Differential Equations and their Applications, DNU:MMF, 2017. Issue 3, № 1, p. 1–19

Theoretical Aspects of Optimal Control
Theory and Partial Differential Equations

AMS subject classifications: 35J92, 58C07

HÖLDER CONTINUITY OF THE RESOLVENT OF ONE CLASS OF BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS ON THE COEFFICIENTS OF GENERALIZED *P*-LAPLACE OPERATOR

T. S. Byra*, P. I. Kogut**
* *Department of Differential Equations, Oles Honchar Dnipro National University, Kazakova, 18, build. 14, Dnipro, 49050 e-mail: tanyabyra8@gmail.com*
** *Department of Differential Equations, Oles Honchar Dnipro National University, Gagarin av., 72, Dnipro, 49010, e-mail: p.kogut@i.ua*

Abstract

We discuss the Hölder continuity property for the inverse mapping that identifies the diffusivity coefficient $\sigma(x)$ in the main part of generalized p-Laplace equation as a function of resolvent operator. In particular, we prove that, within a chosen class of non-smooth admissible functions the resolvent determines the diffusivity in a unique manner and the corresponding inverse mapping is Hölder continuous in the suitable topologies.

Keywords: generalized p-Laplace operator, quasi-linear elliptic equation, resolvent, Hölder continuity

1. Introduction

In this paper we are concerned with the following Dirichlet boundary value problem

$$-\operatorname{div}\left(\sigma\,|\nabla y|^{p-2}\,\nabla y\right) = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$y = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, $N \geq 1$, $f \in W^{-1,q}(\Omega)$ is a given distribution, and σ belongs to the following class of functions

$$\Sigma(\Omega) = \left\{\sigma \in L^\infty(\Omega) : \sigma_0 \leq \sigma(x) \leq \sigma_1^* \text{ a.e. in } \Omega\right\}, \tag{1.3}$$

where $0 < \sigma_0 < \sigma_1^*$ are given real numbers. In what follows we shall call (1.1) the generalized p-Laplace equation. Hereinafter we suppose that $p \in (1, +\infty)$ and $q = p/(p-1)$ is its conjugate exponent.

For two real numbers $1 < p < +\infty$, $1 < q < +\infty$ such that $1/p + 1/q = 1$, we define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$, i.e.

$$W_0^{1,p}(\Omega) := \mathrm{cl}_{\|\cdot\|_{W^{1,p}(\Omega)}} C_0^\infty(\Omega),$$

where

$$\|y\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |y|^p\,dx + \int_\Omega |\nabla y|^p\,dx\right)^{1/p},$$

and endow it with the norm [1, 14]

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)^N} = \left(\int_\Omega |\nabla u|_{\mathbb{R}^N}\,dx\right)^{1/p}, \quad \forall\, u \in W_0^{1,p}(\Omega).$$

Here, ∇u stands for the generalized gradient of $u \in W_0^{1,p}(\Omega)$, i.e. (see [3])

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}\right)^t$$

and

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}\,dx = -\int_\Omega \varphi\frac{\partial u}{\partial x_i}\,dx, \quad \forall\, i = 1, 2, \dots, N, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

We denote by $W^{-1,q}(\Omega)$ the dual space to $W_0^{1,p}(\Omega)$. It is well known that for a bounded domain Ω we have the inclusions

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,q}(\Omega) \text{ for } 2N/(N+2) \le p < \infty$$

with continuous and dense injections. Moreover, the elements of $W^{-1,q}(\Omega)$ are completely described by the following result (see [3, Proposition 9.20]).

Theorem 1.1. *Let $F \in W^{-1,q}(\Omega)$. Then there exist functions $f_0, f_1, \dots, f_N \in L^q(\Omega)$ such that*

$$\langle F, u\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega f_o u\,dx + \sum_{i=1}^N f_i\frac{\partial u}{\partial x_i}\,dx, \quad \forall\, u \in W_0^{1,p}(\Omega)$$

and

$$\|F\|_{W^{-1,q}(\Omega)} = \max_{1\le i\le N} \|f_i\|_{L^q(\Omega)}.$$

Hereinafter, $\langle\cdot,\cdot\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}$ stands for the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$.

It is worth to notice that for any vector field $\vec{v} \in L^q(\Omega)^N$, the divergence $\operatorname{div}\vec{v}$ can be defined as an element of the space $W^{-1,q}(\Omega)$ by the formula

$$\langle \operatorname{div}\vec{v}, \varphi\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = -\int_\Omega (\vec{v}, \nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \tag{1.4}$$

where $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the scalar product of two vectors in $\mathbb{R}^N$.

We recall the well-known notion of generalized solution to the Dirichlet boundary value problem (1.1)–(1.2) which is naturally arise from the so-called weak formulation of the problem (1.1)–(1.2). In our case the weak formulation of (1.1)–(1.2) looks as follows: for given $\sigma \in \Sigma(\Omega)$ and $f \in W^{-1,q}(\Omega)$, find $y \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \sigma|\nabla y|^{p-2} (\nabla y, \nabla\varphi)_{\mathbb{R}^N}\, dx = \langle f, \varphi\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}, \quad \forall\, \varphi \in W_0^{1,p}(\Omega). \tag{1.5}$$

Definition 1.1. We say that $y \in W_0^{1,p}(\Omega)$ is the generalized solution to the Dirichlet boundary value problem (1.1)–(1.2) if, for given $\sigma \in \Sigma(\Omega)$ and $f \in W^{-1,q}(\Omega)$, the distribution $y \in W_0^{1,p}(\Omega)$ is a solution to the variational problem (1.5).

