ON THE WEAKLY-$*$ DENSE SUBSETS IN $L^\infty(\Omega)$

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Abstract. In this paper we study the density property of the compactly supported smooth functions in the space $L^\infty(\Omega)$. We show that this set is dense with respect to the weak-$*$ convergence in variable spaces.

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Let Ω be an open bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. Throughout the paper we suppose that Ω is a measurable set in the sense of Jordan. Let $C_0^\infty(\Omega)$ be the set of smooth functions with a compact support in Ω. It is well known that the set $C_0^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$, that is, the assertion

> "... for any $f \in L^\infty(\Omega)$ can be found a sequence $\{u_k \in C_0^\infty(\Omega)\}_{k=1}^\infty$ such that $u_k \to f$ strongly in $L^\infty(\Omega)$ as $k \to \infty$..."

is not true, in general. So, the main question we are going to study in this paper is the following: how can the density concept of the locally convex space C_0^∞ be interpreted in $L^\infty(\Omega)$? As we will see later it can be done through the concept of the weak-$*$ convergence in the variable spaces.

To begin with, we define the so-called graph-like structure on the domain Ω. Let $\square$ be the following set $\square = [0,1)^2 = [0,1) \times [0,1)$.

Definition 0.1. *We say that the set $\square$ is the cell of periodicity for some graph $\mathcal{F}$ on $\mathbb{R}^2$ if $\square$ contains a "star"-structure such that:*

(i) all edges of this structure have a common point $M \in int\square$; each edge is a line-segment and all end-points of these edges belong to the boundary of $\square$;

(ii) in the set of end-points (vertices) there exist pairs $\underline{(M_i; M_k)}$ such that $x_1^{M_i} = x_1^{M_k}$ or $x_2^{M_i} = x_2^{M_k}$.

As follows from the condition (ii) we admit the existence of isolated vertices in the $\square$-periodic graph $\mathcal{F}$ on $\mathbb{R}^2$. Let $\varepsilon \in E = (0, \varepsilon^0]$ be a small parameter. We assume that ε varies in a strictly decreasing sequence of positive numbers which converge to 0.

Definition 0.2. *We say that $\mathcal{F}_\varepsilon$ is an ε-periodic graph on $\mathbb{R}^2$ if*

$$\mathcal{F}_\varepsilon = \varepsilon\mathcal{F} = \{\varepsilon x \,:\, x \in \mathcal{F}\}.$$

It is clear that the cell of periodicity for Ω_ε is $\varepsilon\square$. Let

$$I^{ed} = \{I_j\ ,\ j = 1, 2, \dots, K\} \tag{0.1}$$

be the set of all edges on $\square$. Let Ω be an open bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary such that

$$\Omega = \{(x_1, x_2) \,:\, x_1 \in \Gamma_1, 0 < x_2 < \gamma(x_1)\}, \tag{0.2}$$

where $\Gamma_1 = (0, a)$, $\gamma \in C^1([0, a])$, and $0 < \gamma_0 = \inf_{x_1 \in [0,a]} \gamma(x_1)$. Then $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_2 = \partial\Omega \setminus \Gamma_1$.

Definition 0.3. *We say that Ω_ε has an ε-periodic graph-like structure if $\Omega_\varepsilon = \Omega \cap \mathcal{F}_\varepsilon$.*

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Our next step is to describe the geometry of the set Ω_ε in terms of so-called singular measures in $\mathbb{R}^2$. To do so, we will follow the Zhikov's approach (see[3]-[5]).

For every segment $I_i \in I^{ed}, i = 1, 2, \ldots, K$ we denote by μ_i its corresponding Lebesgue measure. Now we define the $\square$-periodic Borel measure μ in $\mathbb{R}^2$ as follows

$$\mu = \sum_{i=1}^{K} g_i \cdot \mu_i \qquad \text{on } \square, \tag{0.3}$$

where $g_1, g_2, \ldots, g_K$ are non-negative weights such that $\int_\square d\mu = 1$.

