Asymptotic Analysis of State Constrained Semilinear Optimal Control Problems

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Abstract

The asymptotic behaviour of state constrained semilinear optimal control problems for distributed parameter systems with variable compact control zones is investigated. We derive conditions under which the limiting problems can be made explicit.

Keywords. homogenization, optimal control, state constraints, penalized problem.

AMS subject classifications. 35B27, 49J27

1 Introduction

This paper is concerned with the homogenization of state constrained semilinear optimal control problems for distributed parameter systems with variable compact control zones. In the problems considered here each component of the mathematical model may depend on a small parameter ε (including the supports of control functions). Our aim is to study the limiting behaviour of such problems as $\varepsilon \to 0$. Moreover, we concentrate on L^2-bounded controls.

A similar but simpler problem (with the control zone being independent of ε, and with a fixed nonlinear term appearing in the state equation) has been studied by Conca, Osses and Saint Jean Paulin [10]. Using results of the theory of convex optimization a homogenization result for an optimal control problem has been established in [10] in that the sequence of optimal solutions for the original problem has been shown to converge to an optimal solution of a limit problem. While many investigations in this context, such as [10], consider the indirect method, which concerns

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the asymptotic behavior of the corresponding optimality systems, as the method of choice, we prefer to stay with the original optimal control problem and look for its homogenized limit, for which we establish a solution then. This is typically referred to as the direct method which is based on the concept of Γ-convergence and its variational properties. From this we immediately infer that the above mentioned variational principle (convergence of optimal solutions) holds true for the 'directly obtained' homogenized problems.

As shows the analysis of existing publications for last years, the homogenization of optimal control problems with the distributed parameters became object of steadfast attention of many researchers. For today already enough plenty of the literature is devoted to this theme (e.g. [1, 2, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30, 33, 34, 35, 36, 38]). As a rule, the basic subject of researches is the limiting analysis of various classes optimization problems and optimal control problems for objects which models, including the state equations, cost functional, and possible restrictions on managing parameters, contain dependence on some small parameter $\varepsilon > 0$. This dependence can be defined both regular and singular occurrence of the small parameter (presence of ε-periodic coefficients, presence of ε-periodically perforated zone of controls, quickly oscillating boundaries, etc.). While asymptotic methods for optimal control problems with regular perturbations are full enough developed, optimal control problems with singular perturbations are not investigated in full and asymptotic analysis in these cases is rather not trivial. Plenty of examples in books [5] and [34] (see also [15, 16]) testify to it.

It follows from results available for today that the most typical approach to homogenize optimal control problems with quickly oscillating coefficients or in perforated domains consists in application of the homogenization theory of boundary-value problems for investigation of limiting behaviour of necessary optimality conditions and the corresponding optimal solutions. Thus, the algorithm of asymptotic analysis consists of the following steps: firstly, they write down the necessary optimality conditions for the initial problem; secondly, they look for the corresponding limiting relations and interpret them as necessary optimality conditions for some optimal control problem; finally, using the limiting necessary optimality conditions, they recover an optimal control problem which is called the limiting control problem to the initial one (see e.g.[3, 10, 17, 18, 19, 35, 38]).

However, it should be stressed that such approach is suitable only for enough simple (from the point of view of the control theory) optimal control problems in which there are no restrictions on controls and on state of the system. Attempt to transfer this approach on wider class of optimal control problems was realized in [10], where it is shown that construction of the homogenized optimal control problem is possible only at enough rigid assumptions of structure of the state equation and his dependence on the small parameter.

Along with this, necessary optimality conditions for optimal control problems with restrictions, as a rule, contain variational inequalities or equations in measure terms that very complicates the procedure of homogenization and does it not trivial. On the other hand, the optimal pair (i.e., the optimal control and the corresponding state of system) cannot be named exhaustive characteristic of the problem. Really, every optimal control problem is presented as a set of such mathematical objects: the cost functional, the state equations (i.e. the corresponding boundary-value problem), and existing restrictions on the control functions and state functions. It is natural, that their full identification as a rule is impossible by means of the optimal solution only.

In this connection, another approach, which is based on the theory of Γ-convergence ([4, 5, 6, 7, 8, 9, 13]), is more expedient. His basic idea consists in representation of an initial optimal control problem as some problem of minimization and in subsequent studying its asymptotic behaviour. In a result, the fact that corresponds to the limiting problem of minimization is called the homogenized optimal control problem. It should be stressed here, that according to variational properties of Γ-limits, the given approach keeps convergence of optimal solutions of the initial problem to the similar characteristics of the homogenized optimal control problem. The main difficulty of this approach consists in identification of the effective set of the cost functional of the homogenized optimal control problem.

In this paper we will apply the scheme of direct homogenization, which was developed in [22, 23, 24, 25, 26]. Ideology of Γ-convergence lays in its basis as well. The principal new moment is the representation of an initial optimal control problem in a thick multi-structure as a constrained minimization problem and subsequent studying its asymptotic behaviour. Such an approach allows to reduce the procedure of homogenization to consecutive identification of the set of admissible pairs for the homogenized optimal control problem and then its cost functional.

Let us briefly describe the main result of this paper. In section 1 we give the main notations that will be used for our analysis, and we recall the principal results of Γ-convergence and convergence in Kuratowski's sense.

In section 2 we introduce the optimal control problem and discuss sufficient conditions for its solvability. Further, in section 3 we introduce the notion of L-equivalence on the set of minimization problems and discuss its relation with Γ-convergence. Namely, we show that each of the L-equivalent classes is closed with respect to Γ-convergence.

In section 4 we consider the sequence of penalized problems in homogenization. We prove that for every non-zero penalty parameter μ there exists a variational limit, and we give its analytical structure.

In section 5 we describe the diagram of homogenization for the original problem. Here we give the main result of this chapter, which deals with the commutativity property of the proposed scheme.

The last section is devoted to variational properties of the homogenized problem.

2 Notations and preliminary results

We denote by

Ω	a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary $\partial\Omega$;
S	an $(n-1)$-dimensional manifold in Ω;
E	$=(0,\varepsilon^0)$ an index set;
$\mathcal{H}$	a filter on E, i.e. each of the set H$\in \mathcal{H}$ can be represented in the form $H = E\backslash\Lambda$, where $\Lambda \subset E$ and there is a constant $\delta > 0$ such that $\gamma > \delta$ for all $\gamma \in \Lambda$;
$\mathcal{H}^{\#}$	the family of subsets of E that meet all sets H in $\mathcal{H}$;
$\mathcal{W}$	a family of subsets $\omega_\varepsilon \subset\subset \Omega$ $\forall$ $\varepsilon \in$ E with empty intersection with S;
$\chi_{\omega_\varepsilon}$	the characteristic function of ω_ε;
$w_{H_0^1}$	the weak topology of $H_0^1(\Omega)$;
w_{L^2}	the weak topology of $L^2(\Omega)$;
τ	the topology for $L^2(\Omega) \times H_0^1(\Omega)$ as the product of w_{L^2} and $w_{H_0'}$;
z_d	a fixed element of $L^2(S)$;
ξ_1, ξ_2	fixed elements of $L^2(\Omega)$;
$L^2(\Omega,\omega)$	$= \{u \in L^2(\Omega)\,:\, \operatorname{supp} u \subseteq \omega\}$;
λ, R	positive constants.

Class $\mathcal{A}$. We define the class $\mathcal{A}$ of linear elliptic operators as follows:

$$\mathcal{A} = \left\{A \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right) \text{ satisfying (A1)-(A2)}\right\},$$

where

$(A1)$ $\quad Ay = -\mathrm{div}(\mathbf{A}(x) \bigtriangledown y) + by,\ b > 0$;

$(A2)$ $\quad \mathbf{A}(\cdot)$ is a measurable matrix such that $\lambda \cdot I \le \mathbf{A}(x) \le \lambda^{-1} \cdot I$ for a.e. $x \in \Omega$;

Class $\mathcal{F}$. We define the class of nonlinear monotone operator as follows:

$$\mathcal{F} = \left\{T : L^2(\Omega) \to H^{-1}(\Omega) \qquad \text{satisfying} \qquad (A3) - (A5)\right\},$$

where

(A3) $\| T(y_1) - T(y_2) \|_{H^{-1}(\Omega)} \leq \lambda^{-1} \| y_1 - y_2 \|_{L^2(\Omega)} \big(1 + \| y_1 \|_{L^2(\Omega)} + \| y_2 \|_{L^2(\Omega)}\big)$ for every $y_1, y_2 \in L^2(\Omega)$;

(A4) $\big\langle T(y_1) - T(y_2), y_1 - y_2 \big\rangle_{H_0^1(\Omega)} \geq 0$ for every $\| y_i \|_{H_0^1(\Omega)} \leq R$, $(y_1 \neq y_2)$. Here by $\big\langle f, y \big\rangle_{H_0^1(\Omega)}$ is denotes the value of the functional $f \in H^{-1}(\Omega)$ at the point $y \in H_0^1(\Omega)$;

(A5) The $\mathcal{M}$-property: if $y_n \to y$ weakly in $H_0^1(\Omega), T(y_n) \to \xi$ weakly-$*$ in $H^{-1}(\Omega)$, $\limsup\limits_{n\to\infty} \big\langle T(y_n), y_n \big\rangle_{H_0^1(\Omega)} \leq \big\langle \xi, y \big\rangle_{H_0^1(\Omega)}$, then $T(y) = \xi$.

$\inf\limits_{x\in\Lambda} F(x)$	is the infimum of a function F on a set Λ;
$\Big\langle \inf\limits_{x\in\Lambda} F(x) \Big\rangle$	is a constrained minimization problem defined by the pair $(F; \Lambda)$;
ω_0	is a subset of Ω such that $\omega_0 \cap S = \emptyset$, $\omega_\varepsilon \subseteq \omega_0$ for every $\varepsilon \in E$;
$\mu{-}Li\, Q_\varepsilon$	is a lower topological limit (in Kuratowski's sense) for the filter $\mathcal{H}$ of a collection of subsets $\{Q_\varepsilon\}_{\varepsilon\in E}$ of some Banach space Q with respect to the μ-topology for Q;
$\mu{-}Ls\, Q_\varepsilon$	is a upper topological limit of previous collection of subsets.

In accordance with these definitions, the limit sets $\mu{-}Li\, Q_\epsilon$ and $\mu{-}Ls\, Q_\epsilon$ are defined by the rules:

$$\mu{-}Li\, Q_\varepsilon = \{z \in Q \mid \forall\, V \in \mathcal{N}_\mu(z)\ \exists\, H \in \mathcal{H}, \forall\, \varepsilon \in H : Q_\varepsilon \cap V \neq \emptyset\},$$

$$\mu{-}Ls\, Q_\varepsilon = \big\{z \in Q \mid \forall\, V \in \mathcal{N}_\mu(z)\ \exists\, H \in \mathcal{H}^\#, \forall\, \varepsilon \in H : Q_\varepsilon \cap V \neq \emptyset\big\}.$$

Here $\mathcal{N}_\mu(z)$ denotes the system of all neighborhoods of z in the μ-topology.

However, if the collection of subsets $\{Q_\varepsilon\}_{\varepsilon\in E}$ is uniformly bounded, then the limit sets $\mu{-}Li\, Q_\epsilon$ and $\mu{-}Ls\, Q_\epsilon$ can be defined in another (but equivalent) way, namely

$$\mu{-}Li\, Q_\varepsilon = \Big\{z \in Q \mid \exists\, H \in \mathcal{H}, \exists\, \{z_\varepsilon \in Q_\varepsilon\}_{\varepsilon\in H} : z_\varepsilon \stackrel{\mu}{\longrightarrow} z\Big\},$$

$$\mu{-}Ls\, Q_\varepsilon = \Big\{z \in Q \mid \exists\, H \in \mathcal{H}^\#, \exists\, \{z_\varepsilon \in Q_\varepsilon\}_{\varepsilon\in H} : z_\varepsilon \stackrel{\mu}{\longrightarrow} z\Big\}.$$

If $\mu{-}Li\, Q_\varepsilon = \mu{-}Ls\, Q_\varepsilon$, then this set, denoted as $\mu{-}Lm\, Q_\varepsilon$, is the limit of the collection $\{Q_\varepsilon\}_{\varepsilon\in E}$ in the sense of Kuratowski with respect to the μ-topology of Q.

In addition to the suppositions (A1) - (A5) we assume that the following conditions hold:

(A6) Let χ_* be the weak-$*$ limit of the sequence $\{\chi_{\omega_\varepsilon}\}$, i.e.

$\langle \chi_{\omega_\varepsilon}, \varphi \rangle_{L^1(\Omega)} \to \langle \chi_*, \varphi \rangle_{L^1(\Omega)}, \forall\, \varphi \in L^1(\Omega).$

Then there exists an open set $\omega_0 \subset\subset \Omega$ such that

(i) $\omega_\varepsilon \subseteq \omega_0\,, \forall\, \varepsilon \in E$,

(ii) $\omega_0 = \operatorname{int}(supp\, \chi_*)$,

where by $\operatorname{int}(C)$ we denote the interior of a set C.

(A7) Let $m_* = \chi_* \,|_{\omega_0}$ be the restriction of the function $\chi_* \in L^\infty(\Omega)$ on the set ω_0. Then $m_*^{-1} \in L^\infty(\omega_0)$.

(A8) For the sequence of nonlinear operators $\{T_\varepsilon : H_0^1(\Omega) \to H^{-1}(\Omega)\}_{\varepsilon\in E} \subset \mathcal{F}$ there exists an operator $T_0 \in \mathcal{F}$ such that

$T_\varepsilon(y_\varepsilon) \to T_0(y)$ strongly in $H^{-1}(\Omega)$

for every sequence $\{y_\varepsilon \in H_0^1(\Omega)\}_{\varepsilon\in E}$ weakly converging in $H_0^1(\Omega)$ to y.