For $\sigma \in \Sigma(\Omega)$, let $A_\sigma : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ be the operator given by

$$A_\sigma(y) = -\operatorname{div}\left(\sigma\, |\nabla y|^{p-2}\, \nabla y\right). \tag{1.6}$$

For the reader's convenience, we show in Section 2 that this operator is bounded, strictly monotone, semi-continuous, and coercive. Then by Minty-Browder Theorem (see [5, Theorem 9.14-1], [15, Theorem 2.14]), we deduce that A_σ is an injective operator, i.e. given any $f \in W^{-1,q}(\Omega)$, there exists a unique $u \in W_0^{1,p}(\Omega)$ such that

$$A_\sigma(u) = f.$$

Moreover, the inverse or resolvent operator $R_\sigma = A_\sigma^{-1} : W^{-1,q}(\Omega) \to W_0^{1,p}(\Omega)$ is strictly monotone, bounded, and it maps continuously $W^{-1,q}(\Omega)$ onto $W_0^{1,p}(\Omega)$. Thus, the Dirichlet boundary value problem (1.1)–(1.2), for given $\sigma \in \Sigma(\Omega)$ and $f \in W^{-1,q}(\Omega)$, admits a unique generalized solution $y = R_\sigma(f) \in W_0^{1,p}(\Omega)$ with the following a priori estimate

$$\|y\|_{W_0^{1,p}(\Omega)} = \|R_\sigma(f)\|_{W_0^{1,p}(\Omega)} \le \left(\sigma_0^{-1}\, \|f\|_{W_0^{1,p}(\Omega)}\right)^{q/p}. \tag{1.7}$$

Indeed, using the solution itself y as the test function in (1.5), we have

$$\sigma_0 \int_\Omega |\nabla y|^p\, dx \le \int_\Omega \sigma(x)|\nabla y|^p\, dx = \langle f, \varphi\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \le \|f\|_{W_0^{1,p}(\Omega)}\|y\|_{W_0^{1,p}(\Omega)}$$

and consequently

$$\sigma_0\|y\|^{p-1}_{W_0^{1,p}(\Omega)} \le \|f\|_{W_0^{1,p}(\Omega)}.$$

As we see, the coefficient σ of the equation with generalized p-Laplacian determines uniquely the resolvent R_σ. Our main goal in this paper is to analyze the inverse problem of identifying the diffusivity coefficient $\sigma(x)$ in the principle part of quasi-linear elliptic equation (1.2) as a function of resolvent operator. To do so, we derive some sensitivity estimates for the solutions of the Dirichlet boundary value problem (1.1)–(1.2). In particular, we prove that, within the class of admissible functions $\Sigma(\Omega)$, the resolvent determines the diffusivity coefficient in a unique manner. Furthermore, we prove that the inverse mapping from resolvent to the function σ is Hölder continuous in the suitable topologies.

Our main results ensure the Hölder character of continuity of the inverse map $R_\sigma = A_\sigma^{-1} : W^{-1,q}(\Omega) \to W_0^{1,p}(\Omega)$ and they can be stated as follows.

Theorem 1.2. *For any $\sigma_1, \sigma_2 \in \Sigma$, $f \in W^{-1,q}(\Omega)$, and $p \geq 2$, the following two-sides inequality*

$$2^{2-p}\sigma_0^2 \|f\|^{-1}_{W^{-1,q}(\Omega)} \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|^{p-1}_{W_0^{1,p}(\Omega)} \leq \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leq \sigma_1^* (p-1) \left(\max\left\{1; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)}\right\} \right)^{\frac{p-2}{p-1}} \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)}. \quad (1.8)$$

holds true.

Theorem 1.3. *For any $\sigma_1, \sigma_2 \in \Sigma$, $f \in W^{-1,q}(\Omega)$, and $p \in (1,2)$, the following two-sides inequality*

$$\sigma_0^2 \|f\|^{-1}_{W^{-1,q}(\Omega)} \left(\|R_{\sigma_1}(f)\|^p_{W_0^{1,p}(\Omega)} + \|R_{\sigma_2}(f)\|^p_{W_0^{1,p}(\Omega)} \right)^{\frac{p-2}{p}} \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)} \leq \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leqslant \sigma_1^* \left(\min\left\{1; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)}\right\} \right)^{\frac{p-2}{p-1}} \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)}. \quad (1.9)$$

holds true.

We prove these Theorems in Section 3 and Section 4, respectively.

The results under discussion are close to the questions of stability and sensitivity analysis for boundary value problems and optimization problems associated with them. Stability refers to the continuous behavior of solutions under small perturbations of the problem data, while sensitivity indicates a differentiable dependence (see, [11, 13]). It is worth to notice that, by analogy with a linear case [6], this result plays a key role when applying greedy algorithms to the approximation of parameter-dependent quasi-linear elliptic problems with generalized p-laplacian in an uniform and robust manner, independent of the given source terms (see, for instance, [2, 4, 7]).

2. Preliminaries and Some Auxiliary Results

Let σ be an arbitrary element of the set $\Sigma(\Omega)$. Then, in view of (1.6), the following representation holds true

$$\langle A_\sigma(y), v\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} = \int_\Omega \sigma |\nabla y|^{p-2} (\nabla y, \nabla v)_{\mathbb{R}^N}\, dx. \quad (2.1)$$

We begin this section with the following properties of operator A_σ.

