{d}x \tag{4.52}$$

for any index set $H \in \mathcal{H}^\#$ and any τ-convergent sequence $\{(u_\varepsilon, y_\varepsilon) \in \tilde{\Xi}_\varepsilon\}_{\varepsilon\in H}$.

In order to recover the limit value of the last term in (4.50) we assume that the constrained sets $\mathcal{U}_\varepsilon$ have the form

$$\mathcal{U}_\varepsilon = \left\{ u \in L^2(\Omega) \;\middle|\; \begin{array}{c} \xi_1\chi_\varepsilon \leqslant u \leqslant \xi_2\chi_\varepsilon, \\ u \in \Lambda \subset L^2(\Omega) \end{array} \right\}, \tag{4.53}$$

where $\xi_1, \xi_2 \in L^2(\Omega)$, $\xi_1 \leqslant \xi_2$ almost everywhere, and Λ is a closed convex subset of $L^2(\Omega)$ for which the following property holds: if $u \in \Lambda$ then $\chi_\varepsilon u \in \Lambda$ for every $\varepsilon > 0$.

It is easy to see that for the sequence $\{\mathcal{U}_\varepsilon\}$ there exists a Kuratowski limit $\mathcal{U}_0$ such that

$$\mathcal{U}_0 = \{u = \chi_* v \in L^2(\Omega) \mid \xi_1 \leqslant v \leqslant \xi_2, v \in \Lambda\}. \tag{4.54}$$

As a result we have (see [15]):

(i) $\liminf_{H\ni\varepsilon\to 0}\int_\Omega \chi_\varepsilon (u_\varepsilon)^2\,\mathrm{d}x \geqslant \int_\Omega \chi_*^{-1} u^2\,\mathrm{d}x$ for every sequence $\{u_\varepsilon \in \mathcal{U}_\varepsilon\}_{\varepsilon\in H}$ weakly converging in $L^2(\Omega)$ to some $u \in \mathcal{U}_0$;
(ii) for every $u \in \mathcal{U}_0$ the sequence $\{\widehat{u}_\varepsilon = \chi_\varepsilon \chi_*^{-1} u\}_{\varepsilon\in E}$ is such that $\widehat{u}_\varepsilon \to u$ weakly in $L^2(\Omega)$, $\widehat{u}_\varepsilon \in \mathcal{U}_\varepsilon$ $\forall \varepsilon \in E$, and $\lim_{\varepsilon\to 0}\int_\Omega (\widehat{u}_\varepsilon)^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\langle \chi_\varepsilon, \chi_*^{-2}u^2\rangle_{(L^\infty, L^1)} = \int_\Omega \chi_*^{-1}u^2\,\mathrm{d}x$.

Hence, for the sequence of functionals $\{\int_\Omega \chi_\varepsilon u^2\,\mathrm{d}x\}_{\varepsilon\in E}$ the properties (V2)–(V3) hold. As a result, combining this fact with the relations (4.51)–(4.52), we obtain the following result.

Lemma 4.2. *Assume that the sets $\mathcal{U}_\varepsilon \subset L^2(\Omega)$ have the form* (4.53). *Then under assumptions of Lemma* 4.1 *there exists a variational limit*

$$\Big\langle \inf_{(u,y)\in\Xi_0} I_0(u,y) \Big\rangle \tag{4.55}$$

such that the set Ξ_0 has the representation (4.49) *with $\mathcal{U}_0$ given by* (4.54) *and*

$$I_0(u,y) = \int_\Omega |\nabla y|^2\,\mathrm{d}x + \int_\Omega y^2\,\mathrm{d}\mu_0 + \int_\Omega (y-\chi_* z_d)^2\,\mathrm{d}x + \int_\Omega \chi_*^{-1}u^2\,\mathrm{d}x. \tag{4.56}$$

Now we are in the position to state the main result of this section. We recall that in the framework presented here the variational limit (4.55) is called a homogenized optimal control problem whenever it corresponds to some control problem that can be recovered from the limiting process.