In what follows we will also use the notion of the variational limit for a sequence of constrained minimization problems

$$\left\{ \Big\langle \inf_{x\in Q_\varepsilon} F_\varepsilon(x) \Big\rangle ;\ \varepsilon \in E \right\}. \tag{2.1}$$

We say that a problem $\Big\langle \inf\limits_{x\in Q_0} F_0(x) \Big\rangle$ is a variational limit of (2.1) with respect to some topology μ for Q, if the following conditions (B_1) and (B_2) are satisfied:

(B_1) $Q_0 = \mu\text{–}Lm\, Q_\varepsilon$;

(B_2) $F_0(x) = \Gamma\!-\liminf\limits_{\varepsilon\to 0} \tilde{F}_\varepsilon(x) = \Gamma\!-\limsup\limits_{\varepsilon\to 0} \tilde{F}_\varepsilon(x), \forall\, x \in Q_0$ Here

$$\tilde{F}_\varepsilon(x) = \begin{cases} F_\varepsilon(x),\ x \in Q_\varepsilon \\ +\infty,\ x \in Q\backslash Q_\varepsilon \end{cases}$$

and by $\Gamma\!-\liminf\limits_{\varepsilon\to 0} \tilde{F}_\varepsilon$ and $\Gamma\!-\limsup\limits_{\varepsilon\to 0} \tilde{F}_\varepsilon$ are denoted the Γ-lower and Γ-upper limits (with respect to the μ-topology).

For the sake of easier reference we recall some results that will be used below.

Definition 2.1 *The sequence of functionals $\{F_\varepsilon : Q_\varepsilon \to \mathbf{R}^1 \cup \{+\infty\}\}_{\varepsilon\in E}$ is called μ-coercive if for every $t \in \mathbf{R}$ there exists a μ-compact set $K_t \subset Q$ such that*

$$\bigcup_{\varepsilon\in E} \{x \in Q_\varepsilon : F_\varepsilon(x) \le t\} \subseteq K_t.$$

Theorem 2.1 *Let $\{F_\varepsilon : Q_\varepsilon \to \mathbf{R}^1\}$ be a μ-coercive sequence of functionals. Suppose that for the sequence of sets $\{Q_\varepsilon\}_{\varepsilon\in E}$ there exists a nonempty limit $\mu - LmQ_\varepsilon$. Then the minimization problem*

$$(2.2) \qquad \left\langle \inf_{x\in Q_0} F_0(x) \right\rangle$$

is the variational limit of the sequence (2.1) if and only if the following conditions hold:

(C1) for every $x \in Q_0$ and for every μ-converging to x sequence $\{y_\varepsilon\}_{\varepsilon\in H}$ such that $H \in \mathcal{H}^{\#}$, and $y_\varepsilon \in Q_\varepsilon \forall\, \varepsilon \in H$, we have $F_0(x) \le \liminf_{H\ni\varepsilon\to 0} F_\varepsilon(y_\varepsilon)$;

(C2) for every $x \in Q_0$ there exists an index set $H \in \mathcal{H}$ and a sequence $\{\bar{y}_\varepsilon\}_{\varepsilon\in H}$ such that $\bar{y}_\varepsilon \xrightarrow{\mu} x$, $\bar{y}_\varepsilon \in Q_\varepsilon \ \forall\, \varepsilon \in H$, $F_0(x) \ge \limsup_{H\ni\varepsilon\to 0} F_\varepsilon(\bar{y}_\varepsilon)$.

Theorem 2.2 *Assume that the problem (2.2) is the variational limit of the μ-coercive sequence (2.1), and $F_0 \not\equiv +\infty$ on Q_0. Then the following statements hold:*

(V1) $\min_{x\in Q_o} F_0(x) = \lim_{\varepsilon\to 0} \inf_{x\in Q_\varepsilon} F_\varepsilon(x)$;

(V2) let x_ε^0 be a minimizer of F_ε on Q_ε. If the sequence $\{x_\varepsilon^0 \in Q_\varepsilon\}_{\varepsilon\in E}$ μ-converges to some x^0 (or x^0 is a μ-cluster point of this sequence), then x^0 is a minimizer of F_0 on Q_0, and $F_0(x^0) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon^0)$.

3 Statement of the optimal control problem

We define the optimal control problem with a compact control zone as follows

$$(3.1) \qquad A_\varepsilon y + T_\varepsilon(y) = \chi_{\omega_\varepsilon} u \qquad \text{in } \Omega,$$

$$(3.2) \qquad y \in H_0^1(\Omega),$$

$$(3.3) \qquad u \in U_\varepsilon = \left\{ \begin{array}{c} v \in L^2(\Omega) : \quad \chi_{\omega_\varepsilon}\xi_1 \le v \le \chi_{\omega_\varepsilon}\xi_2 \\ \text{a.e. in } \Omega \end{array} \right\},$$

$$(3.4) \qquad \| \, y \, |_S - z_d \, \|_{L^2(S)} \le \alpha,$$

$$(3.5) \qquad I_\varepsilon(u, y) = \int_{\omega_\varepsilon} u^2(x)dx \to \inf .$$

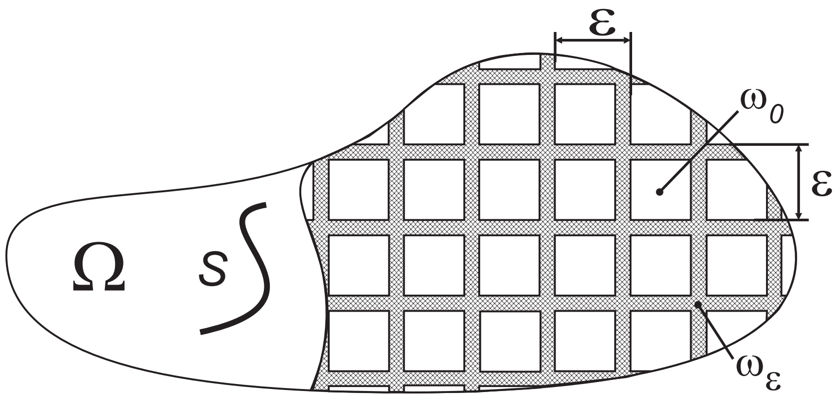

Figure 1: ε-periodic control zone

Here $A_\varepsilon \in \mathcal{A}$, $T_\varepsilon \in \mathcal{F}$ for every $\varepsilon \in E$, $y\mid_S$ is the restriction of the state function y to the manifold S.

We note that by the trace theorem for the Sobolev space $H_0^1(\Omega)$ we have $y\mid_S \in H^{1/2}(S)$, so the state constraints (3.4) is properly defined.

As an example of the above mentioned problem we may consider the control problem of a temperature field on some spatial domain $\Omega \subset \mathbf{R}^2$, where the control zone has an ε-periodic lattice structure on some part of the domain (see Fig.1).

It is easy to see that in the limit as $\varepsilon \to 0$ the sets $\{\omega_\varepsilon\}$ coincide with the whole subdomain $\{\omega_0\}$. However, the weak-$*$ limit of its characteristic functions $\{\chi_\varepsilon\}$ doesn't coincide with the characteristic function of ω_0. Indeed for the $\{\omega_\varepsilon\}$ on Fig.1 we have

$$\chi_{\omega_\varepsilon} \xrightarrow{*} \chi_* = \begin{cases} \int_\square \chi_{\omega_\square}(x)dx = 1-(1-\delta)^2, & x \in \omega_0, \\ 0, & x \notin \omega_0, \end{cases}$$

and $\chi_* < \chi_{\omega_0}$ on Ω.

Definition 3.1 *The control-state-pair $(u,y) \in L^2(\Omega) \times H_0^1(\Omega)$ is called admissible if (u,y) satisfies the restriction (3.1) - (3.5). We denote by Ξ_ε the set of all admissible pairs for the fixed $\varepsilon \in E$, i.e.*

$$\Xi_\varepsilon := \left\{ (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{ll} A_\varepsilon y + T_\varepsilon(y) = \chi_{\omega_\varepsilon} u & \text{in } \Omega, \\ \| y\mid_S - z_d \|_{L^2(S)} \le \alpha, & \\ \chi_{\omega_\varepsilon}\xi_1 \le u \le \chi_{\omega_\varepsilon}\xi_2 & a.e. \text{ in } \Omega \end{array} \right\} \tag{3.6}$$

In order to obtain the sufficient conditions of solvability of the original optimal control problem (3.1)-(3.5) for every $\varepsilon \in E$, we consider the following penalized problem

$$\Big\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y)\Big\rangle, \tag{3.7}$$

where

$$\mathcal{N}_\varepsilon = \left\{ (u,y) \in L^2(\Omega)\times H_0^1(\Omega) \;\middle|\; \begin{array}{ll} A_\varepsilon y + T_\varepsilon(y) = \chi_{\omega_\varepsilon} u & \text{in } \Omega, \\ \chi_{\omega_\varepsilon}\xi_1 \le u \le \chi_{\omega_\varepsilon}\xi_2 & \text{a.e. in } \Omega \end{array} \right\} \tag{3.8}$$

$$I_{\varepsilon,\mu}(u,y) = \int_{\omega_\varepsilon} u^2 dx + \mu^{-1}\big[\nu\left(\alpha - \| \, y\,|_S - z_d \,\|_{L^2(S)}\right)\big]^2, \tag{3.9}$$

μ is the penalty parameter, $\nu : \mathbf{R}^1 \to \mathbf{R}^1_+$ is a convex monotonically decreasing function, such that ν is strictly monotone on $\mathbf{R}^1_-$ and $\nu(t) = 0 \ \forall t \ge 0$.

The following lemma will be used to prove existence of solutions for the original problem (3.1)-(3.5).

Lemma 3.1 *The penalized problem (3.7) has a solution $(u^\mu_\varepsilon, y^\mu_\varepsilon) \in \mathcal{N}_\varepsilon$ for every $\mu > 0$ and $\varepsilon \in E$.*

Proof. We consider a minimizing sequence for (3.7), i.e.

$$I_{\varepsilon,\mu}\left(u^n_\varepsilon, y^n_\varepsilon\right) \longrightarrow \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y).$$

As follows from [31] and (A1)-(A4), the operators

$$B_\varepsilon(y) = A_\varepsilon y + T_\varepsilon(y)$$

are invertible and bounded. Therefore for every bounded sequence $\{u^n_\varepsilon \in L^2(\Omega;\omega_\varepsilon)\}$ there is a bounded sequence of corresponding solutions $\{y^n_\varepsilon = y_\varepsilon(u^n_\varepsilon) \in H^1_0(\Omega)\}_{n\in\mathbb{N}}$. Since the functional $I_{\varepsilon,\mu}$ is τ-coercive and lower τ-semicontinuous on $\mathcal{N}_\varepsilon$, we deduce that the sequence $\{(u^n_\varepsilon, y^n_\varepsilon)\}_{n\in\mathbb{N}}$ is a bounded. Hence we may conclude that there exists a pair $(\bar u_\varepsilon, \bar y_\varepsilon)$ such that

$$\left.\begin{array}{rcl} u^n_\varepsilon \to \bar u_\varepsilon & \text{weakly in} & L^2(\Omega), \\ y^n_\varepsilon \to \bar y_\varepsilon & \text{weakly in} & H^1_0(\Omega), \\ \liminf\limits_{n\to\infty} I_{\varepsilon,\mu}(u^n_\varepsilon, y^n_\varepsilon) & \ge & I_{\varepsilon,\mu}(\bar u_\varepsilon, \bar y_\varepsilon). \end{array}\right\} \tag{3.10}$$

It remains to prove that $(\bar u_\varepsilon, \bar y_\varepsilon) \in \mathcal{N}_\varepsilon$. As follows from (3.8) we have

$$u^n_\varepsilon = \chi_{\omega_\varepsilon}\, u^n_\varepsilon\ , \ \forall n \in \mathbb{N}, \forall \varepsilon \in E,$$

i.e. $u_\varepsilon^n \in L^2(\Omega;\omega_\varepsilon)$. On the other hand the set

$$U_\varepsilon = \left\{ v \in L^2(\Omega) \,:\, \chi_{\omega_\varepsilon} \cdot \xi_1 \le v \le \chi_{\omega_\varepsilon} \cdot \xi_2 \quad \text{a.e. in } \Omega \right\}$$

is w_{L^2}-closed. Hence we have $\bar{u}_\varepsilon \in U_\varepsilon$.

It remains to verify that for the limit pair $(\bar{u}_\varepsilon, \bar{y}_\varepsilon)$ the following equality holds

$$A_\varepsilon \bar{y}_\varepsilon + T_\varepsilon(\bar{y}_\varepsilon) = \chi_{\omega_\varepsilon} \bar{u}_\varepsilon \text{ in } \Omega.$$

Indeed, it is easy to see that

$$A_\varepsilon y_\varepsilon^n \to A_\varepsilon \bar{y}_\varepsilon \text{ weakly-}* \text{ in } H^{-1}(\Omega).$$

Therefore

$$T_\varepsilon(y_\varepsilon^n) = \chi_{\omega_\varepsilon} u_\varepsilon^n - A_\varepsilon y_\varepsilon^n \to \chi_{\omega_\varepsilon} \bar{u}_\varepsilon - A_\varepsilon \bar{y}_\varepsilon \tag{3.11}$$

weakly-$*$ in $H^{-1}(\Omega)$ as well.