Proposition 2.1. For every $\sigma \in \Sigma(\Omega)$ the operator $A_\sigma : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is bounded, i.e.

$$\|A_\sigma(u)\|_{W^{-1,q}(\Omega)} \leq C \|u\|_{W_0^{1,p}(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega),$$

where

$$\|A_\sigma\| \leq C = \|\sigma\|_{L^\infty(\Omega)}.$$

Proof. Indeed, by definition of the operator norm and the set $\Sigma(\Omega)$, we have

$$\begin{aligned}
\|A_\sigma\| &= \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \|A_\sigma(u)\|_{W^{-1,q}(\Omega)} = \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \sup_{\|u\|_{W_0^{1,p}(\Omega)}} \langle A_\sigma(u), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\
&= \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \sup_{\|u\|_{W_0^{1,p}(\Omega)}} \int_\Omega \sigma|\nabla u|^{p-2}\left(\nabla u, \nabla v\right)_{\mathbb{R}^N}\, dx \\
&\le \|\sigma\|_{L^\infty(\Omega)} \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \sup_{\|u\|_{W_0^{1,p}(\Omega)}} \int_\Omega |\nabla u|^{p-1}|\nabla v|\, dx \\
&\le \|\sigma\|_{L^\infty(\Omega)} \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \sup_{\|u\|_{W_0^{1,p}(\Omega)}} \left(\int_\Omega |\nabla u|^p\, dx\right)^{(p-1)/p} \left(\int_\Omega |\nabla v|^p\, dx\right)^{1/p} \\
&= \|\sigma\|_{L^\infty(\Omega)} \sup_{\|u\|_{W_0^{1,p}(\Omega)}\le 1} \sup_{\|u\|_{W_0^{1,p}(\Omega)}} \|u\|^{p-1}_{W_0^{1,p}(\Omega)}\|v\|_{W_0^{1,p}(\Omega)} = \|\sigma\|_{L^\infty(\Omega)}.
\end{aligned}$$

□

Proposition 2.2. For every $\sigma\in\Sigma(\Omega)$ the operator $A_\sigma : W_0^{1,p}(\Omega)\to W^{-1,q}(\Omega)$ is coercive, namely

$$\langle A_\sigma(u), u\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \ge \sigma_0\|u\|^p_{W_0^{1,p}(\Omega)}, \quad \forall\, u\in W_0^{1,p}(\Omega).$$

Proof. This inequality immediately follows from the definition of the set $\Sigma(\Omega)$ and the following equality

$$\langle A_\sigma(u), u\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u|^p\, dx.$$

□

Proposition 2.3. For every $\sigma\in\Sigma(\Omega)$ and $p\in(1,\infty)$ the operator $A_\sigma : W_0^{1,p}(\Omega)\to W^{-1,q}(\Omega)$ is strictly coercive, i.e.

$$\langle A_\sigma(u)-A_\sigma(v), u-v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \ge 0, \quad \forall\, u.v\in W_0^{1,p}(\Omega), \tag{2.2}$$

$$\langle A_\sigma(u)-A_\sigma(v), u-v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = 0 \implies u=v. \tag{2.3}$$

Proof. To begin with, we make use the following two well-know inequalities. For any $a,b\in\mathbb{R}$ and $p\ge 2$

$$\left(|a|^{p-2}\, a - |b|^{p-2}\, b\right)(a-b) \ge 2^{2-p}\left|a-b\right|^p; \tag{2.4}$$

For any $a,b\in\mathbb{R}$ and $1<p\le 2$

$$\left(|a|^{p-2}\, a - |b|^{p-2}\, b\right)(a-b) \ge \left(\,|a|+|b|\,\right)^{p-2}\left|a-b\right|^2. \tag{2.5}$$

Since for arbitrary functions $u,v\in W_0^{1,p}(\Omega)$, we have

$$\langle A_\sigma(u)-A_\sigma(v), u-v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega \sigma\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla u-\nabla v\right)_{\mathbb{R}^N}\, dx,$$

where
$$|\nabla u| = \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2} \quad \text{and} \quad (\nabla u, \nabla v)_{\mathbb{R}^N} = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i},$$
it follows from the Cauchy-Schwarz inequality that
$$\begin{aligned}\left\langle A_\sigma(u) - A_\sigma(v), u - v \right\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} &= \int_\Omega \sigma |\nabla u|^p \, dx + \int_\Omega \sigma |\nabla u|^p \, dx \\ &\quad - \int_\Omega \sigma |\nabla u|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^N} \, dx - \int_\Omega \sigma |\nabla v|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^N} \, dx \\ &\geq \int_\Omega \sigma |\nabla u|^{p-2} \left(|\nabla u|^2 - |\nabla u||\nabla v| \right) dx - \int_\Omega \sigma |\nabla v|^{p-2} \left(|\nabla v||\nabla u| - |\nabla v|^2 \right) dx \\ &= \int_\Omega \sigma \left(|\nabla u|^{p-2}|\nabla u| - |\nabla v|^{p-2}|\nabla v|, |\nabla u| - |\nabla v| \right) dx. \end{aligned} \tag{2.6}$$
Hence, in accordance with (2.4)–(2.5), we can deduce from (2.6) the following estimates

For $p \in (1, 2]$**:**
$$\begin{aligned}\left\langle A_\sigma(u) - A_\sigma(v), u - v \right\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} &\geq \int_\Omega \sigma \left(|\nabla u| + |\nabla v| \right)^{p-2} \left(|\nabla u| - |\nabla v| \right)^2 dx \\ \geq \sigma_0 \int_\Omega \left(|\nabla u| + |\nabla v| \right)^{p-2} \left(|\nabla u| - |\nabla v| \right)^2 dx &\overset{\text{by } (2.5)}{\geq} 0, \quad \forall\, u, v \in W_0^{1,p}(\Omega); \end{aligned} \tag{2.7}$$