Theorem 4.1. *Let $\xi_1, \xi_2, z_d \in L^2(\Omega)$ be such that $\xi_1 \leqslant \xi_2$ a.e. in Ω, and let $\{f_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$ be a sequence weakly converging to some function $f_0 \in L^2(\Omega)$. Assume that the sequence of open subsets $\{\Omega_\varepsilon \subset \Omega\}_{\varepsilon\in E}$ is such that*

$$\chi_\varepsilon \to \chi_* \quad \textit{weakly-}{*} \textit{ in } L^\infty(\Omega),\ \chi_*^{-1} \in L^\infty(\Omega).$$

Then there exists a non-negative Borel measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$ such that for the family of optimal control problems

$$\begin{cases} -\Delta y = f_\varepsilon + u, & \text{in } \mathcal{D}'(\Omega_\varepsilon),\\ y \in H_0^1(\Omega_\varepsilon), \chi_\varepsilon\xi_1 \leqslant u \leqslant \chi_\varepsilon\xi_2,\ u \in \Lambda,\\ I_\varepsilon(u,y) = \int_{\Omega_\varepsilon} |\nabla y|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} (y - z_d)^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} u^2\,\mathrm{d}x \to \inf \end{cases} \tag{4.57}$$

there exists a homogenized problem as $\varepsilon \to 0$ with the following representation

$$\begin{cases} -\Delta y + \mu_0 y = f_0 + u & \text{in } V'_{\mu_0},\\ y \in V_{\mu_0}, \chi_*\xi_1 \leqslant u \leqslant \chi_*\xi_2 \text{ a.e. in } \Omega, \chi_*^{-1}u \in \Lambda,\\ I_0(u,y) = \int_\Omega |\nabla y|^2\,\mathrm{d}x + \int_\Omega y^2\,\mathrm{d}\mu_0 + \int_\Omega (y - \chi_* z_d)^2\,\mathrm{d}x + \int_\Omega \chi_*^{-1}u^2\,\mathrm{d}x \to \inf. \end{cases} \tag{4.58}$$

In addition, if $(u_\varepsilon^0, y_\varepsilon^0)$ and (u^0, y^0) are the optimal solutions of the problems (4.57) *and* (4.58)*, respectively, then*

$$\tilde{y}_\varepsilon^0 \to y^0 \quad \text{weakly in } H_0^1(\Omega), \qquad \tilde{u}_\varepsilon^0 \to u^0 \quad \text{weakly in } L^2(\Omega), \tag{4.59}$$

$$\chi_\varepsilon(\tilde{u}_\varepsilon^0 - \chi_*^{-1}u^0) \to 0 \quad \text{strongly in } L^2(\Omega), \tag{4.60}$$

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \to I_0(u^0, y^0). \tag{4.61}$$

Proof. The validity of these statements including the variational properties (4.59), (4.61) immediately follows from Lemma 4.1, Lemma 4.2, Definition 3.2 and Theorem 3.1. As for the property (4.60) we note that as $\chi_\varepsilon \to \chi_*$ weakly-$*$ in $L^\infty(\Omega)$ and

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (u_\varepsilon^0)^2\,\mathrm{d}x = \int_\Omega \chi_*^{-1}(u^0)^2\,\mathrm{d}x$$

one gets

$$\int_\Omega (\chi_\varepsilon(\tilde{u}_\varepsilon^0 - \chi_*^{-1}u^0))^2\,\mathrm{d}x = \int_\Omega (\tilde{u}_\varepsilon^0)^2\,\mathrm{d}x - 2\int_\Omega \tilde{u}_\varepsilon^0 \chi_*^{-1} u^0\,\mathrm{d}x + \int_\Omega \chi_*^{-2}(u^0)^2 \chi_\varepsilon\,\mathrm{d}x \to 0.$$

This concludes the proof. □

We conclude the paper with the example of an optimal control problem on a perforated domain with Dirichlet conditions on the boundary and show that the γ^Δ-limit measure $\mu_0 \in \mathcal{M}_2^0(\Omega)$ in the limit problem (4.58) can be recovered. We consider the case where the domain Ω is perforated by an increasing number of holes of critical size. That is the type of perforated domains which was proposed by Cioranescu and Murat in their celebrated paper [6].