Thus the support of the measure μ is the union of all edges $I_i \in I^{ed}$, each of which is a 1-dimensional manifold in $\mathbb{R}^2$. Since the homothetic contraction of the plane at ε^{-1} takes the grid $\mathcal{F}$ to $\mathcal{F}_\varepsilon = \varepsilon\mathcal{F}$, we introduce a "scaling" ε-periodic measure μ_ε as follows

$$\mu_\varepsilon(B) = \varepsilon^2 \mu(\varepsilon^{-1}B) \text{ for every Borel set } B \subset \mathbb{R}^2. \tag{0.4}$$

Then

$$\int_{\varepsilon\square} d\mu_\varepsilon = \varepsilon^2 \int_\square d\mu = \varepsilon^2.$$

Hence the measure μ_ε is weakly convergent to the Lebesgue measure $\mathcal{L}^2$, that is

$$d\mu_\varepsilon \rightharpoonup dx \Leftrightarrow \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} \varphi d\mu_\varepsilon = \int_{\mathbb{R}^2} \varphi\, dx \tag{0.5}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^2)$ (see Zhikov [3] for a proof).

We define the space $L^\infty(\Omega, d\mu_\varepsilon)$ in the way: $y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)$ if and only if y_ε is a μ_ε-measurable function on Ω and there exists a constant $M > 0$ such that $|y_\varepsilon(x)| \leq M$ μ_ε-every where in Ω.

Definition 0.4. *We say that a sequence $\{y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)\}_{\varepsilon\to 0}$ is uniformly bounded if $\sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} < +\infty$.*

Definition 0.5. *A uniformly bounded sequence $\{y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)\}_{\varepsilon\to 0}$ is said to be weakly-$*$ convergent in the variable space $L^\infty(\Omega, d\mu_\varepsilon)$ to $y \in L^\infty(\Omega)$ if*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi y_\varepsilon\, d\mu_\varepsilon = \int_\Omega \varphi y\, dx \qquad \text{for every } \varphi \in C_0^\infty(\Omega)$$

(in the symbols $y_\varepsilon \overset{*}{\rightharpoonup} y$).

We begin with the following result:

Theorem 0.6. *Let $\{y_\varepsilon\}_{\varepsilon\to 0}$ be any bounded sequence in the variable space $L^\infty(\Omega, d\mu_\varepsilon)$. Then this sequence is relatively compact with respect to the weak-$*$ convergence in $L^\infty(\Omega, d\mu_\varepsilon)$.*

Proof. Let us set

$$l_\varepsilon(\varphi) = \int_\Omega y_\varepsilon \varphi\, d\mu_\varepsilon \quad \varphi \in C_0^\infty(\Omega).$$

Then, by the Hölder inequality, we have

$$|l_\varepsilon(\varphi)| \leq \int_\Omega |y_\varepsilon||\varphi|\, d\mu_\varepsilon \leq \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} \int_\Omega |\varphi|\, d\mu_\varepsilon. \tag{0.6}$$

Hence

$$|l_\varepsilon(\varphi)| \le \|y_\varepsilon\|_{L^\infty(\Omega,\, d\mu_\varepsilon)} \|\varphi\|_{C(\Omega)} \mu_\varepsilon(K),$$

where by K we denote a support of φ in Ω. Since $d\mu_\varepsilon \rightharpoonup dx = d\mathcal{L}^2$ in the space of Radone measures and

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(K) \le \mathcal{L}^2(K) \quad \text{for every compact subset of } \Omega$$

(see Zhikov [3]), it follows that

$$|l_\varepsilon(\varphi)| \le 2\|\varphi\|_{C(\Omega)} \mu(K) \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\, d\mu_\varepsilon)}$$

for $\varepsilon > 0$ small enough. On the other hand, the set

$$T(K) = \{\varphi \in C_0^\infty(\Omega),\ \operatorname{supp}\varphi \subseteq K\}$$

is separable with respect to the norm $\|\varphi\|_{C(\Omega)}$. Then, due to the Cantor diagonal method, it can be easy proved that the sequence $\{l_\varepsilon(\cdot)\}_{\varepsilon\to 0}$ consists a subsequence which is pointwise convergent on $T(K)$. As a result, there exists a subsequence of values $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} l_{\varepsilon_j}(\varphi) = l(\varphi) \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{0.7}$$

Taking into account the inequality (0.6), we conclude

$$|l(\varphi)| \le \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\, d\mu_\varepsilon)} \lim_{\varepsilon\to 0} \int_\Omega |\varphi|\, d\mu_\varepsilon = \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega,\, d\mu_\varepsilon)} \int_\Omega |\varphi|\, dx.$$

So, $l(\cdot)$ is the linear continuous functional on $L^1(\Omega)$. Hence, the following representation holds true

$$l(\varphi) = \int_\Omega v\varphi\, dx,$$

where v is some element of $L^\infty(\Omega)$. Thus, in view of (0.7), v is a weak-$*$ limit of the subsequence $\{y_{\varepsilon_j}\}_{j=1}^\infty$ in the variable space $L^\infty(\Omega, d\mu_\varepsilon)$. □

Now we are in a position to state the main result of our paper.