For $p \in (2, \infty)$**:**
$$\begin{aligned}\left\langle A_\sigma(u) - A_\sigma(v), u - v \right\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} &\geq 2^{2-p} \int_\Omega \sigma \left| |\nabla u| - |\nabla v| \right|^p dx \\ \geq \sigma_0 \int_\Omega \left| |\nabla u| - |\nabla v| \right|^p dx &\overset{\text{by } (2.4)}{\geq} 0, \quad \forall\, u, v \in W_0^{1,p}(\Omega). \end{aligned} \tag{2.8}$$

As a result, the properties (2.2)–(2.3) are the direct consequence of inequalities (2.7)–(2.8). □

Proposition 2.4. For every $\sigma \in \Sigma(\Omega)$ and $p \in (1, \infty)$ the operator $A_\sigma : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is radially continuous, i.e. for any elements $u, v \in W_0^{1,p}(\Omega)$ the mapping $t \mapsto \langle A_\sigma(u + tv), v \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}$ is continuous.

Proof. In view of the representation (2.1), we see that
$$\langle A_\sigma(u + tv), v \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} = \int_\Omega \sigma |\nabla u + t\nabla v|^{p-2} \left(\nabla u + t\nabla v, \nabla v \right)_{\mathbb{R}^N} dx$$
Then
$$\begin{aligned}\Delta(t) &:= \langle A_\sigma(u + tv), v \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} - \langle A_\sigma(u), v \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\ &= t \int_\Omega \sigma |\nabla u + t\nabla v|^{p-2} |\nabla v|^2 \, dx + \int_\Omega \sigma \left(|\nabla u + t\nabla v|^{p-2} - |\nabla u|^{p-2} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} \, dx \\ &= I_1(t) + I_2(t), \end{aligned}$$

where the first term $I_1(t)$ can be estimated by the Hölder inequality as follows

$$\begin{aligned} I_1(t) &\le \|\sigma\|_{L^\infty(\Omega)} 2^{p-3} \left(t \int_\Omega |\nabla u|^{p-2} |\nabla v|^2 \, dx + t^{p-1} \int_\Omega |\nabla v|^p \, dx \right) \\ &\le \sigma_1^* 2^{p-3} \left(t \|u\|^{p-2}_{W_0^{1,p}(\Omega)} \|v\|^2_{W_0^{1,p}(\Omega)} + t^{p-1} \|v\|^p_{W_0^{1,p}(\Omega)} \right), \quad \forall\, u, v \in W_0^{1,p}(\Omega). \end{aligned} \tag{2.9}$$

Since $p \in (1, \infty)$, it follows from (2.9) that

$$\lim_{t \to 0} I_1(t) = 0. \tag{2.10}$$

As for the asymptotic behaviour of the second term $I_2(t)$ as $t \to 0$, we make use of the following well-known inequalities (see [12])

- if $p > 3$, then

$$\left| |a|^{p-2} - |b|^{p-2} \right| \le (p-2) \left(|a| + |b| \right)^{p-3} |a - b| \,, \quad \forall\, a, b \in \mathbb{R}; \tag{2.11}$$

- if $2 \le p \le 3$, then

$$\left| |a|^{p-2} - |b|^{p-2} \right| \le |a - b|^{p-2}, \quad \forall\, a, b \in \mathbb{R}. \tag{2.12}$$

It is clear from (2.12) that

- if $1 < p < 2$, then

$$\left| |a|^{p-2} - |b|^{p-2} \right| \le \frac{\left| |a|^{2-p} - |b|^{2-p} \right|}{|a|^{2-p} |b|^{2-p}} \le \frac{|a - b|^{2-p}}{|a|^{2-p} |b|^{2-p}}, \quad \forall\, a, b \in \mathbb{R}. \tag{2.13}$$

As a result, we have:

(i) if $p > 3$, then

$$\begin{aligned} I_2(t) &\le t\sigma_1^*(p-2) \int_\Omega \left(|\nabla u + t\nabla v| + |\nabla u| \right)^{p-3} |\nabla u| |\nabla v|^2 \, dx \\ &\le t\sigma_1^*(p-2) \int_\Omega \left(t|\nabla v| + 2|\nabla u| \right)^{p-3} |\nabla u| |\nabla v|^2 \, dx \\ &\le t\sigma_1^*(p-2) 2^{p-4} \int_\Omega \left(t^{p-3} |\nabla v|^{p-1} |\nabla u| + 2^{p-3} |\nabla u|^{p-2} |\nabla v|^2 \right) dx \\ &\le tC \left(\|v\|^{p-1}_{W_0^{1,p}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)} + \|u\|^{p-2}_{W_0^{1,p}(\Omega)} \|v\|^2_{W_0^{1,p}(\Omega)} \right); \end{aligned} \tag{2.14}$$

(ii) if $2 \le p \le 3$, then

$$I_2(t) \le t^{p-2} \sigma_1^* \int_\Omega |\nabla v|^{p-2} |\nabla u| |\nabla v| \, dx \le t^{p-2} \sigma_1^* \|v\|^{p-1}_{W_0^{1,p}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)}; \tag{2.15}$$

(iii) if $1 < p < 2$, then

$$
\begin{aligned}
I_2(t) &\le t^{2-p}\sigma_1^* \int_\Omega \left| \frac{1}{|\nabla u + t\nabla v|^{2-p}} - \frac{1}{|\nabla u|^{2-p}} \right| |\nabla u||\nabla v|\,dx \\
&\overset{\text{by (2.13)}}{\le} t^{2-p}\sigma_1^* \int_\Omega \frac{|\nabla v|^{2-p}}{|\nabla v|^{2-p}|\nabla u + t\nabla v|^{2-p}} |\nabla u||\nabla v|\,dx \\
&\le t^{2-p}\sigma_1^* \int_\Omega |\nabla u + t\nabla v|^{p-2}|\nabla u||\nabla v|\,dx \le \quad \text{(for t small enough)} \\
&\le \left(\frac{t}{1-t}\right)^{2-p} \sigma_1^* \int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx \\
&\le \left(\frac{t}{1-t}\right)^{2-p} \sigma_1^* \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)}.
\end{aligned} \tag{2.16}
$$

□

Our next intension is to establish a couple of auxiliary inequalities that will be applicable in our further analysis.