Now we pass to the limit as $n \to \infty$ in the following equality

$$\left\langle T_\varepsilon(y_\varepsilon^n), y_\varepsilon^n \right\rangle_{H_0^1(\Omega)} = \left\langle \chi_{\omega_\varepsilon} u_\varepsilon^n, y_\varepsilon^n \right\rangle_{L^2(\Omega)} - \left\langle A_\varepsilon y_\varepsilon^n, y_\varepsilon^n \right\rangle_{H_0^1(\Omega)}.$$

Using the strong convergence of y_ε^n to $\bar{y}_\varepsilon$ in $L^2(\Omega)$ and the weak convergence of u_ε^n to $\bar{u}_\varepsilon$, we get

$$\left\langle \chi_{\omega_\varepsilon} u_\varepsilon^n, y_\varepsilon^n \right\rangle_{L^2(\Omega)} \to \left\langle \chi_{\omega_\varepsilon} \bar{u}_\varepsilon^n, \bar{y}_\varepsilon^n \right\rangle_{L^2(\Omega)}.$$

Further, by assumption (A1) the expressions

$$\int_\Omega \| p(x) \|_{\mathbf{R}^n}^2 \, dx \qquad \text{and} \qquad \int_\Omega (\mathcal{A}_\varepsilon(x) p(x),\ p(x))_{\mathbf{R}^n} \, dx$$

define equivalent norms in $[L^2(\Omega)]^n$. Hence, by the property of lower w_{L^2}-semicontinuity of the norm in $[L^2(\Omega)]^n$ we find that

$$\begin{aligned} \liminf_{n\to\infty} \left\langle A_\varepsilon y_\varepsilon^n, y_\varepsilon^n \right\rangle_{H_0^1(\Omega)} &= \liminf_{n\to\infty} \int_\Omega (\mathcal{A}_\varepsilon(x) \nabla y_\varepsilon^n, \nabla y_\varepsilon^n)_{\mathbf{R}^n} \, dx \\ \ge \int_\Omega (\mathcal{A}_\varepsilon(x) \nabla \bar{y}_\varepsilon, \nabla \bar{y}_\varepsilon)_{\mathbf{R}^n} \, dx &= \left\langle A_\varepsilon \bar{y}_\varepsilon, \bar{y}_\varepsilon \right\rangle_{H_0^1(\Omega)} \end{aligned}$$

for every sequence $\{y_\varepsilon^n\}_{n\in\mathbb{N}}$ such that

$$y_\varepsilon^n \to \bar{y}_\varepsilon \qquad \text{weakly in } H_0^1(\Omega).$$

Therefore

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle T_\varepsilon(y_\varepsilon^n), y_\varepsilon^n\right\rangle_{H_0^1(\Omega)} &= \limsup_{n\to\infty}\left[\left\langle \chi_{\omega_\varepsilon} u_\varepsilon^n, y_\varepsilon^n\right\rangle_{L^2(\Omega)} - \left\langle A_\varepsilon y_\varepsilon^n, y_\varepsilon^n\right\rangle_{H_0^1(\Omega)}\right] \\
&= \left\langle \chi_{\omega_\varepsilon} \bar{u}_\varepsilon, \bar{y}_\varepsilon\right\rangle_{L^2(\Omega)} - \liminf_{n\to\infty}\left\langle A_\varepsilon y_\varepsilon^n, y_\varepsilon^n\right\rangle_{H_0^1(\Omega)} \\
&\le \left\langle \chi_{\omega_\varepsilon} \bar{u}_\varepsilon, \bar{y}_\varepsilon\right\rangle_{L^2(\Omega)} - \left\langle A_\varepsilon \bar{y}_\varepsilon, \bar{y}_\varepsilon\right\rangle_{H_0^1(\Omega)} \\
&= \left\langle \chi_{\omega_\varepsilon} \bar{u}_\varepsilon - A_\varepsilon \bar{y}_\varepsilon, \bar{y}_\varepsilon\right\rangle_{H_0^1(\Omega)}. \qquad (3.12)
\end{aligned}$$

Using (3.11), (3.12), and the $\mathcal{M}$-property for the operators T_ε, we obtain

$$T_\varepsilon(\bar{y}_\varepsilon) = \chi_{\omega_\varepsilon}\bar{u}_\varepsilon - A_\varepsilon \bar{y}_\varepsilon\,,$$

that is the limit pair $(\bar{u}_\varepsilon, \bar{y}_\varepsilon)$ belongs to the set $\mathcal{N}_\varepsilon$. Thus in view of (3.10) we deduce that $(\bar{u}_\varepsilon, \bar{y}_\varepsilon)$ is an optimal pair for the penalized problem (3.7). The lemma is proved. ∎

Lemma 3.2 *Assume that for the given $\varepsilon \in E$ the set Ξ_ε is nonempty. Then every sequence of optimal solutions $\{(u_\varepsilon^\mu, y_\varepsilon^\mu)\}_{\mu\to 0}$ for (3.7) has a subsequence such that (still denoted by the suffix μ)*

$$(u_\varepsilon^\mu, y_\varepsilon^\mu) \xrightarrow{\ \tau\ } (u_\varepsilon^0, y_\varepsilon^0),$$

where $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair for the original problem (3.1)–(3.5).

Proof. Let $\{(u_\varepsilon^\mu, y_\varepsilon^\mu)\}_{\mu\to 0}$ be a sequence of optimal pairs for penalized problems (3.7). Since the set Ξ_ε is nonempty it implies

$$I_{\varepsilon,\mu}(u_\varepsilon^\mu, y_\varepsilon^\mu) \le I_{\varepsilon,\mu}(u^*, y^*) = I_\varepsilon(u^*, y^*) = C, \qquad (3.13)$$

for any pair $(u^*, y^*) \in \Xi_\varepsilon$, where the constant C doesn't depend on μ.

Hence, by analogy with the previous proof, we deduce that the sequence $\{(u_\varepsilon^\mu, y_\varepsilon^\mu)\}_{\mu\to 0}$ is bounded. Therefore there exists a pair $(u_\varepsilon^0, y_\varepsilon^0) \in \mathcal{N}_\varepsilon$ such that

$$\begin{aligned}
u_\varepsilon^\mu &\to u_\varepsilon^0 \quad \text{weakly in} \quad L^2(\Omega)\\
y_\varepsilon^\mu &\to y_\varepsilon^0 \quad \text{weakly in} \quad H_0^1(\Omega),
\end{aligned}$$

where we used the closeness of the set $\mathcal{N}_\varepsilon$ with respect to the τ-topology.

We proceed to show that the following inequality holds true

$$\| y_\varepsilon^0 |_S - z_d \|_{L^2(S)} \le \alpha. \qquad (3.14)$$

Indeed, as follows from (3.13) there exists a positive constant c independent of μ such that

$$\left[\nu\left(\alpha - \| y_\varepsilon^\mu |_S - z_d \|_{L^2(S)}\right)\right]^2 \le c\cdot\mu.$$

Since the embedding $H^{1/2}(S) \subset L^2(S)$ is compact and $y_\varepsilon^\mu \mid_S \in H^{1/2}(S)$ for every $\mu > 0$, it follows that

$$\left[\nu\left(\alpha - \| y_\varepsilon^0 \mid_S - z_d \|_{L^2(S)}\right]^2 = \limsup_{\mu\to 0} \left[\nu\left(\alpha - \| y_\varepsilon^0 \mid_S - z_d \|_{L^2(S)}\right]^2 \le c \cdot \lim_{\mu\to 0} \mu = 0$$

Thus inequality (3.14) holds. This, in turn, implies that for limit pair $(u_\varepsilon^0, y_\varepsilon^0)$ we have

$$(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon.$$

Now it remains to verify that this pair is an optimal solution for (3.1)-(3.5). Assume the conversion then there exists a pair $(u^*, y^*) \in \Xi_\varepsilon$ such that

$$I_\varepsilon(u^*, y^*) < I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0).$$

However, by (3.14) and the property of lower τ-semicontinuity of the functional $I_{\varepsilon,\mu}$, we have

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = I_{\varepsilon,\mu}(u_\varepsilon^0, y_\varepsilon^0) \le \liminf_{\mu\to 0}(u_\varepsilon^\mu, y_\varepsilon^\mu) \le I_{\varepsilon,\mu}(u^*, y^*) = I_\varepsilon(u^* y^*).$$

This contradiction proves Lemma 3.2. ■

Now, as an obvious consequence of previous lemmas, we have the following result.

Theorem 3.1 *The optimal control problem (3.1)-(3.5) is solvable if and only it is regular, i.e. the set of admissible pairs Ξ_ε is nonempty.*

Remark 3.1 *As follows from the arguments above, each of the sets Ξ_ε is closed with respect to the τ-topology.*

4 On the equivalence of constrained minimization problems

This section deals with some aspects of the concept variational convergence applied to different classes of constrained minimization problems. These results will be used in the next sections where we will focus on a homogenization procedure.

Let $(X, \|\cdot\|)$ be a Banach space, μ the weak topology of X. We define the class $\mathcal{P}(X)$ of all constrained minimization problems of the form

$$\left\langle \inf_{x\in\Lambda} F(x) \right\rangle,$$

where $F : \Lambda \to \mathbf{R} \cup \{+\infty\} \stackrel{\triangle}{=} \overline{\mathbf{R}}$ is a lower μ-semicontinuous functional and Λ is a τ-closed subset of X.

Let us introduce the following binary relation $\langle L; \mathcal{P} \rangle$ on $\mathcal{P}$:

$$\Big\langle \inf_{x\in\Lambda_1} F_1(x) \Big\rangle L \Big\langle \inf_{x\in\Lambda_2} F_2(x) \Big\rangle$$

if and only if there exists a set $Q \subset X$ such that

$$\text{(R1)} \qquad \mathrm{Dom}(F_1) \cap \Lambda_1 = Q = \mathrm{Dom}(F_2) \cap \Lambda_2 \tag{4.1}$$

$$\text{(R2)} \qquad F_1(x) = F_2(x) \quad \text{for every} \quad x \in Q. \tag{4.2}$$

Here $\mathrm{Dom}(F) = \{x \in X : F(x) < +\infty\}$.

It is easily seen that $\langle L; \mathcal{P} \rangle$ is an equivalence relation.

Definition 4.1 *We say that two sequences of constrained minimization problems*

$$\Big\{ \big\langle \inf_{x\in\,\Lambda_\varepsilon} F_\varepsilon(x) \big\rangle \Big\}_{\varepsilon\in E} \quad \text{and} \quad \Big\{ \big\langle \inf_{x\in\,W_\varepsilon} G_\varepsilon(x) \big\rangle \Big\}_{\varepsilon\in E} \tag{4.3}$$

are the jointly L-equivalent if

$$\big\langle \inf_{x\in\Lambda_\varepsilon} F_\varepsilon(x) \big\rangle L \big\langle \inf_{x\in W_\varepsilon} G_\varepsilon(x) \big\rangle$$

for every $\varepsilon \in E$.

In what follows, we will use the following notations:

$$P_\Lambda F(x) = \begin{cases} F(x), & x \in \Lambda, \\ +\infty, & x \in X \setminus \Lambda, \end{cases}$$

$$\mathbf{1}_\Lambda(x) = \begin{cases} 0, & x \in \Lambda, \\ +\infty, & x \in X \setminus \Lambda. \end{cases}$$

Now we are in the position to state the main theorem of this section.

Theorem 4.1 *Assume that the sequences (4.3) are jointly L-equivalent. Moreover, assume that for the sequence*

$$\Big\{ \big\langle \inf_{x\in\,\Lambda_\varepsilon} F_\varepsilon(x) \big\rangle \Big\}_{\varepsilon\in E} \tag{4.4}$$

there exists a variational limit (with respect to the μ-topology) $\left\{\left\langle \inf_{x\in\Lambda_0} F_0(x)\right\rangle\right\}$, *and that the sequence of functionals* $\left\{F_\varepsilon : \Lambda_\varepsilon \to \bar{\mathbf{R}}\right\}_{\varepsilon\in E}$ *is uniformly bounded below, i.e.*

$$F_\varepsilon(x) \geq \gamma \quad \forall\, x \in \Lambda_\varepsilon, \quad \forall\, \varepsilon \in E,\ \gamma > -\infty.$$

Then the constrained minimization problems

$$\left\langle \inf_{x\in\mu-Li\,W_\varepsilon} \Gamma\!-\limsup[P_{W_\varepsilon} G_\varepsilon](x)\right\rangle,$$

$$\left\langle \inf_{x\in\mu-Ls\,W_\varepsilon} \Gamma\!-\liminf[P_{W_\varepsilon} G_\varepsilon](x)\right\rangle,$$

$$\text{and } \left\langle \inf_{x\in\Lambda_0} F_0(x)\right\rangle$$

belong to the same class of L-equivalence.