THEOREM 0.7. *For any element $y \in L^\infty(\Omega)$ there can be found a sequence of smooth functions $\{y_\varepsilon \in C_0^\infty(\Omega)\}_{\varepsilon>0}$ satisfying the conditions:*

$$|y_\varepsilon| \le \|y\|_{L^\infty(\Omega)} \textit{ for every } \varepsilon \in E; \quad y_\varepsilon \overset{*}{\rightharpoonup} y \textit{ in } L^\infty(\Omega, d\mu_\varepsilon) \textit{ as } \varepsilon \to 0. \tag{0.8}$$

Proof. Let y be any element of of $L^\infty(\Omega)$. We set $c = \|y\|_{L^\infty(\Omega)}$. Since $L^\infty(\Omega) \subset L^2(\Omega)$ and the space of smooth functions $C^\infty(\Omega)$ is dense in $L^2(\Omega)$ it follows that there is a sequence $\{y_\varepsilon \in C^\infty(\Omega)\}$ satisfying the conditions:

$$|y_\varepsilon| \le c \text{ for every } \varepsilon \in E; \quad \| y_\varepsilon - y \|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi y_\varepsilon\, dx = \int_\Omega \varphi y\, dx \ \text{ for every } \varphi \in C_0(\overline{\Omega}). \tag{0.9}$$

Further we note that $y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)$ (as a smooth function) and hence $|y_\varepsilon| \le c\ \mu_\varepsilon$-almost everywhere. We have to show that $y_\varepsilon \to y$ weakly-$*$ in $L^\infty(\Omega, d\mu_\varepsilon)$, i.e.

$$\int_\Omega \varphi y_\varepsilon\, d\mu_\varepsilon \to \int_\Omega \varphi y\, dx \quad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2). \tag{0.10}$$

We partition the domain Ω into the sets $\varepsilon\Box_j$, where $\Box_j$ is periodic covering of $\mathbb{R}^2$ by the cell $\Box$. Then

$$\int_\Omega \varphi y_\varepsilon\, d\mu_\varepsilon = \sum_j \int_{\varepsilon\Box_j} \varphi y_\varepsilon\, d\mu_\varepsilon + \sum \int_{\Omega\cap\varepsilon\Box_j} \varphi y_\varepsilon\, d\mu_\varepsilon, \tag{0.11}$$

where the second sum is calculated over the set of the 'boundary' squares such that $\varepsilon\Box_j \cap \partial\Omega \ne \emptyset$. By Mean Value Theorem, for each index j there exist points x_j in the cells $\varepsilon\Box_j$ such that

$$\int_{\varepsilon\Box_j} \varphi y_\varepsilon\, d\mu_\varepsilon = \varphi(x_j)\, y_\varepsilon(x_j) \int_{\varepsilon\Box_j} d\mu_\varepsilon = \varphi(x_j)\, y_\varepsilon(x_j)\, \varepsilon^2 \int_\Box d\mu = \varphi(x_j)\ y_\varepsilon(x_j)\, \varepsilon^2 \quad \forall\, j.$$

Then in view of (0.11), we get

$$\begin{aligned}\int_\Omega \varphi y_\varepsilon\, d\mu_\varepsilon = \left(\sum_j \varphi(x_j)\ y_\varepsilon(x_j)\, \varepsilon^2 - \int_\Omega \varphi y_\varepsilon\, dx\right) + \sum \int_{\Omega\cap\varepsilon\Box_j} \varphi y_\varepsilon\, d\mu_\varepsilon + \int_\Omega \varphi y_\varepsilon\, dx \\ = I_1 + I_2 + \int_\Omega \varphi y_\varepsilon\, dx.\end{aligned} \tag{0.12}$$

Note that

$$|I_2| = \left|\sum \int_{\Omega\cap\varepsilon\Box_j} \varphi y_\varepsilon\, d\mu_\varepsilon\right| \le \sup_{j\in D(\varepsilon)} \left(\sup_{x\in\,\Omega\cap\varepsilon\Box_j} |\varphi|\,|y_\varepsilon|\right) \varepsilon^2\, D(\varepsilon) \le c\|\varphi\|_{C(\Omega)} \varepsilon^2\, D(\varepsilon),$$

where $D(\varepsilon)$ is the quantity of the 'boundary' squares, and $\varepsilon^2\, D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$. Hence $I_2 \to 0$ as ε tends to zero.