Proposition 2.5. Let $p \ge 2$ be a given exponent. Then the following inequality

$$
\Big| |a|^{p-2}\,a - |b|^{p-2}\,b \Big| \le (p-1)\max\{|a|, |b|\}^{p-2} \Big| a - b \Big| \tag{2.17}
$$

holds true for any $a, b \in \mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ be arbitrary values. We set $I = \Big| |a|^{p-2}\,a - |b|^{p-2}\,b \Big|$. Then

$$
\begin{aligned}
I &= \Big| |a|^{p-1} \operatorname{sign} a - |b|^{p-1} \operatorname{sign} b \Big| \\
&\le \max\{|a|, |b|\}^{p-1} \left| 1 - \operatorname{sign} a \cdot \operatorname{sign} b \cdot \left(\frac{\min\{|a|, |b|\}}{\max\{|a|, |b|\}} \right)^{p-1} \right|.
\end{aligned}
$$

Since $1 \pm x^\alpha \le \alpha(1 \pm x)$ for all $\alpha \in [1; \infty)$ and for $0 \le x \le 1$, it follows that

$$
\begin{aligned}
I &\le (p-1)\max\{|a|, |b|\}^{p-1} \left| 1 - \operatorname{sign} a \cdot \operatorname{sign} b \cdot \frac{\min\{|a|, |b|\}}{\max\{|a|, |b|\}} \right| \\
&= (p-1)\max\{|a|, |b|\}^{p-2} \Big| \max\{|a|, |b|\} - \operatorname{sign} a \cdot \operatorname{sign} b \cdot \min\{|a|, |b|\} \Big| \\
&\le (p-1)\max\{|a|, |b|\}^{p-2} \left| |a| - \frac{\operatorname{sign} b}{\operatorname{sign} a} |b| \right| \\
&= (p-1)\max\{|a|, |b|\}^{p-2} \Big| |a| \cdot \operatorname{sign} a - |b| \cdot \operatorname{sign} b \Big| \\
&= (p-1)\max\{|a|, |b|\}^{p-2} \Big| a - b \Big|.
\end{aligned}
$$

□

Proposition 2.6. Let $1 < p \leq 2$. Then the estimate

$$\Big| |a|^{p-2} a - |b|^{p-2} b \Big| \leq \min\{|a|, |b|\}^{p-2} \Big| a - b \Big| \tag{2.18}$$

holds true for any $a, b \in \mathbb{R}$.

Proof. Let a, b be arbitrary elements of $\mathbb{R}$. Then, in the framework of notations using in the proof of Proposition 2.5, we can estimate the expression I as follows

$$\begin{aligned} I &= \Big| |a|^{p-1} \operatorname{sign} a - |b|^{p-1} \operatorname{sign} b \Big| \\ &\leq \min\{|a|, |b|\}^{p-1} \left| \left(\frac{\max\{|a|, |b|\}}{\min\{|a|, |b|\}} \right)^{p-1} - \operatorname{sign} a \cdot \operatorname{sign} b \right|. \end{aligned}$$

To conclude the proof, it remains to take into account the fact that

$$x^\alpha \leq x \quad \forall\, 0 < \alpha \leqslant 1 \quad \text{and} \quad \forall x \geq 1,$$

and the following chain of inequalities

$$\begin{aligned} I &\leq \min\{|a|, |b|\}^{p-1} \left| \frac{\max\{|a|, |b|\}}{\min\{|a|, |b|\}} - \operatorname{sign} a \cdot \operatorname{sign} b \right| \\ &= \min\{|a|, |b|\}^{p-2} \Big| \max\{|a|, |b|\} - \operatorname{sign} a \cdot \operatorname{sign} b \cdot \min\{|a|, |b|\} \Big| \\ &\leq \min\{|a|, |b|\}^{p-2} \left| |a| - \frac{\operatorname{sign} b}{\operatorname{sign} a} |b| \right| = \min\{|a|, |b|\}^{p-2} \Big| |a| \cdot \operatorname{sign} a - |b| \cdot \operatorname{sign} b \Big| \\ &= \min\{|a|, |b|\}^{p-2} \Big| a - b \Big|. \end{aligned}$$

□

The following lemma plays a crucial role in substantiation of the main results of our paper.

Lemma 2.1. *Let $\gamma \in L^\infty(\Omega)$ be such that*

$$\pm \int_\Omega \gamma |\nabla u|^p \, dx \leq C \quad \text{for any } u \in W_0^{1,p}(\Omega) \text{ satisfying } \|u\|_{W_0^{1,p}(\Omega)} = 1 \tag{2.19}$$

for some constant $C > 0$. Then

$$\|\gamma\|_{L^\infty(\Omega)} \leq C. \tag{2.20}$$

Proof. Following in many aspects the line of proof of Lemma 2.2 in [6], we define the function $\varphi_\varepsilon(r)$ by the rule

$$\varphi_\varepsilon(r) = \begin{cases} 0 & \text{when } r \leq 0, \\ r & \text{when } 0 < r < \varepsilon, \\ \varepsilon & \text{when } \varepsilon \leq r, \end{cases}$$

where $\varepsilon > 0$ is a given value.