We define the perforated domain as follows

$$\Omega_\varepsilon = \Omega \cap Q_\varepsilon, \quad Q_\varepsilon = \mathbb{R}^n \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

Here we assume that $\mathbb{R}^n$ ($n \geqslant 2$) is covered by cubes P_i^ε of size 2ε and T_i^ε is a ball of radius $a_\varepsilon = \exp(-C_0/\varepsilon^2)$ if $n = 2$ and $a_\varepsilon = C_0\varepsilon^{n/n-2}$ if $n \geqslant 3$, centered at the very center of the cube P_i^ε.

For given $f \in L^2(\Omega)$ and each value of $\varepsilon \in (0;\varepsilon^0]$ we consider the following optimal control problem on Ω_ε

$$\begin{cases} -\Delta y = f + u \quad \text{in } \mathcal{D}'(\Omega_\varepsilon), \\ y \in H_0^1(\Omega_\varepsilon), \qquad \|u\|_{L^2(\Omega)} \leqslant 2, \\ -(\operatorname{meas}\Omega)^{-1/2}\chi_\varepsilon \leqslant \chi_\varepsilon u, \quad \text{a.e. in } \Omega, \\ I_\varepsilon(u,y) = \int_{\Omega_\varepsilon} |\nabla y|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} u^2\,\mathrm{d}x \to \inf. \end{cases} \tag{4.62}$$

It is easy to see that the weak-$*$ limit in $L^\infty(\Omega)$ of the characteristic functions $\chi_{\Omega_\varepsilon}$ is equal to $\chi_* = 1$ a.e. in Ω. Thanks to Theorem 4.1 there is a homogenized problem for the family (4.62) which can be represented in the form

$$\begin{aligned} &-\Delta y + \mu_0 y = f + u \quad \text{in } V'_{\mu_0}, \\ &y \in V_{\mu_0}, \|u\|_{L^2(\Omega)} \leqslant 2, -(\operatorname{meas}\Omega)^{-1/2} \leqslant u \quad \text{a.e. in } \Omega, \\ &I_0(u,y) = \int_\Omega |\nabla y|^2\,\mathrm{d}x + \int_\Omega y^2\,\mathrm{d}\mu_0 + \int_\Omega u^2\,\mathrm{d}x \to \inf. \end{aligned} \tag{4.63}$$

In order to recover the measure $\mu_0 \in \mathcal{M}_0^2(\Omega)$ we use the following result (see [5,6]).

Theorem 4.2 (Casado-Diaz [5], Cioranescu and Murat [6]). *If there exists a sequence* $\{\omega_\varepsilon \in H^1(\Omega)\}_{\varepsilon\in E}$ *such that*

$$\omega_\varepsilon = 0 \text{ on the holes } T_i^\varepsilon, \quad 1 \leqslant i \leqslant n(\varepsilon), \qquad \omega_\varepsilon \to 1 \quad \text{weakly in } H^1(\Omega),$$

then there is a measure $\mu_* \in \mathcal{M}_0^2(\Omega)$ *satisfying: for any* $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ *and for any* $v \in H_0^1(\Omega)$ *such that* $\tilde{v}_\varepsilon = \chi_\varepsilon v$ *and* $\tilde{v}_\varepsilon \to v$ *weakly in* $H_0^1(\Omega)$ *we have*

$$\langle -\Delta\omega_\varepsilon, v_\varepsilon\rangle \to \int_\Omega v\,\mathrm{d}\mu_*. \tag{4.64}$$

The sequence $\{\omega_\varepsilon\}_{\varepsilon\in E}$ was constructed in [6] for the above mentioned case of perforated domains $\{\Omega_\varepsilon\}$ and it was proven that the condition (4.64) holds true if μ_* is defined as

$$\mathrm{d}\mu_* = \frac{\pi}{2}\frac{1}{C_0}\,\mathrm{d}x \quad \text{if } n = 2, \qquad \mathrm{d}\mu_* = S_n(n-2)C_0^{n-2}/2^n\,\mathrm{d}x \quad \text{if } n \geqslant 3,$$

where S_n is the surface of the unit sphere in $\mathbb{R}^n$ and C_0 is a positive constant.