Proof. First of all, from the definition of variational limits we may infer for the sequence of functional $\{P_{\Lambda_\varepsilon} F_\varepsilon : X \to \bar{\mathbf{R}}\}_{\varepsilon\in E}$ that there exists a Γ-limit such that

$$\Gamma\!-\lim[P_{\Lambda_\varepsilon} F_\varepsilon](x) = P_{\Lambda_0} F_0(x)\ ,\ \forall x \in X.$$

Since the sequences (4.3) are jointly L-equivalent, it follows that

$$P_{\Lambda_\varepsilon} F_\varepsilon(x) = P_{W_\varepsilon} G_\varepsilon(x)\ \ ,\ \ \forall\, x \in X,\ \forall\, \varepsilon \in E. \tag{4.5}$$

Therefore, the sequence $\left\{P_{W_\varepsilon} G_\varepsilon : X \to \bar{\mathbf{R}}\right\}_{\varepsilon\in E}$ is Γ-convergent as well. Moreover,

$$\Gamma\!-\lim P_{\Lambda_\varepsilon} F_\varepsilon = \Gamma\!-\lim P_{W_\varepsilon} G_\varepsilon \qquad \text{on X}. \tag{4.6}$$

It remains to verify that there exists a set Q_0 satisfying the following condition

$$Q_0 = \mathrm{Dom}(F_0) \cap \Lambda_0 = \mathrm{Dom}\,(\Gamma\!-\lim[P_{W_\varepsilon} G_\varepsilon]) \cap \mu\!-\!Li\,W_\varepsilon. \tag{4.7}$$

Using the uniformly boundedness from below of the function $\left\{F_\varepsilon : \Lambda_\varepsilon \to \bar{\mathbf{R}}\right\}_{\varepsilon\in E}$, equality (4.5), and the properties of Γ-limits (see [11]), we obtain

$$\Gamma\!-\lim\left[P_{W_\varepsilon} G_\varepsilon\right](x) \geq \gamma + \Gamma\!-\limsup \mathbf{1}_{Q_\varepsilon}(x)\,,\ \forall x \in X,$$

where the sets Q_ε are defined as

$$Q_\varepsilon = \mathrm{Dom}(F_\varepsilon) \cap \Lambda_\varepsilon = \mathrm{Dom}(G_\varepsilon) \cap W_\varepsilon\,,\ \forall \varepsilon \in E. \tag{4.8}$$

As follows from the Γ-limit properties we have

$$\Gamma\!-\limsup \mathbf{1}_{Q_\varepsilon}(x) = \mathbf{1}_{\mu-Li\,Q_\varepsilon}.$$

Hence

$$\emptyset \neq \mathrm{Dom}\,(\Gamma\!-\lim[P_{W_\varepsilon}G_\varepsilon]) = \mathrm{Dom}(P_{\Lambda_0}F_0) \subset \mu\!-\!Li\,Q_\varepsilon. \tag{4.9}$$

Using (4.8) we see that

$$\begin{aligned} \mu - LiQ_\varepsilon &\subseteq \mu\!-\!Lm\,\Lambda_\varepsilon \cap \mu\!-\!Ls\,(\mathrm{Dom}(F_\varepsilon)) \\ &= \Lambda_0 \cap \mu\!-\!Ls\,(\mathrm{Dom}(F_\varepsilon))\,, \\ \mu\!-\!Li\,Q_\varepsilon &\subset \mu\!-\!Li\,W_\varepsilon \cap \mu\!-\!Ls\,(\mathrm{Dom}(G_\varepsilon))\,. \end{aligned}$$

Then from (4.9) we deduce that

$$\begin{aligned} \mathrm{Dom}\,(\Gamma\!-\lim[P_{W_\varepsilon}G_\varepsilon]) &\subseteq \mu\!-\!Li\,W_\varepsilon, \\ \mathrm{Dom}\,(P_{\Lambda_0}) &\subseteq \Gamma_0. \end{aligned}$$

Thus, in view of (4.6) the equality (4.7) holds true. ∎

Let's consider the following application of this theorem.

Remark 4.1 *Let $\left\{I_\varepsilon : X \to \overline{\mathbf{R}}\right\}_{\varepsilon\in E}$ be a sequence of uniformly bounded below functionals, and let $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ and $\{\mathit{\Pi}_\varepsilon\}_{\varepsilon\in E}$ be two sequences of subsets of X such that*

$$\Xi_\varepsilon = \mathit{\Pi}_\varepsilon \cap W \quad \text{for every } \varepsilon \in E,$$

where W is some fixed subset.

Then it is easy to see that the following sequences of constrained minimization problems

$$\left\{\left\langle \inf_{x\in \Xi_\varepsilon} I_\varepsilon(x)\right\rangle\right\}_{\varepsilon\in E} \quad \text{and} \quad \left\{\left\langle \inf_{x\in \mathit{\Pi}_\varepsilon} (I_\varepsilon(x) + \mathbf{1}_W(x))\right\rangle\right\}_{\varepsilon\in E}$$

are the jointly L-equivalent.

Assume that for the sequence $\left\{\left\langle \inf_{x\in \mathit{\Pi}_\varepsilon} (I_\varepsilon(x) + \mathbf{1}_W(x))\right\rangle\right\}_{\varepsilon\in E}$ there exists a variational limit

$$\left\langle \inf_{x\in \mathit{\Pi}_0} J_0(x)\right\rangle. \tag{4.10}$$

Then, by Theorem 3.1, there exists a variational limit for the sequence

$$\left\{\left\langle \inf_{x\in \Xi_\varepsilon} I_\varepsilon(x)\right\rangle\right\}_{\varepsilon\in E}$$

such that this one is the L-equivalent to (4.10).

5 Variational limit of the sequences of penalized problems

The aim of this section is to find, for every fixed $\mu > 0$, the variational limit for the following sequence of penalized problems

$$(5.1) \qquad \left\{ \left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y) \right\rangle \right\}_{\varepsilon\in E},$$

where

$$(5.2) \qquad I_{\varepsilon,\mu}(u,y) = \int\limits_{\omega_\varepsilon} u^2 dx + \mu^{-1}\Big[\nu\left(\alpha - \| \, y\,|_S - z_d \, \|_{L^2(S)}\right)\Big]^2,$$

$$(5.3) \qquad \mathcal{N}_\varepsilon = \left\{ (u,y) \in L^2(\Omega)\times H_0^1(\Omega) \;\middle|\; \begin{array}{l} A_\varepsilon y + T_\varepsilon(y) = \chi_{\omega_\varepsilon} u \text{ in } \Omega \\ \chi_{\omega_\varepsilon}\cdot\xi_1 \le u \le \chi_{\omega_\varepsilon}\xi_2 \text{ a.e. in } \Omega \end{array} \right\}$$

and the function $\nu : \mathbf{R}^1 \to \mathbf{R}^1_+$ is defined in section 2.

Remark 5.1 *Let $\{u_\varepsilon \in L^2(\Omega)\}_{\varepsilon\in E}$ be a sequence such that*

$$u_\varepsilon \to u_0 \text{ weakly in } L^2(\Omega).$$

It is well known that this limit may doesn't coincide with the weak limit of $\{\chi_{\omega_\varepsilon} u_\varepsilon\}_{\varepsilon\in E}$. Assume that

$$\chi_{\omega_\varepsilon} u_\varepsilon \to u^* \qquad \text{weakly in } L^2(\Omega).$$

Note that under the above mentioned assumptions we have $\operatorname{supp} u^* \subseteq \omega_0$. Indeed, since the equality

$$\int\limits_\Omega \chi_{\omega_\varepsilon} u_\varepsilon \varphi \, dx = \int\limits_\Omega \chi_{\omega_0}\cdot\chi_{\omega_\varepsilon} u_\varepsilon \varphi \, dx$$

holds for every $\varphi \in L^2(\Omega)$ it follows that

$$\Big\langle \chi_{\omega_\varepsilon} u_\varepsilon, \chi_{\omega_0}\varphi \Big\rangle_{L^2(\Omega)} \to \int\limits_\Omega \chi_{\omega_0}\cdot u^*\varphi \, dx, \ \forall\, \varphi \in L^2(\Omega).$$

On the other hand

$$\left\langle \chi_{\omega_\varepsilon} u_\varepsilon, \varphi \right\rangle_{L^2(\Omega)} \to \int\limits_\Omega u^*\varphi \, dx \,, \, \forall\, \varphi \in L^2(\Omega).$$

It follows that $u^* = \chi_{\omega_0} u^*$ in $L^2(\Omega)$, hence $u^* = 0$ a.e. in $\Omega\backslash\omega_0$. i.e. $u^* \in L^2(\Omega;\omega_0)$. Moreover, in view of (A7) each of a cluster point u^* of the sequence $\{\chi_{\omega_\varepsilon} u_\varepsilon\}_{\varepsilon\in E}$ can be represented in the form $u^* = \chi_* \tilde{u}$, where $\tilde{u} \in L^2(\Omega)$ and $\tilde{u} \neq u_0$.

Remark 5.2 *Note that for every $\varepsilon \in E$ the operators*

$$B_\varepsilon(y) = A_\varepsilon y + T_\varepsilon(y)\,;\; B_\varepsilon : H_0^1(\Omega) \to H^{-1}(\Omega)$$

belong to the class $\mathcal{F}$. But thanks to (A1) these operators are uniformly coercive. Therefore, the set of such operators is compact with respect to the G-convergence (see [37],[32]), i.e. there exists an coercive operator $B_0 : H_0^1(\Omega) \to H^{-1}(\Omega)$ satisfying the conditions (A3)-(A4) such that

$$gr(B_0) = \nu{-}Lm\; gr(B_\varepsilon), \tag{5.4}$$

where ν is the product of the strong topology of $H^{-1}(\Omega)$ and the weak topology of $H_0^1(\Omega)$, and by $gr(B_0)$ is denoted the graph of operator B_0, i.e.

$$gr(B_0) = \left\{(f, y) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;:\; B_0(y) = f\right\}.$$

Therefore, it appears reasonable that the class of above mentioned operators

$$\mathcal{B} = \left\{B : H_0^1(\Omega) \to H^{-1}(\Omega) \,|\, B = A + T\,,\, \forall\, A \in \mathcal{A},\, \forall\, T \in \mathcal{F}\right\}$$

contains the G-limit operator B_0 of the sequence $\{A_\varepsilon + T_\varepsilon\}_{\varepsilon \in E}$..

Remark 5.3 *As follows from our assumptions, the sequence of penalized problems (5.1) is uniformly regular, i.e. $\mathcal{N}_\varepsilon \neq \emptyset$ for every $\varepsilon \in E$. Moreover, the corresponding sequence of admissible pairs*

$$\left\{\mathcal{N}_\varepsilon \subset L^2(\Omega) \times H_0^1(\Omega)\right\}_{\varepsilon \in E} \tag{5.5}$$

is uniformly bounded in $L^2(\Omega; \omega_0) \times H_0^1(\Omega)$. The last assertion immediately follows from the property of the class $\mathcal{B}$. Indeed, in view of [31], each of the operators B_ε has an inverse such that

$$\begin{aligned}
\| B_\varepsilon^{-1}(f) \|_{H_0^1(\Omega)} &\leq \nu_1 \| f \|_{H^{-1}(\Omega)} + \nu_2,\ \forall\, f \in H^{-1}(\Omega),\\
\| B_\varepsilon^{-1}(f) - B_\varepsilon^{-1}(g) \|_{H_0^1(\Omega)} &\leq \nu_2 \left(1 + \| f \|_{H^{-1}(\Omega)} + \| g \|_{H^{-1}(\Omega)}\right)\\
&\quad \times \| f - g \|_{H^{-1}(\Omega)},\ \forall\, f, g \in H^{-1}(\Omega),
\end{aligned}$$

where the constants ν_i don't depend on ε.

Remark 5.4 *Hereinafter we will assume that the assumptions (A1)-(A8) hold true.*

The following result deals with the existence and identification of the Kuratowski's limit of the sequence $\{\mathcal{N}_\varepsilon \subset L^2(\Omega; \omega_\varepsilon) \times H_0^1(\Omega)\}_{\varepsilon \in E}$.

Lemma 5.1 *The sequence of admissible pairs* $\{\mathcal{N}_\varepsilon\}_{\varepsilon\in E}$ *for the penalized problems (5.1) has a limit in the Kuratowski sense with respect to the* τ*-topology such that*

$$\mathcal{N}_0 = \left\{ (u,y) \in L^2(\Omega)\times H_0^1(\Omega) \,\middle|\, \begin{array}{rcl} A_0 y + T_0(y) & = & \chi_{\omega_0} u \\ \chi_*\xi_1 \le & u & \le \chi_*\xi_2 \end{array} \right\} \tag{5.6}$$

where $A_0 = -div(\boldsymbol{A}^{hom}(x)\nabla) + b$ *is a* G*-limit of the linear operators* $\{A_\varepsilon\}_{\varepsilon\in E}$*, and* χ_* *is a weak-*$*$ *limit of* $\{\chi_{\omega_\varepsilon}\}$*.*

Proof. Thanks to the Remark 5.3 we may use the definition of the Kuratowski's limit in the terms of τ-convergent sequences (see section 2).

Let H be any element of the grid $\mathcal{H}^\#$, and let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in H}$ be a sequence of pairs such that

$$(u_\varepsilon, y_\varepsilon) \in \mathcal{N}_\varepsilon \quad \forall\, \varepsilon \in H,$$
$$(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u,y).$$

We have to show that a limit pair (u,y) belongs to the set $\mathcal{N}_0$. Indeed, since for every $\varepsilon \in H$ we have

$$\chi_{\omega_\varepsilon}\cdot\xi_1 \le u_\varepsilon \le \chi_{\omega_\varepsilon}\cdot\xi_2 \text{ a.e. in } \Omega, \tag{5.7}$$

it follows that, passing to the limit in (5.7) as $H \ni \varepsilon \to 0$, and using property (A6), we obtain

$$\chi_*\cdot\xi_1 \le u \le \chi_*\cdot\xi_2 \quad \text{a.e. in } \Omega,$$
$$u \in L^2(\Omega;\omega_0) \Rightarrow u = \chi_{\omega_0} u.$$

On the other hand we have

$$(u_\varepsilon, y_\varepsilon) \in \mathrm{gr}(B_\varepsilon)\ , \forall\, \varepsilon \in H,$$

$$\nu{-}Lm\ \mathrm{gr}(B_\varepsilon) = \mathrm{gr}\ (B_0),$$

because the class $\mathcal{B}$ is closed with respect to the G-convergence. Hence, using the fact that

$$(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u,y) \quad \text{implies} \quad (u_\varepsilon, y_\varepsilon) \xrightarrow{\nu} (u,y),$$

we deduce: $(u,y) \in \ \mathrm{gr}(B_0)$.