Now we show that $I_1 \to 0$. To do so, we note that

$$\begin{aligned}|I_1| &= \left|\sum_j \varphi(x_j)\ y_\varepsilon(x_j)\, \varepsilon^2 - \int_\Omega \varphi y_\varepsilon\, dx\right| \\ &\le \left|\sum_j \left(\varphi(x_j)\ y_\varepsilon(x_j) - \frac{1}{\varepsilon^2}\int_{\varepsilon\Box_j} \varphi y_\varepsilon\, dx\right)\varepsilon^2\right| + \left|\sum \int_{\Omega\cap\varepsilon\Box_j} \varphi y_\varepsilon\, dx\right| \\ &\le \sum_j \left|\varphi(x_j) y_\varepsilon(x_j) - \frac{1}{\varepsilon^2}\int_{\varepsilon\Box_j} \varphi y_\varepsilon\, dx\right| \varepsilon^2 + c\|\varphi\|_{C(\Omega)} \varepsilon^2\, D(\varepsilon).\end{aligned}$$

Let us suppose the converse, that is,

$$\lim_{\varepsilon\to 0}\sum_j \left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|\varepsilon^2>0.$$

Since Ω is bounded, it is contained in a number of squares $\varepsilon\Box_j$ smaller than C/ε^2, where C does not depend on ε. So, there exist a constant $C^*>0$ and a value $\varepsilon^*>0$ such that

$$\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|\ge C^* \tag{0.13}$$

(for an infinite number of indices j for every fixed ε). Hence the extremely wild oscillations is present in the sequence $\{\varphi y_\varepsilon\}$. However (see [1],[2]), if we have the very rapid fluctuations in the functions $\{\varphi y_\varepsilon\}$, then the convergence $\varphi y_\varepsilon\to\varphi y$ almost everywhere in Ω is excluded. This fact immediately reflects the failure of the strong convergence $\varphi y_\varepsilon\to\varphi y$ in $L^2(\Omega)$ as $\varepsilon\to 0$. Indeed, by the initial assumptions, we have

$$\begin{aligned}|y_\varepsilon|\le c \ \text{ for every } \ \varepsilon\in E, \quad \varphi y_\varepsilon\rightharpoonup\varphi y \ \text{ in } \ L^1(\Omega),\\ \text{and} \ \ \|\varphi y_\varepsilon-\varphi y\|_{L^1(\Omega)}\to 0 \ \text{ as } \ \varepsilon\to 0 \ \text{ for any } \ \varphi\in C_0^\infty(\mathbb{R}^2).\end{aligned}$$

Let A be any subset of Ω with $|A|\ne 0$. Then, by Valadier's Theorem [2], $\varphi y_\varepsilon\to\varphi y$ strongly if and only if the following criterion is satisfied: $\forall\,\delta>0\ \exists\,\varepsilon^0>0$, $\exists\,B\subset A$ with $|B|\ne 0$ such that

$$|B|^{-1}\int_B\left|\varphi y_\varepsilon-|B|^{-1}\int\limits_B\varphi y_\varepsilon\,dx\right|dx<\delta \quad \forall\,\varepsilon<\varepsilon^0.$$

Hence, for any $\varepsilon<\varepsilon^0$ there is a square $\varepsilon\Box_j\subset B$ such that

$$\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\left|\varphi y_\varepsilon-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|dx<\delta.$$

Since the functions φy_ε are continuous and uniformly bounded it follows that for any point x_j of $\varepsilon\Box_j$ satisfying the condition

$$\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\ne 0$$

there can be found a constant $A_*>0$ satisfying

$$\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\left|\varphi y_\varepsilon-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|dx=A_*\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|.$$

Hence

$$\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|<A_*^{-1}\delta$$

and we come into conflict with (0.13). So, our supposition was wrong and we get

$$\lim_{\varepsilon\to 0}\sum_j \left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon\Box_j}\varphi y_\varepsilon\,dx\right|\varepsilon^2=0.$$

As a result, we have $I_1 \to 0$. Thus, summing up the results obtained above and the relations (0.12),(0.9), we come to the desired identity (0.10). □

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