A different approach for the construction of the functions $\{\omega_\varepsilon\}_{\varepsilon\in E}$ can be found in [1,7,20,24].

Now we prove that $\mu_* = \mu_0$ in (4.62). For any fixed $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$ we consider the variational formulation of the original problem taking as test function $\omega_\varepsilon\varphi$ where $\varphi \in \mathcal{D}(\Omega)$. One gets

$$\begin{aligned} \int_\Omega (f+u)\omega_\varepsilon\varphi\,\mathrm{d}x &= \int_\Omega \varphi(\nabla\tilde{y}_\varepsilon, \nabla\omega_\varepsilon)\,\mathrm{d}x + \int_\Omega \omega_\varepsilon(\nabla\tilde{y}_\varepsilon,\nabla\varphi)\,\mathrm{d}x \\ &= \int_\Omega \omega_\varepsilon(\nabla\tilde{y}_\varepsilon,\nabla\varphi)\,\mathrm{d}x + \langle -\Delta\omega_\varepsilon, \varphi\tilde{y}_\varepsilon\rangle - \int_\Omega \tilde{y}_\varepsilon(\nabla\omega_\varepsilon,\nabla\varphi)\,\mathrm{d}x. \end{aligned}$$

Then, using (4.64) and the facts that

$$\begin{aligned}&\omega_\varepsilon \to 1 \quad \text{strongly in } L^2(\Omega), \qquad \nabla\omega_\varepsilon \to \nabla 1 = 0 \quad \text{weakly in } \big(L^2(\Omega)\big)^n,\\ &\tilde{y}_\varepsilon \to y \quad \text{weakly in } H_0^1(\Omega),\end{aligned}$$

and by passing to the limit as $\varepsilon \to 0$ we get

$$\int_\Omega (f+u)\varphi \,\mathrm{d}x = \int_\Omega (\nabla y, \nabla\varphi)\,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}\mu_* \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Hence the limit function $y \in H_0^1(\Omega)$ satisfies the equation

$$-\Delta y + \mu_* y = f + u \quad \text{in } \mathcal{D}'(\Omega). \tag{4.65}$$

Since the function $y \in H_0^1(\Omega)$ satisfies also the state Eq. (4.63) it follows that due to the uniqueness result (see Lemma 5.4 in [11]) we have $\mu_0 = \mu_*$. Thus, in order to summarize, the homogenized optimal control problem for the family (4.62) is unique and has the following form (for $n = 2$)

$$\begin{cases} -\Delta y + \pi/(2C_0) y = f + u \quad \text{in } \mathcal{D}'(\Omega),\\ y \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} \leqslant 2,\ u \geqslant -(\text{meas}\,\Omega)^{-1/2} \quad \text{a.e. in } \Omega,\\ I_0(u,y) = \int_\Omega |\nabla y|^2 \,\mathrm{d}x + \pi/(2C_0)\int_\Omega y^2 \,\mathrm{d}x + \int_\Omega u^2 \,\mathrm{d}x \to \inf. \end{cases} \tag{4.66}$$

In view of the variational properties of homogenized problems (see Theorem 4.1) we may add that the unique solution $(u^0, y^0) \in L^2(\Omega) \times H_0^1(\Omega)$ of the limit problem (4.66) satisfies the following conditions

$$\begin{aligned}&\tilde{y}^0_\varepsilon \to y^0 \quad \text{weakly in } H_0^1(\Omega), \qquad \tilde{u}^0_\varepsilon \to u^0 \quad \text{weakly in } L^2(\Omega),\\ &\int_{\Omega_\varepsilon} \big(u^0_\varepsilon - u^0\big)^2 \,\mathrm{d}x \to 0, \qquad I_\varepsilon\big(u^0_\varepsilon, y^0_\varepsilon\big) \to I_0\big(u^0, y^0\big),\end{aligned}$$

where $\{(u^0_\varepsilon, y^0_\varepsilon) \in L^2(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)\}$ is the sequence of the optimal pairs for the family (4.62).

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