Now we have to prove the representation

$$B_0(y) = A_0 y + T_0(y).$$

Indeed, by the definition of G-limit (see [32]) we have

$$y_\varepsilon \to y = B_0^{-1}(\chi_{\omega_0} u) \text{ weakly in } H_0^1(\Omega)$$

However,

$$y_\varepsilon = A_\varepsilon^{-1}\left(\chi_{\omega_\varepsilon} u_\varepsilon - T_\varepsilon(y_\varepsilon)\right), \ \forall\, \varepsilon \in E,$$
$$\chi_{\omega_\varepsilon} u_\varepsilon - T_\varepsilon(y_\varepsilon) \to \chi_{\omega_0} u - T_0(y) \text{ strongly in } H^{-1}(\Omega),$$
$$A_\varepsilon \xrightarrow{G} A_0.$$

Therefore,

$$A_\varepsilon^{-1}\left(\chi_{\omega_\varepsilon} u_\varepsilon - T_\varepsilon(y_\varepsilon)\right) \to A_0^{-1}\left(\chi_{\omega_0} u - T_0(y)\right) \text{ weakly in } H_0^1(\Omega).$$

Thus, on the one hand

$$y_\varepsilon \longrightarrow y \text{ weakly in } H_0^1(\Omega),$$

while on the other hand

$$y_\varepsilon \longrightarrow A_0^{-1}(\chi_{\omega_0} u - T_0(y)) \text{ weakly in } H_0^1(\Omega).$$

Consequently,

$$y = A_0^{-1}(\chi_{\omega_0} u - T_0(y)), \text{ i.e.}$$
$$A_0 y + T_0(y) = \chi_{\omega_0} u \text{ in } \Omega.$$

Thus, we have shown that the limit pair (u, y) satisfies the inclusion $(u, y) \in \mathcal{N}_0$.

Now we establish the reverse statement, namely for any pair $(u, y) \in \mathcal{N}_0$ there is a set $H \in \mathcal{H}$ and a sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon \in H}$ such that

$$(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \mathcal{N}_\varepsilon \ \forall\, \varepsilon \in H,$$

$$(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \xrightarrow{\tau} (u, y).$$

Let (u, y) be a fixed pair of $\mathcal{N}_0$. Let $\Phi : L^\infty(\omega_0) \to L^\infty(\Omega)$ be the following extension operator

$$\Phi f = \begin{cases} f(x) & , \quad x \in \omega_0, \\ +1 & , \quad x \in \Omega \backslash \omega_0. \end{cases}$$

Then for the function $m_* = \chi_* \mid_{\omega_0} \in L^\infty(\omega_0)$, that was defined in (A7), we have

$$\Phi m_* \in L^\infty(\Omega)\,,\ (\Phi m_*)^{-1} \in L^\infty(\Omega).$$

Now, for a given pair $(u, y) \in \mathcal{N}_0$ we construct the sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon \in H}$ as follows:

$$\left.\begin{array}{lcll} H & = & E & \\ \bar{u}_\varepsilon & = & (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} u & \text{for every } \varepsilon \in E, \\ \bar{y}_\varepsilon & = & B_\varepsilon^{-1}(\bar{u}_\varepsilon) & \text{for every } \varepsilon \in E. \end{array}\right\} \tag{5.8}$$

Since $\chi_* \cdot (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} = \chi_{\omega_\varepsilon}$ we immediately see that

$$\chi_{\omega_\varepsilon} \cdot \xi_1 \le \bar{u}_\varepsilon \le \chi_{\omega_\varepsilon} \cdot \xi_2\ ,\ \forall\, \varepsilon \in E.$$

Hence each of the pairs (5.8) is admissible, i.e. $(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \mathcal{N}_\varepsilon$ for every $\varepsilon \in E$. It remains to verify that the sequence $\{\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon\in E}$ τ-converges to the fixed pair (u, y). For this we note that for every function $\varphi \in L^2(\Omega)$ we have

$$\int_\Omega (\Phi m_*)^{-1} u\varphi \cdot \chi_{\omega_\varepsilon}\, dx \to \int_\Omega (\Phi m_*)^{-1} \chi_* \varphi u\, dx = \int_{\omega_0} m_*^{-1} \chi_* \varphi u\, dx = \int_{\omega_0} u\varphi\, dx,$$

that is

$$u \in L^2(\Omega; \omega_0) \text{ and } \bar{u}_\varepsilon \to u \text{ weakly in } L^2(\Omega).$$

Further, by the properties of G-limits we deduce: if $\bar{u}_\varepsilon \to u$ strongly in $H^{-1}(\Omega)$ and $B_\varepsilon \xrightarrow{G} B_0 = A_0 + T_0$, then

$$\bar{y}_\varepsilon = B_\varepsilon^{-1}(\bar{u}_\varepsilon) \to y_0 \text{ weakly in } H_0^1(\Omega),$$

where $y_0 = B_0^{-1}(u)$. However $(u, y) \in \mathcal{N}_0$, hence $y = B_0^{-1}(u)$. Then, using Remark 4.3 we obtain that these functions y_0 and y must coincided in $H_0^1(\Omega)$, i.e.

$$\| y_0 - y \|_{H_0^1(\Omega)} = 0.$$

Thus (u, y) is the weak limit of $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \mathcal{N}_\varepsilon\}_{\varepsilon\in E}$. As a result we have proved that the set $\mathcal{N}_0$ is in fact the Kuratowski's limit of $\{\mathcal{N}_\varepsilon\}_{\varepsilon\in E}$, i.e.

$$\mathcal{N}_0 = \tau\text{–}Lm\, \mathcal{N}_\varepsilon.$$

The lemma is proved. ∎

To establish the main result of this section we give the following lemma.

Lemma 5.2 *For every bounded sequence $\{u_\varepsilon\}_{\varepsilon\in E} \subset L^2(\Omega)$ such that $\chi_{\omega_\varepsilon} \to u_*$ weakly in $L^2(\Omega)$, the following inequality holds*

$$\liminf_{\varepsilon\to 0} \int_\Omega \chi_{\omega_\varepsilon} u_\varepsilon^2\, dx \geq \int_{\omega_0} m_*^{-1} (u_*)^2\, dx. \tag{5.9}$$

Proof. Since the inequality

$$\int_\Omega g^2\, dx - \int_\Omega f^2\, dx \geq 2 \int_\Omega (g - f)\, dx \tag{5.10}$$

holds for every $f, g \in L^2(\Omega)$ we put in (5.10)

$$g = \chi_{\omega_\varepsilon} u_\varepsilon \ , \ f = (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} u_*.$$

Using the fact that $\chi_{\omega_\varepsilon}^2 = \chi_{\omega_\varepsilon}$, $u_* = \chi_{\omega_0} u_*$, and $(\Phi m_*)^{-1} \chi_{\omega_\varepsilon} \to (\Phi m_*)^{-1} \chi_* = \chi_{\omega_0}$ weakly-$*$ in $L^\infty(\Omega)$ we pass to the limit in (5.10) as $\varepsilon \to 0$. For the right-hand side of the above inequality we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (\Phi m_*)^{-1} u_* [\chi_{\omega_\varepsilon} u_\varepsilon - (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} u_*]\, dx$$

$$= \int_{\Omega} (\Phi m_*)^{-1} u_* [u_* - (\Phi m_*)^{-1} \chi_* u_*] \, dx$$

$$= \int_{\Omega} (\Phi m_*)^{-1} u_* [u_* - u_*] \, dx = 0.$$

It follows that

$$\liminf_{\varepsilon \to 0} \int_{\Omega} \chi_{\omega_\varepsilon} u_\varepsilon^2 \, dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega} [(\Phi m_*)^{-1} u_*]^2 \chi_{\omega_\varepsilon} \, dx.$$

However,

$$\int_{\Omega} [(\Phi m_*)^{-1} u_*]^2 \chi_{\omega_\varepsilon} \, dx \to \int_{\Omega} [\Phi m_*]^{-2} (u_*)^2 \chi_* \, dx = \int_{\Omega} (\Phi m_*)^{-1} \chi_{\omega_0} u_*^2 = \int_{\omega_0} m_*^{-1} u_*^2 \, dx.$$

Thus the relation (5.9) is an immediate consequence of these observations which concludes the proof. ■

We are now in the position to state the main result of this section.

Theorem 5.1 *Assume that the hypotheses (A1)-(A8) are satisfied. Then for the sequence of penalized problems (5.1) there exists the variational τ-limit*

$$\left\langle \inf_{(u,y) \in \mathcal{N}_0} I_\mu^{hom}(u,y) \right\rangle \tag{5.11}$$

as $\varepsilon \to 0$ (for every fixed $\mu > 0$) such that

$$I_\mu^{hom}(u,y) = \int_{\omega_0} m_*^{-1} u^2 \, dx + \mu^{-1} [\nu \left(\alpha - \| \, y \, |_S - z_d \, \|_{L^2(S)} \right)]^2 \tag{5.12}$$

for every $(u,y) \in \mathcal{N}_0$, where the set $\mathcal{N}_0$ is defined in (5.6).

Proof. We will closely follow the lines of proof of the Theorem 2.1. First of all we prove the equality

$$\Gamma - \lim_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} \left[\int_{\omega_\varepsilon} u^2 \, dx \right] = \int_{\omega_0} m_*^{-1} u^2 \, dx \, , \, \forall \, (u,y) \in \mathcal{N}_0. \tag{5.13}$$

Indeed, in view of Lemma 5.2 we see that the following inequality

$$\int_{\omega_0} m_*^{-1} u^2 \, dx \leq \liminf_{H \ni \varepsilon \to 0} \int_{\omega_\varepsilon} u_\varepsilon^2 \, dx$$

holds for every sequence $\{(u_\varepsilon, y_\varepsilon) \in \mathcal{N}_\varepsilon\}_{\varepsilon\in H}$ such that $H \in \mathcal{H}^\#$ and $(u_\varepsilon, y_\varepsilon)$ τ-converges to some pair (u, y).

On the other hand, for every fixed pair $(u^0, y^0) \in \mathcal{N}_0$ we may construct the sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon\in E}$ such that

$$\bar{u}_\varepsilon = (\Phi m_*)^{-1}\chi_{\omega_\varepsilon} u^0 \;\forall\; \varepsilon \in E,$$
$$\bar{y}_\varepsilon = B_\varepsilon^{-1}(\bar{u}_\varepsilon); \forall\; \varepsilon \in E.$$

It is easy to see that

$$\begin{aligned} \bar{u}_\varepsilon &\to u^0 \quad \text{weakly in} \quad L^2(\Omega),\\ \bar{y}_\varepsilon &\to y^0 \quad \text{weakly in} \quad H_0^1(\Omega) \text{ (by properties of G-limit).} \end{aligned}$$

In addition we have

$$\int_{\omega_\varepsilon} \bar{u}_\varepsilon^2 dx = \int_\Omega (\Phi m_*)^{-2}\chi_{\omega_\varepsilon}(u^0)^2 dx$$
$$\to \int_\Omega (\Phi m_*)^{-1}\chi_{\omega_0}(u^0)^2 dx = \int_{\omega_0} m_*^{-1}(u^0)^2 dx.$$

Thus by virtue of the Theorem 2.1 the equality (5.13) holds true.

In conclusion we note that the map

$$G : (u, y) \to \left[\nu\left(\alpha - \| \, y|_S - z_d \, \|_{L^2(S)}\right)\right]^2$$

is τ-continuous because of the embedding $H_0^1(\Omega) \to L^2(S)$ is compact. Therefore, by the properties of Γ-limit, we have

$$\Gamma - \lim P_{\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y) = \Gamma - \lim P_{\mathcal{N}_\varepsilon}\Big[\int_{\omega_\varepsilon} u^2 dx\Big] + \mu^{-1}\left[\nu\left(\alpha - \| \, y|_S - z_d \, \|_{L^2(S)}\right)\right]^2.$$

This concludes the proof. ∎

6 Homogenization procedure for optimal control problems with state constraints

This section is devoted to the proof of commutativity of the following homogenization diagram

$$\text{(6.1)} \qquad \begin{array}{ccc} \Big\langle \inf\limits_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y)\Big\rangle & \xrightarrow{\varepsilon\to 0} & \Big\langle \inf\limits_{(u,y)\in\mathcal{N}_0} I_\mu^{hom}(u, y)\Big\rangle \\ \mu \to 0\Big\downarrow & & \Big\downarrow \mu \to 0 \\ \Big\langle \inf\limits_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y)\Big\rangle & \xrightarrow{\varepsilon\to 0} & \Big\langle \inf\limits_{(u,y)\in\Xi_{hom}} I^{hom}(u, y)\Big\rangle \end{array}$$

Here:

(i) $\left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y)\right\rangle$ is the penalized problem, that was defined in (5.2)-(5.3);

(ii) $\left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_{\varepsilon}(u,y)\right\rangle$ is the original optimal control problem (3.5) -(3.6);

(iii) $\left\langle \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_{\mu}(u,y)\right\rangle$ is the homogenized penalized problem (5.6),(5.12).