As follows from this definition, $\varphi_\varepsilon(r)$ is a continuous function and such that

$$\frac{d\varphi_\varepsilon(r)}{dr} = \chi_{(0;\varepsilon)}(r),$$

where $\chi_{(0;\varepsilon)}(r)$ is the characteristic function of the interval $(0;\varepsilon)$.

Let $x_0 \in \Omega$ and ε_0 be sufficiently small in such way that $B(x_0,\varepsilon_0) \subset \Omega$. For $\varepsilon \in (0;\varepsilon_0)$ we define $u_{x_0,\varepsilon}$ as follows

$$u_{x_0,\varepsilon}(x) = (|B(x_0,\varepsilon_0)|)^{-\frac{1}{p}} \varphi_\varepsilon(|x-x_0|). \tag{2.21}$$

Then we have

$$\nabla u_{x_0,\varepsilon}(x) = (|B(x_0,\varepsilon_0)|)^{-\frac{1}{p}} \chi_{B(x_0,\varepsilon)}(x) \frac{x-x_0}{|x-x_0|}.$$

So,

$$\begin{aligned} \|u_{x_0,\varepsilon}\|^p_{W_0^{1,p}(\Omega)} &= \|\nabla u_{x_0,\varepsilon}\|^p_{L^p(\Omega)^N} = \int_\Omega |\nabla u_{x_0,\varepsilon}|^p\, dx \\ &= \int_\Omega \left(|B(x_0,\varepsilon)|^{-\frac{1}{p}}\right)^p \left|\chi_{B(x_0,\varepsilon)}(x)\right|^p dx \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} dx = 1. \end{aligned}$$

Therefore, $\|u_{x_0,\varepsilon}\|^p_{W_0^{1,p}(\Omega)} = 1$ and by Lebesgue's differential theorem

$$\int_\Omega \gamma(x)\, |\nabla u_{x_0,\varepsilon}|^p\, dx = \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} \gamma(x)\, dx \xrightarrow[\varepsilon\to 0]{} \gamma(x_0)$$

for almost each $x_0 \in \Omega$.

Taking this result into account, we can define a sequence $\{u_n^\pm\}$ in $W_0^{1,p}(\Omega)$ such that $\|u_n^\pm\|_{W_0^{1,p}(\Omega)} = 1$ for each $n \in \mathbb{N}$, and

$$\lim_n \int_\Omega \pm\gamma(x) \left|\nabla u_n^\pm\right|^p dx = \pm\gamma(x_0).$$

Since

$$\int_\Omega \pm\gamma \left|\nabla u_n^\pm\right|^p dx \le C$$

by (2.19), it follows that

$$\pm\gamma(x_0) \le C \quad \text{for almost each} \quad x_0 \in \Omega.$$

Hence, $|\gamma(x_0)| \le C$ for almost each $x_0 \in \Omega$, and this implies the desired inclusion (2.20). □

3. Proof of the Theorem 1.2

We divide the proof onto two steps and give it in the form of two lemmas — Lemma 3.1 and Lemma 3.2. We begin with the first inequality in (1.8), namely,

$$2^{2-p}\sigma_0^2 \|f\|^{-1}_{W^{-1,q}(\Omega)} \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|^{p-1}_{W_0^{1,p}(\Omega)} \le \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)}. \tag{3.1}$$

Its validity is a direct consequence of the following lemma.

Lemma 3.1. *Let $f \in W^{-1,q}(\Omega)$ be a given distribution. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be two arbitrary coefficients of diffusivity and let $u_1 = R_{\sigma_1}(f)$ and $u_2 = R_{\sigma_2}(f)$ be the corresponding generalized solutions to the Dirichlet problem (1.1)–(1.2). Then the inequality*

$$\|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \le 2^{\frac{p-2}{p-1}} \; \sigma_0^{\frac{2}{1-p}} \; \|f\|_{W^{-1,q}(\Omega)}^{\frac{q}{p}} \; \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)}^{\frac{q}{p}} \tag{3.2}$$

holds true for any $p \ge 2$.

Proof. Indeed, for any $v \in W_0^{1,p}(\Omega)$, we have

$$\langle A_{\sigma_1}(u_1), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \langle A_{\sigma_2}(u_2), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}$$

or

$$\int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2}\nabla u_1, \nabla v\right) dx = \int_\Omega \sigma_2 \left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx. \tag{3.3}$$

In view of the obvious relations

$$\begin{aligned}
&\int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx \\
&= \int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2}\nabla u_1, \nabla v\right) dx - \int_\Omega \sigma_1 \left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx \\
&\overset{\text{by (3.3)}}{=} \int_\Omega \sigma_2 \left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx - \int_\Omega \sigma_1 \left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx \\
&= \int_\Omega (\sigma_2 - \sigma_1)\left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx,
\end{aligned}$$

we have

$$\int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx = \int_\Omega (\sigma_2 - \sigma_1)\left(|\nabla u_2|^{p-2}\nabla u_2, \nabla v\right) dx. \tag{3.4}$$

Setting $v = u_1 - u_2$ in the identity (3.4), we see that

$$\int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\right) dx \overset{\text{by (2.4)}}{\ge} 2^{2-p}\,\sigma_0 \,\|u_1 - u_2\|^p_{W_0^{1,p}(\Omega)}.$$