First of all we show that for every $\varepsilon \in E$

$$\left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y)\right\rangle \to \left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_{\varepsilon,\mu}(u,y)\right\rangle$$

as $\mu \to 0$ in the variational sense (see definition (B1)-(B2)). Indeed, let us introduce the following notations:

$$\Lambda = \left\{(u,y) \in L^2(\Omega)\times H^1_0(\Omega) : \| \, y|_S - z_d \, \|_{L^2(S)} \leq \alpha\right\} \tag{6.2}$$

$$\mathbf{1}_\Lambda(u,y) = \begin{cases} 0, & (u,y)\in\Lambda, \\ +\infty, & (u,y)\notin\Lambda. \end{cases} \tag{6.3}$$

Since the sequence of functionals

$$\left\{ I_{\varepsilon,\mu}(u,y) = I_\varepsilon(u,y) + \mu\Big[\nu\left(\alpha - \| \, y|_S - z_d \, \|_{L^2(S)}\right)\Big]^2 \right\}_{\mu>0}$$

is monotonically increasing as $\mu \to 0$ it follows that for every $\varepsilon \in E$ there exists a Γ-limit as $\mu \to 0$ for the corresponding extensions $\{P_{\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y)\}_{\mu>0}$ which can be represented in the form (see[11], [1])

$$\begin{aligned} \Gamma - \lim_{\mu\to 0} P_{\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y) &= \sup_{\mu\to 0} P_{\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y) \\ \triangleq J_\varepsilon(u,y) &= \begin{cases} I_\varepsilon(u,y) + \mathbf{1}_\Lambda(u,y), & (u,y)\in\mathcal{N}_\varepsilon \\ +\infty, & (u,y)\notin\mathcal{N}_\varepsilon \end{cases} \end{aligned}$$

In order to verify this relation, it is easy to see that the following equalities hold for every $\varepsilon \in E$.

$$\mathrm{Dom}(J_\varepsilon)\cap\mathcal{N}_\varepsilon = \mathrm{Dom}(I_\varepsilon)\cap\Xi_\varepsilon = \Xi_\varepsilon,$$
$$J_\varepsilon(u,y) = I_\varepsilon(u,y) \ \ \forall\,(u,y)\in\Xi_\varepsilon.$$

Hence, the constrained minimization problems

$$\left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_{\varepsilon}(u,y)\right\rangle \ \text{and} \ \left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} J_{\varepsilon}(u,y)\right\rangle \tag{6.4}$$

are L-equivalent. However the last problem is the variational limit of the penalized problems. Hence, we infer that $\left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)\right\rangle$ is the variational limit as $\mu \to 0$ of the sequence $\left\{\left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u,y)\right\rangle\right\}_{\mu>0}$ up to L-equivalence.

By the same arguments we may state that for the sequence of minimization problems

$$\left\{\left\langle \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\mu(u,y)\right\rangle\right\}_{\mu>0}$$

there exists the variational limit as ε tends to zero

$$\left\langle \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\Lambda(u,y)\right\rangle \tag{6.5}$$

which has the following representation

$$I^{hom}_\Lambda(u,y) = \int_{\omega_0} m_*^{-1}u^2dx + \mathbf{1}_\Lambda(u,y). \tag{6.6}$$

It is easy to see that the limit problem (6.5) is L-equivalent to the following one

$$\left\langle \inf_{(u,y)\in\Xi_{hom}} I^{hom}(u,y)\right\rangle, \tag{6.7}$$

where

$$\left.\begin{aligned} I^{hom}(u,y) &= \int_{\omega_0} m_*^{-1}u^2dx, \\ \Xi_{hom} &= \mathcal{N}_0\cap\Lambda. \end{aligned}\right\} \tag{6.8}$$

Thus, the prime object of our consideration in this section is to show that the constrained minimization problem (6.7) is L-equivalent to the variational limit of the original sequence $\left\{\langle \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)\rangle\right\}_{\varepsilon\in E}$ as $\varepsilon\to 0$.

We start with the following remarks.

Remark 6.1 *As it was shown before, the minimization problems (6.4) belong to the same class of L-equivalence for every $\varepsilon \in E$. Therefore, for the following analysis it is convenient to find a variational limit for the jointly L-equivalent sequence*

$$\left\{\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} (I_\varepsilon(u,y) + \mathbf{1}_\Lambda(u,y))\rangle\right\}_{\varepsilon\in E}. \tag{6.9}$$

Remark 6.2 *We note that the homogenized problem (6.5) has a sense only, if the original optimal control problem (3.1)-(3.5) is uniformly regular, i.e. $\Xi_\varepsilon \neq \phi$ for all $\varepsilon \in E$. Otherwise it may happen that*

$$\mathcal{N}_0\cap\Lambda = \emptyset.$$

It is the fundamental difference between the problem (3.1)-(3.5) and the simpler problem that was considered in [10].

Now we establish some auxiliary results.

Lemma 6.1 *Let (u^0, y^0) be any pair of $\mathcal{N}_0$, H be an index set such that $H \in \mathcal{H}^{\#}$, and let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon \in H}$ be a sequence that satisfies the following conditions*

$$(u_\varepsilon, y_\varepsilon) \in \mathcal{N}_\varepsilon \ \forall\, \varepsilon \in H, \ \textit{and}\ (u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u^0, y^0). \tag{6.10}$$

Then we have

$$I_\Lambda^{hom}(u^0, y^0) \le \liminf_{H \ni \varepsilon \to 0} J_\varepsilon(u_\varepsilon, y_\varepsilon), \tag{6.11}$$

where I_Λ^{hom} is defined in (6.6) and

$$J_\varepsilon(u, y) = I_\varepsilon(u, y) + \boldsymbol{1}_\Lambda(u, y)\ ,\ \forall \varepsilon \in H.$$

Proof. Since $J_\varepsilon(u, y) = \sup\limits_{\mu \to 0} I_{\varepsilon, \mu}(u, y)$ for every $\varepsilon \in E$ and for every $(u, y) \in \mathcal{N}_\varepsilon$ it follows that

$$J_\varepsilon(u, y) \ge I_{\varepsilon, \mu}(u, y)\ \forall\ \varepsilon \in E \forall\ (u, y) \in \mathcal{N}_\varepsilon\ \forall\ \mu > 0$$

Hence this inequality will be true, if we pass to the Γ-limit as $\varepsilon \to 0$, i.e.

$$\Gamma\!-\liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \ge \Gamma\!-\liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} I_{\varepsilon, \mu}(u, y). \tag{6.12}$$

However, as follows from the Theorem 5.1 we have

$$\begin{aligned} &\Gamma\!-\liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} I_\varepsilon(u, y) = \Gamma\!-\lim_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} I_{\varepsilon, \mu}(u, y) \\ &= P_{\mathcal{N}_0} I_\mu^{hom}(u, y)\ , \quad \forall (u, y) \in L^2(\Omega) \times H_0^1(\Omega). \end{aligned} \tag{6.13}$$

Therefore,

$$\Gamma\!-\liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \ge I_\mu^{hom}(u, y)$$

for every $(u, y) \in \mathcal{N}_0$.

Since this inequality holds true for every $\mu > 0$, it follows that

$$\Gamma\!-\liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \ge \sup_{\mu \to 0} I_\mu^{hom}(u, y) = I_\Lambda^{hom}(u, y), \tag{6.14}$$

where the functional I_Λ^{hom} is defined in (6.6).

We fix some pair (u^0, y^0) and take a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon \in H}$ satisfying the conditions (6.10). Then based on the properties of lower Γ-limits (see[11]), we have

$$\Gamma - \liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u^0, y^0) \le \liminf_{H \ni \varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u_\varepsilon, y_\varepsilon) = \liminf_{H \ni \varepsilon \to 0} J_\varepsilon(u_\varepsilon, y_\varepsilon). \tag{6.15}$$

Thus the relation (6.11) is an immediate consequence of (6.14) and (6.15). ■

In complete analogy with Theorem 5.1 we may prove the following result.

Lemma 6.2 *Assume that assumptions (A6)-(A7) are satisfied. Then for the sequence of functionals* $\left\{P_{\mathcal{N}_\varepsilon} I_\varepsilon : L^2(\Omega) \times H_0^1(\Omega) \to \bar{\mathbf{R}}\right\}_{\varepsilon \in E}$ *there exists the* Γ*-limit* $\Gamma - \lim_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} I_\varepsilon : L^2(\Omega) \times H_0^1(\Omega) \to \bar{\mathbf{R}}$ *such that*

$$\Gamma - \lim_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} I_\varepsilon = P_{\mathcal{N}_0} I^{hom},$$

where the functional I^{hom} *was defined in (6.8).*

In the following we prove the Γ-convergence of the sequence of the functionals

$$\left\{P_{\mathcal{N}_\varepsilon} J_\varepsilon = P_{\mathcal{N}_\varepsilon}(I_\varepsilon + \mathbf{1}_\Lambda) : L^2(\Omega) \times H_0^1(\Omega) \to \bar{\mathbf{R}}\right\}_{\varepsilon \in E} \tag{6.16}$$

with respect to the τ-topology and find an analytical representation for its Γ-limit.

To this end we have to prove the validity of the following inequalities

$$\Gamma - \liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \geq P_{\mathcal{N}_0} I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y), \tag{6.17}$$

$$\Gamma - \limsup_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \leq P_{\mathcal{N}_0} I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y), \tag{6.18}$$

Indeed, if these inequalities hold true, then we immediately deduce that for the sequence (6.16) there exists a Γ-limit such that

$$\Gamma - \lim_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon = P_{\mathcal{N}_0} I^{hom} + \mathbf{1}_\Lambda.$$

It is easy to see that the inequality (6.17) is valid, as, in view of (6.12)-(6.13), we have

$$\Gamma - \liminf_{\varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon \geq P_{\mathcal{N}_0} I_\mu^{hom} \text{ for every } \mu > 0 \tag{6.19}$$

So, passing to the supremum in (6.19) as $\mu \to 0$ and using the fact that

$$\sup_{\mu \to 0} P_{\mathcal{N}_0} I_\mu^{hom} = P_{\mathcal{N}_0} I_\Lambda^{hom} = P_{\mathcal{N}_0} I^{hom} + \mathbf{1}_\Lambda,$$

we obtain the result required.

Before the proof of the inequality (6.18) we need to introduce some assumptions and notations.

We denote by:

1. $\operatorname{ri}\Lambda$ the relative inferior of the set Λ with respect to the space $L^2(\Omega) \times H_0^1(\Omega)$, i.e. $\operatorname{ri}\Lambda = \left\{(u, y) \in L^2(\Omega) \times H_0^1(\Omega) \,\middle|\, \| y|_S - z_d \|_{L^2(S)} < \alpha \right\}$;

2. $cl_\tau Q$ the τ-closure of a set $Q \subset L^2(\Omega) \times H_0^1(\Omega)$ which is defined as follows: some pair (u, y) belongs to $cl_\tau Q$ if and only if (u,y) is a τ-limit point of a sequence $\{u_n, y_n)\}_{n\in N}$ such that $(u_n, y_n) \in Q$ for every $n \in N$;

3. $X_1 = \Big\{(u, y) \in L^2(\Omega) \times H_0^1(\Omega) \Big| (u, y) \notin \mathcal{N}_0 \text{ or } (u, y) \notin \Lambda\Big\}$;

4. $X_2 = \Big\{(u, y) \in L^2(\Omega) \times H_0^1(\Omega) \Big| (u, y) \in \mathcal{N}_0 \text{ and } (u, y) \in \mathrm{ri}\,\Lambda\Big\}$;

5. $X_3 = \Big\{(u, y) \in L^2(\Omega) \times H_0^1(\Omega) \Big| (u, y) \in \mathcal{N}_0 \text{ and } (u, y) \in \Lambda \setminus \mathrm{ri}\,\Lambda\Big\}$;

Hypotheses (A9):

there exists a τ-convergent sequence of admissible pais $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ such that its limit belongs to the set $\mathrm{ri}\,\Lambda$.

Hypotheses (A10):

$$cl_\tau(\mathcal{N}_0 \cap \mathrm{ri}\,\Lambda) = \mathcal{N}_0 \cap \Lambda. \tag{6.20}$$

Lemma 6.3 *Assume that the hypotheses (A1)-(A10) are satisfied. Then the inequality (6.18) holds true.*

Proof. Since the main space $L^2(\Omega) \times H_0^1(\Omega)$ can be divided into three parts $X_1 \cup X_2 \cup X_3$, we prove the validity of inequality (6.18) on every set X_i, separately.

Step 1. Let (u, y) be any pair of X_1. Then

$$P_{\mathcal{N}_0} I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y) = +\infty.$$

Hence, the inequality (6.18) is obvious for this pair.

Step 2. Suppose that $(u, y) \in X_2$. In accordance with hypothesis (A9) it is possible because in our case $X_2 \neq \emptyset$. Then

$$(u, y) \in \mathcal{N}_0 \quad \text{and} \quad \| y|_S - z_d \|_{L^2(S)} < \alpha. \tag{6.21}$$

We construct the sequence $\{(\bar{u}_\varepsilon, \bar{y}_\varepsilon)\}_{\varepsilon\in E}$ as follows

$$\bar{u}_\varepsilon = (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} u, \quad \forall\, \varepsilon \in E,$$
$$\bar{y}_\varepsilon = B_\varepsilon^{-1}(\bar{u}_\varepsilon), \qquad \forall\, \varepsilon \in E.$$

Then by analogy with proof of Theorem 5.1 it is easy to show that

$$(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \mathcal{N}_\varepsilon\,, \forall \varepsilon \in E,$$
$$(\bar{u}_\varepsilon, \bar{y}_\varepsilon \xrightarrow{\tau} (u, y). \tag{6.22}$$

As immediately follows from (6.21) and (6.22), there is a set $H \in \mathcal{H}^{\#}$ such that

$$(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \mathrm{ri}\,\Lambda \text{ for every } \varepsilon \in H,$$

i.e.

$$\| \bar{y}_\varepsilon|_S - z_d \|_{L^2(S)} < \alpha \ \forall \varepsilon \in H.$$

Thus, for every $\varepsilon \in H$ we have $(\bar{u}_\varepsilon, \bar{y}_\varepsilon) \in \Xi_\varepsilon$. This implies

$$\begin{aligned}
\limsup_{H \ni \varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(\bar{u}_\varepsilon, \bar{y}_\varepsilon) &= \limsup_{H \ni \varepsilon \to 0} \int_\Omega (\bar{u}_\varepsilon)^2 dx \\
&= \lim_{H \ni \varepsilon \to 0} \int_\Omega (\Phi m_*)^{-2} \chi_{\omega_\varepsilon} u^2 dx = \int_{\omega_0} m_*^{-1} u^2 dx \\
&= I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y) = P_{\mathcal{N}_0} I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y).
\end{aligned}$$

At the same time for other sequences $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon \in H}$ (here $H \in \mathcal{H}^{\#}$) which is τ-convergent to (u, y) we will have

$$\limsup_{H \ni \varepsilon \to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u_\varepsilon, y_\varepsilon) \geq \int_{\omega_0} m_*^{-1} u^2 dx = P_{\mathcal{N}_0} I^{hom}(u, y) + \mathbf{1}_\Lambda(u, y).$$

Thus inequality (6.18) holds.