On the other hand we can provide the following estimation

$$\begin{aligned}
&\int_\Omega (\sigma_2 - \sigma_1)\left(|\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1 - u_2)\right) dx \\
&\le \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \int_\Omega \left|\left(|\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1 - u_2)\right)\right| dx \\
&\le \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \left(\int_\Omega \left||\nabla u_2|^{p-2}\nabla u_2\right|^q dx\right)^{\frac{1}{q}} \left(\int_\Omega |\nabla(u_1 - u_2)|^p\, dx\right)^{\frac{1}{p}} \\
&= \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla u_2|^p\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla(u_1 - u_2)|^p\, dx\right)^{\frac{1}{p}} \\
&= \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \|u_2\|^{p-1}_{W_0^{1,p}(\Omega)} \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \\
&\overset{by (1.7)}{\le} \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)}\; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)} \|u_1 - u_2\|_{W_0^{1,p}(\Omega)}.
\end{aligned}$$

Hence, in view of the above chain of inequalities, we deduce

$$2^{2-p}\ \sigma_0 \left\| u_1 - u_2 \right\|^p_{W_0^{1,p}(\Omega)} \le \left\| \sigma_2 - \sigma_1 \right\|_{L^\infty(\Omega)}\ \sigma_0^{-1} \left\| f \right\|_{W^{-1,q}(\Omega)} \left\| u_1 - u_2 \right\|_{W_0^{1,p}(\Omega)}.$$

As a result, we arrive at the expected estimate (3.2) that can be expressed in other notations as follows

$$\left\| R_{\sigma_1}(f) - R_{\sigma_2}(f) \right\|^{p-1}_{W_0^{1,p}(\Omega)} \le 2^{p-2}\ \sigma_0^{-2} \left\| f \right\|_{W^{-1,q}(\Omega)} \left\| \sigma_2 - \sigma_1 \right\|_{L^\infty(\Omega)}. \tag{3.5}$$

□

In order to prove the second inequality in (1.8), we make use of the following auxiliary result.

Proposition 3.1. Let u, v be given elements of $W_0^{1,p}(\Omega)$. Let $\sigma \in \Sigma$ be a fixed diffusivity coefficient. Then the relation

$$\begin{aligned} \left\| A_\sigma(u) - A_\sigma(v) \right\|_{W^{-1,q}(\Omega)} &\le (p-1) \left\| \sigma \right\|_{L^\infty(\Omega)} \\ &\times \left(\max \left\{ \left\| u \right\|_{W_0^{1,p}(\Omega)}, \left\| v \right\|_{W_0^{1,p}(\Omega)} \right\} \right)^{p-2} \left\| u - v \right\|_{W_0^{1,p}(\Omega)}. \end{aligned} \tag{3.6}$$

is valid for any $p \ge 2$.

Proof. We note that the following estimate

$$\begin{aligned} &\left\langle A_\sigma(u) - A_\sigma(v), \varphi \right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega \sigma \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \varphi \right) dx \\ &\le \left\| \sigma \right\|_{L^\infty(\Omega)} \left(\int_\Omega \left| |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right|^q dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}} \\ &= \left\| \sigma \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega \left| |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\overset{by (2.17)}{\le} \left\| \sigma \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega \left((p-1) \left(\max \left\{ |\nabla u|, |\nabla v| \right\} \right)^{p-2} |\nabla u - \nabla v| \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &= \left\| \sigma \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{W_0^{1,p}(\Omega)} (p-1) \left(\int_\Omega \left(\max \left\{ |\nabla u|, |\nabla v| \right\} \right)^{\frac{p(p-2)}{p-1}} |\nabla (u - v)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\le \left\| \sigma \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{W_0^{1,p}(\Omega)} (p-1) \left(\int_\Omega \left(\max \left\{ |\nabla u|, |\nabla v| \right\} \right)^p dx \right)^{\frac{p-2}{p}} \left(\int_\Omega |\nabla (u - v)|^p \, dx \right)^{\frac{1}{p}} \\ &= \left\| \sigma \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{W_0^{1,p}(\Omega)} (p-1) \left(\max \left\{ \left\| u \right\|_{W_0^{1,p}(\Omega)}, \left\| v \right\|_{W_0^{1,p}(\Omega)} \right\} \right)^{p-2} \left\| u - v \right\|_{W_0^{1,p}(\Omega)} \end{aligned}$$

holds true for any $\varphi \in W_0^{1,p}(\Omega)$. Then to deduce (3.6), it remains to notice that

$$\left\| A_\sigma(u) - A_\sigma(v) \right\|_{W^{-1,q}(\Omega)} = \sup_{\left\| \varphi \right\|_{W_0^{1,p}(\Omega)} \le 1} \left| \left\langle A_\sigma(u) - A_\sigma(v), \varphi \right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \right|.$$

□

Before proceeding further, we utilize the following representation

$$A_{\sigma_1}(u) - A_{\sigma_2}(u) = A_{\sigma_2}\left(R_{\sigma_2}\left(A_{\sigma_1}(u)\right)\right) - A_{\sigma_2}\left(R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\right). \tag{3.7}$$

which is valid for each $u \in W_0^{1,p}(\Omega)$ and arbitrary $\sigma_1, \sigma_2 \in \Sigma(\Omega)$.

Indeed, let $u, v \in W_0^{1,p}(\Omega)$ be such that, for given $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ and $f \in W^{-1,q}(\Omega)$,

$$A_{\sigma_1}(u) = f \quad \text{and} \quad A_{\sigma_2}(u) = f.$$

Then, $R_{\sigma_1}(f) = u$, $R_{\sigma_2}(f) = v$ and, as a result, we have

$$A_{\sigma_2}\left(R_{\sigma_2}\left(A_{\sigma_1}(u)\right)\right) = A_{\sigma_2}\left(R_{\sigma_2}(f)\right) = A_{\sigma_2}(v) = f = A_{\sigma_1}(u);$$

$$A_{\sigma_2}\left(R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\right) = A_{\sigma_2}\left(R_{\sigma_1}(f)\right) = A_{\sigma_2}(u).$$

We are now in a position to show that the second inequality in (1.8) is valid.