Step 3. Let (u^*, y^*) be any pair of X_3, i.e

$$(u^*, y^*) \in \mathcal{N}_0 \cap (\Lambda \backslash \mathrm{ri}\,\Lambda).$$

Then in view of the supposition (A10) there is a sequence

$$\{(u_n, y_n) \in \mathcal{N}_0 \cap \mathrm{ri}\,\Lambda\}_{n \in \mathbb{N}}$$

such that

$$(u_n, y_n) \xrightarrow{\tau} (u^*, y^*). \tag{6.23}$$

For every $n \in \mathbb{N}$ we construct the sequence $\{(u_n^\varepsilon, y_n^\varepsilon) \in \mathcal{N}_\varepsilon\}_{\varepsilon \in E}$, where

$$\begin{aligned}
u_n^\varepsilon &= (\Phi m_*)^{-1} \chi_{\omega_\varepsilon} \cdot u_n \\
y_n^\varepsilon &= B_\varepsilon^{-1}(u_n^\varepsilon).
\end{aligned}$$

It is easy to see that

$$(u_n^\varepsilon, y_n^\varepsilon) \xrightarrow[\varepsilon \to 0]{\tau} (u_n, y_n) \quad \text{for every } n \in \mathbb{N}. \tag{6.24}$$

Since each of the pairs (u_n, y_n) belongs to $\mathrm{ri}\,\Lambda$, it follows that for every $n \in \mathbb{N}$ there is a set $H_n \in \mathcal{H}^{\#}$ such that

$$(u_n^\varepsilon, y_n^\varepsilon) \in \mathrm{ri}\,\Lambda \quad \forall\, \varepsilon \in H_n,\ \forall\, n \in \mathbb{N}.$$

Thanks to this, and in view of (6.23)-(6.24), we can construct a numerical sequence $\{\varepsilon_n; \varepsilon_n \to 0\}_{n\in\mathbb{N}}$ such that

$$(u_n^{\varepsilon_n}, y_n^{\varepsilon_n}) \in \mathcal{N}_{\varepsilon_n} \ \cap \ \mathrm{ri}\,\Lambda\ ,\ \forall n \in \mathbb{N},$$
$$(u_n^{\varepsilon_n}, y_n^{\varepsilon_n}) \xrightarrow[n\to\infty]{\tau} (u^*, y^*).$$

From this we immediately have

$$\begin{aligned} & u_n \longrightarrow u^* && \text{weakly in } L^2(\Omega), \\ & u_n^{\varepsilon_n} = (\Phi m_*)^{-1}\chi_{\omega_{\varepsilon_n}} u_n \to u^* && \text{weakly in } L^2(\Omega), \\ & (\Phi m_*)^{-1}\chi_{\omega_{\varepsilon_n}} \to 1 && \text{weakly-}* \text{ in } L^\infty(\Omega). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_\Omega (u_n^{\varepsilon_n})^2 dx = \lim_{n\to\infty} \int_\Omega (\Phi m_*)^{-2}\chi_{\omega_{\varepsilon_n}} u_n^2 dx = \int_{\omega_0} m_*^{-1}(u^*)^2 dx.$$

Using this relation and the fact that

$$(u_n^{\varepsilon_n}, y_n^{\varepsilon_n}) \in \mathcal{N}_{\varepsilon_n} \ \cap \ \mathrm{ri}\,\Lambda \ \forall\, n \in \mathbb{N},$$

we obtain

$$\begin{aligned} \limsup_{n\to\infty} P_{\mathcal{N}_\varepsilon} J_{\varepsilon_n}(u_n^{\varepsilon_n}, y_n^{\varepsilon_n}) &= \limsup_{n\to\infty} \int_\Omega (u_n^{\varepsilon_n})^2 dx \\ &= \int_{\omega_0} m_*^{-1}(u^*)^2 dx = P_{\mathcal{N}_0} I^{hom}(u^*, y^*) + \mathbf{1}_\Lambda(u^*, y^*). \end{aligned}$$

On the other hand, by the main property of upper Γ-limit (see[1]), we have

$$\limsup_{n\to\infty} P_{\mathcal{N}_{\varepsilon_n}} J_{\varepsilon_n}(u_n^{\varepsilon_n}, y_n^{\varepsilon_n}) \geq \Gamma\!-\limsup_{\varepsilon\to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u^*, y^*).$$

From this and previous equality we immediately obtain the required inequality (6.18). This completes the proof. ∎

Thus, because (6.17)-(6.18) hold and the set $\mathcal{N}_0$ is the Kuratowski limit of the sequence of admissible pairs $\{\mathcal{N}_\varepsilon\}_{\varepsilon\in E}$ (see Lemma 5.1), the following statement is obvious.

Theorem 6.1 *Assume that the initial conditions (A1)-(A10) are satisfied. Then for the sequence of constrained minimization problems*

$$\left\{ \left\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} J_\varepsilon(u, y) \right\rangle \right\}_{\varepsilon\in E} \tag{6.25}$$

there exists the variational limit which has the following representation:

$$\left\langle \inf_{(u,y)\in\mathcal{N}_0} \left[I^{hom}(u, y) + \boldsymbol{1}_\Lambda(u, y) \right] \right\rangle. \tag{6.26}$$

Since the constrained minimization problem (6.26) is L-equivalent to

$$\left\langle \inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u,y)\right\rangle$$

we may give the following important consequences of this theorem.

Corollary 6.1.1 *For the sequence of admissible pairs in the original optimal control problem (3.1)-(3.5) $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ there exists a limit with respect to the τ-topology (in the Kuratowski sense) such that*

$$\tau{-}Lm\,\Xi_\varepsilon = \Xi^{hom} = \mathcal{N}_0\cap\Lambda.$$

So, we have

$$\Xi^{hom} = \left\{ (u,y)\in L^2(\Omega)\times H_0^1(\Omega) \;\middle|\; \begin{array}{rcl} A_0 y + T_0(y) & = & \chi_{\omega_0} u, \\ \chi_*\xi_1 \leq & u & \leq \chi_*\xi_2, \\ \| \, y|_S - z_d \, \|_{L^2(S)} & \leq & \alpha. \end{array} \right\} \tag{6.27}$$

Corollary 6.1.2 *Under the supposition of the Theorem 6.1 the homogenization diagram (5.1) is commutative.*

Remark 6.3 *We note that by the properties of the Kuratowski limits, for any set $C\subset L^2(\Omega)\times H_0^1(\Omega)$, we have*

$$\tau{-}Lm\,C = cl_\tau\,C.$$

We may rewrite assumption (A10) as

$$\textit{(A10)}\qquad \tau{-}Lm(\mathcal{N}_0\cap ri\,\Lambda) = \mathcal{N}_0\cap\Lambda.$$

At the same time, in the general case, we always have the following inclusion

$$\tau - Lm(\mathcal{N}_0\cap ri\,\Lambda) \subseteq \mathcal{N}_0\cap\Lambda. \tag{6.28}$$

If we suppose that the inclusion (6.28) is strict then the result of variational convergence for the sequence (6.25) will have another form. Assume that the condition (A10) is not valid which means that there is a pair (u^0,y^0) such that

$$(u^0,y^0)\in\mathcal{N}_0\cap(\Lambda\backslash ri\,\Lambda)$$
$$\textit{and}\quad (u^0,y^0)\notin \tau{-}Lm(\mathcal{N}_0\cap ri\,\Lambda).$$

Then, as follows from Γ*-limit properties, we have*

$$\Gamma-\limsup_{\varepsilon\to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon \geq \Gamma-\limsup_{\varepsilon\to 0} \boldsymbol{1}_{\mathcal{N}_\varepsilon\cap\Lambda} = \boldsymbol{1}_{\tau-Li(\mathcal{N}_\varepsilon\cap\Lambda)}. \tag{6.29}$$

However, in view of the proof of Lemma 6.3, the following inclusion holds

$$\tau-Li(\mathcal{N}_\varepsilon\cap\Lambda) \subseteq \tau - Lm(\mathcal{N}_0\cap ri\,\Lambda).$$

From this and (6.29) we conclude

$$\Gamma-\limsup_{\varepsilon\to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon(u^0,y^0) = +\infty.$$

On the other hand we have the obvious relation

$$P_{\mathcal{N}_0} I^{hom}(u^0,y^0) + \boldsymbol{1}_\Lambda(u^0,y^0) = \int_{\omega_0} m_*^{-1}(u^0)^2 dx \; < \; +\infty.$$

Thus, the equality

$$\Gamma - \limsup_{\varepsilon\to 0} P_{\mathcal{N}_\varepsilon} J_\varepsilon = P_{\mathcal{N}_0} I^{hom} + \boldsymbol{1}_\Lambda$$

holds true only on the set $Dom(\Gamma-\limsup P_{\mathcal{N}_\varepsilon} J_\varepsilon)$*, where*

$$Dom(\Gamma - \limsup P_{\mathcal{N}_\varepsilon} J_\varepsilon) = cl_\tau(\mathcal{N}_0\cap ri\,\Lambda) = \tau-Lm(\mathcal{N}_0\cap ri\,\Lambda).$$

Taking this and the inequality (6.18) into account we may conclude that the variational limit for the sequence (6.25) exists and that it can be represented in the form

$$\Big\langle \inf_{(u,y)\in\tau-Lm(\mathcal{N}_0\cap \mathrm{ri}\,\Lambda)} I^{hom}(u,y)\Big\rangle. \tag{6.30}$$

Thus, the minimization problem (6.26) can be interpreted only as a relaxed problem to (6.30) but not as a homogenized one. It is easy to see that these problems are linked by the following relation

$$\inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u,y) \leq \inf_{(u,y)\in Lm(\mathcal{N}_0\cap \mathrm{ri}\,\Lambda)} I^{hom}(u,y).$$

Nevertheless, we have the following theorem.

Theorem 6.2 *For the relaxed optimal control problem (6.26) the following relation holds true*

$$\inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u,y) = \inf_{(u,y)\in Lm(\mathcal{N}_0\cap ri\,\Lambda)} I^{hom}(u,y). \tag{6.31}$$

The proof of this result will be given in the next section.

Remark 6.4 *We note that in the linear case (when $T_\varepsilon \equiv 0$) each of the sets $\mathcal{N}_\varepsilon$ is a closed subset of some linear manifold in $L^2(\Omega) \times H_0^1(\Omega)$. Therefore the hypothesis (A10) is satisfied thanks to the Theorem 2 from [21].*

Remark 5.5. Actually, the fulfilment of the hypothesis (A9) means that each of the sets Ξ_ε has to include the admissible pairs $(u_\varepsilon, y_\varepsilon)$ for which can be found a constant σ $(0 < \sigma < \alpha)$ such that

$$\| y_\varepsilon|_S - z_d \|_{L^2(S)} \leq \alpha - \sigma \text{ for every } \varepsilon \in E.$$

At the same time, as we will see in the next section, the equality

$$\lim_{\varepsilon\to 0} \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y) = \inf_{(u,y)\in\tau - Lm(\mathcal{N}_0 \cap \mathrm{ri}\,\Lambda)} I^{hom}(u,y) = \inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u,y)$$

and the τ-convergence of the optimal solutions $(u_\varepsilon^0, y_\varepsilon^0)$ for the original problems to an optimal solution of the relaxed problem (6.26) holds true under the following assumption (the weak analogy of the hypothesis (A9)):

(A9$^{\text{new}}$) the optimal control problem (3.1)-(3.5) is uniformly regular, i.e. $\Xi_\varepsilon \neq \emptyset$ for every $\varepsilon \in E$.

7 Variational properties of the homogenized problem

As follows from Theorem 6.1, the limiting constrained minimization problem (6.26) corresponds to the following optimal control problem

$$A_0 y + T_0(y) = \chi_{\omega_0} u, \tag{7.1}$$

$$\chi_* \cdot \xi_1 \leq u \leq \chi_* \cdot \xi_2 \text{ a.e. in } \Omega, \tag{7.2}$$

$$\| y|_S - z_d \|_{L^2(S)} \leq \alpha, \tag{7.3}$$

$$I^{hom}(u,y) = \int_{\omega_0} m_*^{-1} \cdot u^2 dx \to \inf. \tag{7.4}$$

In the sequel this problem is denoted as the homogenized optimal control problem.

As it was shown in section 2 under hypotheses (A1)-(A5) and (A9) the original problem (3.1)-(3.5) has an optimal solution $(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ for every $\varepsilon \in E$. Since this sequence is bounded in $L^2(\Omega) \times H_0^1(\Omega)$ we may suppose that there exists a pair (u^0, y^0) such that

$$(u_\varepsilon^0, y_\varepsilon^0) \xrightarrow{\tau} (u^0, y^0).$$

Now we state the following result which concerns the variational properties of homogenized problem (7.1)-(7.4).

Theorem 7.1 *Assume that conditions (A1)-(A10) are satisfied. Then the following statements hold:*

(i) the sequence of optimal pairs $\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ *τ-converges to* (u^0, y^0) *as* $\varepsilon \to 0$*, where* (u^0, y^0) *is an optimal pair for homogenized problem (7.1)-(7.5);*

(ii)

$$\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y) \to \min_{(u,y)\in\Xi^{hom}} I^{hom}(u,y) = I^{hom}(u^0,y^0); \tag{7.5}$$

(iii)

$$\int_\Omega (u^0_\varepsilon)^2 dx \to \int_{\omega_0} m_*^{-1}(u^0)^2 dx; \tag{7.6}$$

$$u^0_\varepsilon - (\Phi m_*)^{-1}\chi_{\omega_\varepsilon} u^0 \to 0 \quad \text{strongly in } L^2(\Omega). \tag{7.7}$$

Proof. Since the problem (7.1)-(7.5) corresponds to the variational limit of the original optimal control problem (3.1)-(3.5) it follows that the statements (i) and (ii) are direct consequences of the variational properties of Γ-limits (see Theorem 2.2).

The statement (7.6) immediately follows from (ii). Thus we have to prove (7.7). Indeed, using (7.6) and the arguments to proof Lemma 5.1, we obtain

$$\| u^0_\varepsilon - (\Phi m_*)^{-1}\chi_{\omega_\varepsilon} u^0 \|^2_{L^2(\Omega)} =$$
$$\int_\Omega (u^0_\varepsilon)^2 dx - 2\int_\Omega u^0_\varepsilon (\Phi m_*)^{-1} u^0 dx + \int_\Omega (\Phi m_*)^{-1}\chi_{\omega_\varepsilon}(u^0)^2 dx \to$$
$$\to \int_\Omega m_*^{-1}(u^0)^2 dx - 2\int_\Omega m_*^{-1}(u^0)^2 dx + \int_\Omega m_*^{-1}(u^0)^2 dx = 0$$

The theorem is proved. ∎

Now we may return to the Theorem 6.2 from the previous section. In fact it is possible to establish a stronger result than it was done in Theorem 6.2.

Theorem 7.2 *Assume that the supposition (A1)-(A8) and* $(A9^{new})$ *are satisfied. Then there is a set* $H \in \mathcal{H}$ *such that* *(i) the sequence of optimal pairs* $\{(u^0_\varepsilon, y^0_\varepsilon)\}_\varepsilon$ *for the original optimal control problem (3.1)-(3.5)* τ*-converges to* (u^0, y^0) *as* $H \ni \varepsilon \to 0$*, where* (u^0, y^0) *is an optimal pair for the relaxed problem (6.26);*

(ii) $\inf\limits_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) \equiv I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \stackrel{H\ni\varepsilon\to 0}{\longrightarrow} I^{hom}(u^0, y^0) = \inf\limits_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u, y)$*;*

(iii) $$\int\limits_\Omega (u^0_\varepsilon)^2 dx \to \int\limits_{\omega_0} m_*^{-1}(u^0)^2 dx;$$

$$u^0_\varepsilon - (\Phi m_*)^{-1}\chi_{\omega_\varepsilon} u^0 \to 0 \textit{ strongly in } L^2(\Omega).$$

Proof. As follows from the diagram (5.1) and variational properties of Γ-limits we have

$$(X1): \ \Big\langle \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y)\Big\rangle \ \stackrel{\varepsilon\to 0}{\longrightarrow} \ \Big\langle \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\mu(u, y)\Big\rangle,$$
$$(X2): \ \Big\langle \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\mu(u, y)\Big\rangle \ \stackrel{\mu\to 0}{\longrightarrow} \ \Big\langle \inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u, y)\Big\rangle,$$

Let $\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon\in E}$ be the sequence of optimal pairs for the original problem (3.1)-(3.5). Then each of these pairs is an optimal for the corresponding penalized problem, i.e.

$$I_{\varepsilon,\mu}(u^0_\varepsilon, y^0_\varepsilon) = \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y), \ \forall\varepsilon \in E, \ \forall\mu > 0.$$

Indeed, let $(u^0_{\varepsilon,\mu}, y^0_{\varepsilon,\mu})$ be an optimal pair for the penalized problem. Then

$$I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \le I_\varepsilon(u^0_{\varepsilon,\mu}, y^0_{\varepsilon,\mu}) \le I_{\varepsilon,\mu}(u^0_{\varepsilon,\mu}, y^0_{\varepsilon,\mu})$$
$$\le I_{\varepsilon,\mu}(u^0_\varepsilon, y^0_\varepsilon) \equiv I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon).$$

Therefore,

$$I_{\varepsilon,\mu}(u^0_\varepsilon, y^0_\varepsilon) = I_{\varepsilon,\mu}(u^0_{\varepsilon,\mu}, y^0_{\varepsilon,\mu}),$$

i.e.

$$I_{\varepsilon,\mu}(u^0_\varepsilon, y^0_\varepsilon) = \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y).$$

Since this sequence of optimal pairs (that is $\{(u^0_\varepsilon, y^0_\varepsilon)\}$) is τ-compact, it follows that there exists a set $H \in \mathcal{H}$ and a pair (u^0, y^0) such that

$$(u^0_\varepsilon, y^0_\varepsilon) \stackrel{\tau}{\longrightarrow} (u^0, y^0) \text{ as } H \ni \varepsilon \to 0.$$

Thus, as follows from (X1), we immediately have:

$$\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u, y) = \inf_{(u,y)\in\mathcal{N}_\varepsilon} I_{\varepsilon,\mu}(u, y) \to \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\mu(u, y),$$

$$I^{hom}_\mu(u^0, y^0) = \inf_{(u,y)\in\mathcal{N}_0} I^{hom}_\mu(u, y), \ \forall\mu > 0.$$

Hence the pair $(u^0, y^0) \in \mathcal{N}_0$ is an optimal solution for every limiting penalized problem

$$\Big\langle \inf_{(u,y)\in\mathcal{N}_0} I_\mu^{hom}(u,y) \Big\rangle.$$

Therefore, by variational property of the convergence (X2) we obtain that

$$I^{hom}(u^0, y^0) = \inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I^{hom}(u,y),$$

$$\inf_{(u,y)\in\mathcal{N}_0} I_\mu^{hom}(u,y) \to \inf_{(u,y)\in\mathcal{N}_0\cap\Lambda} I_\varepsilon(u,y) \text{ as } \mu \to 0.$$

Thus the properties (i)-(ii) are proved. The verification of the last property (iii) can be done as in Theorem 7.1. This concludes the proof of the theorem. ∎

Remark 7.1 *It is easy to see that the relaxed problem (6.26) can be written in the form of optimal control problem (7.1)-(7.4). Thus if the assumptions (A9)-(A10) are not valid and instead of them ($A9^{new}$) holds true, then this problem preserves all variational properties of the homogenized problem*

$$\Big\langle \inf_{(u,y)\in cl_\tau(\mathcal{N}_0\cap ri\Lambda)} I^{hom}(u,y) \Big\rangle.$$

Remark 7.2 *As follows from the previous section if the set of admissible pairs $\mathcal{N}_0$ for the control problem (7.1)-(7.4) satisfies the condition*

$$Lm(\mathcal{N}_0 \cap ri\Lambda) \subset \mathcal{N}_0 \cap \Lambda,$$

then the limit problem (6.30) may not be recovered in the form of some optimal control problem. This fact is the distinguishing feature of the state constrained optimal control problems in homogenization. The next fundamental difference between such problems and unconstrained problems is related to the following fact. Since (3.1)-(3.5) is the result of the homogenization of the family (3.1)-(3.5) it follows that for every $\delta > 0$ there is an $\varepsilon^0 > 0$ such that

$$\int_\Omega \varphi(u_\varepsilon^0 - u^0)dx < \delta \text{ for every } \varphi \in L^2(\Omega) \text{ and for all } \varepsilon < \varepsilon^0,$$

where u_ε^0 and u^0 are optimal controls for the original and homogenized problems, respectively. Assume that $\chi_* < 1$ and u^0 is equal to $\chi_*\xi_1$, i.e. u^0 is admissible control function for the problem (7.1)-(7.5). However we cannot use this function as a suboptimal control in (3.1)-(3.5) for any $\varepsilon < \varepsilon^0$, because $\chi_{\omega_\varepsilon} u^0 = \chi_{\omega_\varepsilon}\chi_*\xi_1 < \chi_{\omega_\varepsilon}\xi_1$.

In this case we should construct suboptimal controls for the original problem as follows

$$\tilde{u}_\varepsilon = (\Phi m_*)^{-1}\chi_{\omega_\varepsilon} u^0.$$

This conclusion immediately follows from the property (7.7).

Example. Let Ω be a bounded connected open domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, let ω_0 be an open nonempty connected subdomain of Ω. Also, let F be a 1-periodic domain in $\mathbf{R}^2$ with a lattice structure such that the cell of periodicity $\square$ has the same form as on Fig.2. Let S be a 1-dimensional manifold in $\Omega\backslash\bar{\omega}_0$.

To formulate our optimal control problem we introduce the ε-periodic lattice structure F_ε on $\mathbf{R}^2$ as $F_\varepsilon = \varepsilon F$. We denote by ω_ε the sets $F_\varepsilon \cap \omega_0$ and by $a(\cdot)$ a $\square$-periodic function of $L^\infty(\mathbf{R}^2)$ such that

$$a\left(\frac{x}{\varepsilon}\right) = \begin{cases} \beta, & x \in \Omega \cap F_\varepsilon \quad (\beta > 0), \\ \gamma, & x \in \Omega\backslash F_\varepsilon \quad\ (\gamma >). \end{cases}$$

We define the optimal control problem as follows

$$-\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla y\right) = \chi_{\omega_\varepsilon} u \text{ in } \Omega, \tag{7.8}$$

$$y = 0 \text{ on } \partial\Omega, \tag{7.9}$$

$$\| \, y|_S - z_d \, \|_{L^2(S)} \leq \alpha, \tag{7.10}$$

$$\chi_{\omega_\varepsilon} \cdot \xi_1 \leq \ u \ \leq \chi_{\omega_\varepsilon} \cdot \xi_2 \text{ a.e. in } \Omega, \tag{7.11}$$

$$I_\varepsilon = \int\limits_{\omega_\varepsilon} u^2 dx \to \inf. \tag{7.12}$$

Here: $\xi_1, \xi_2 \in L^2(\Omega),\ \xi_2 - \xi_1 \geq 0$ a.e. in $\Omega, z_d \in L^2(S)$.

As follows from our results above, we may recover the homogenized problem under the following supposition: there exists a sequence of admissible pairs $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon\in E}$ such that

1) $(u_\varepsilon, y_\varepsilon)$ satisfies the conditions (7.8)-(7.11) for every $\varepsilon \in E (E = (0;1])$;
2) $(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u_0, y_0)$, where $\| \, y_0|_S - z_d \, \|_{L^2(S)} < \alpha$.

In the result the homogenized optimal control problem has the form

$$\left.\begin{array}{ll} a_* \triangle y = \chi_{\omega_0} u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ \chi_* \xi_1 \leq u \leq \chi_* \xi_2 & \text{a.e.in } \Omega, \\ \| \, y|_S - z_d \, \|_{L^2(S)} \leq \alpha, & \\ I^{hom} = \int\limits_{\omega_0} m_*^{-1} u^2 dx \to \inf & \end{array}\right\} \tag{7.13}$$

where $\chi_* = 2\delta - \delta^2$ in ω_0 and $\chi_* = 0$ in $\Omega\backslash\omega_0$, $m_* = 2\delta - \delta^2$, $a_* = [\beta^{-1}(2\delta - \delta^2) + \gamma^{-1}(1-\delta)^2]^{-1}$.

In this connection we note that under the supposition 1)–2) the original problem (7.8)-(7.12) has a unique solution $(u_\varepsilon^0, y_\varepsilon^0)$ for every $\varepsilon \in E$ such that

$$u_\varepsilon^0 \to u^0 \text{ weakly in } L^2(\Omega),$$

$$y_\varepsilon^0 \to y^0 \text{ weakly in} H_0^1(\Omega),$$

$$u_\varepsilon^0 - (2\delta - \delta^2)^{-1}\chi_{\omega_\varepsilon} u^0 \to 0 \text{ strongly in } L^2(\Omega),$$

$$\int\limits_{\omega_\varepsilon} (u_\varepsilon^0)^2 dx \to \int\limits_{\omega_0} (u^0)^2 dx \cdot (2\delta - \delta^2)^{-1},$$

where (u^0, y^0) is the unique optimal pair for the homogenized problem (7.13).

Remark 7.3 *It is easy to see that the homogenized problem (7.13) has no limit as* $\delta \to 0$. *Within the framework of this theory we may give the following explanation of this fact.*

First of all we note that, after passing to the limit in the original problem ($\varepsilon \neq 0$*) as* $\delta \to 0$ *(thickness vanishes), we obtain an optimal control problem with a graph-like support for the control. Therefore, the admissible controls do not belong to* $L^2(\Omega)$ *in this case. Thus, for every* $\varepsilon \in E$ *each of the sets of admissible pairs* Ξ_ε *is empty in* $L^2(\Omega) \times H_0^1(\Omega)$. *This implies that for the limit analysis of such problems, as the parameter* ε *tends to 0, we should then work with a weaker topology on the "control-state" space, for example with the product of the weak topologies in* $H^1(\Omega)$ *and* $H_0^1(\Omega)$.

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