Lemma 3.2. *Let $f \in W^{-1,q}(\Omega)$ be a given distribution. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be two arbitrary coefficients of diffusivity and let $u_1 = R_{\sigma_1}(f)$ and $u_2 = R_{\sigma_2}(f)$ be the corresponding generalized solutions to the Dirichlet problem (1.1)–(1.2). Then the following estimate*

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \le \sigma_1^*(p-1)\left(\max\left\{1; \sigma_0^{-1}\|f\|_{W^{-1,q}(\Omega)}\right\}\right)^{\frac{p-2}{p-1}} \\ \times \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)}\end{aligned} \tag{3.8}$$

holds true for any $p \ge 2$.

Proof. To begin with, we make use of the following representation

$$\langle A_{\sigma_1}(u) - A_{\sigma_2}(u), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega (\sigma_1 - \sigma_2)\,|\nabla u|^{p-2}(\nabla u, \nabla v)\, dx. \tag{3.9}$$

Setting

$$I_1(u,v) = \langle A_{\sigma_1}(u) - A_{\sigma_2}(u), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)},$$
$$I_2(u,v) = \int_\Omega (\sigma_1 - \sigma_2)\,|\nabla u|^{p-2}(\nabla u, \nabla v)\, dx,$$

we see that

$$\begin{aligned} I_1(u,u) &\overset{\text{by (3.7)}}{=} \langle A_{\sigma_2}\left(R_{\sigma_2}\left(A_{\sigma_1}(u)\right)\right) - A_{\sigma_2}\left(R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\right), u\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\ &\le \|A_{\sigma_2}\left(R_{\sigma_2}\left(A_{\sigma_1}(u)\right)\right) - A_{\sigma_2}\left(R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\right)\|_{W^{-1,q}(\Omega)}\|u\|_{W_0^{1,p}(\Omega)} \\ &\overset{\text{by(3.6)}}{\le} (p-1)\|\sigma_2\|_{L^\infty(\Omega)}\|R_{\sigma_2}\left(A_{\sigma_1}(u)\right) - R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\|_{W_0^{1,p}(\Omega)}\|u\|_{W_0^{1,p}(\Omega)} \\ &\times\left(\max\left\{\|R_{\sigma_2}\left(A_{\sigma_1}(u)\right)\|_{W_0^{1,p}(\Omega)}, \|R_{\sigma_1}\left(A_{\sigma_1}(u)\right)\|_{W_0^{1,p}(\Omega)}\right\}\right)^{p-2}. \end{aligned} \tag{3.10}$$

Since $A_{\sigma_1}(u) = f$, $A_{\sigma_2}(u) = f$, $R_{\sigma_1}(A_{\sigma_1}(u)) = u$, and

$$\|R_{\sigma_2}(A_{\sigma_1}(u))\|_{W_0^{1,p}(\Omega)} \overset{\text{by (1.7)}}{\le} \left(\sigma_0^{-1}\|A_{\sigma_1}(u)\|_{W^{-1,q}(\Omega)}\right)^{\frac{q}{p}} = \left(\sigma_0^{-1}\|f\|_{W^{-1,q}(\Omega)}\right)^{\frac{q}{p}},$$

it follows that the estimate (3.10) can be reduced to the following one

$$\begin{aligned} I_1(u,u) \le (p-1)\,\|\sigma_2\|_{L^\infty(\Omega)}\,\|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)} \\ \times \left(\max\left\{\left(\sigma_0^{-1}\|f\|_{W^{-1,q}(\Omega)}\right)^{\frac{q}{p}}, \|u\|_{W_0^{1,p}(\Omega)}\right\}\right)^{p-2} \|u\|_{W_0^{1,p}(\Omega)}. \end{aligned} \tag{3.11}$$

Then, in view of relation (3.9), we can obtain the estimate for the term $I_2(u,u)$. Indeed, in this case we have

$$\begin{aligned} I_2(u,u) = \int_\Omega (\sigma_1 - \sigma_2)\,|\nabla u|^p\,dx \le \sigma_1^*(p-1)\left(\max\left\{1; \sigma_0^{-1}\,\|f\|_{W^{-1,q}(\Omega)}\right\}\right)^{\frac{p-2}{p-1}} \\ \times \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)} \end{aligned} \tag{3.12}$$

provided $p \ge 2$ and $\|u\|_{W_0^{1,p}(\Omega)} = 1$.

As a result, the estimate (3.8) is a direct consequence of Lemma 2.1 and inequality (3.12). □

4. Proof of the Theorem 1.3

The first inequality of the Theorem 1.3 is contained in the following lemma.

Lemma 4.1. *Let $f \in W^{-1,q}(\Omega)$ be a given distribution. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be two arbitrary coefficients of diffusivity and let $u_1 = R_{\sigma_1}(f)$ and $u_2 = R_{\sigma_2}(f)$ be the corresponding generalized solutions to the Dirichlet problem (1.1)–(1.2). Then the inequality*

$$\begin{aligned} &\|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \\ &\le \sigma_0^{-2}\,\|f\|_{W^{-1,q}(\Omega)}\,\|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)}\left(\|\nabla u_1\|^p_{W_0^{1,p}(\Omega)} + \|\nabla u_2\|^p_{W_0^{1,p}(\Omega)}\right)^{\frac{2-p}{p}}. \end{aligned} \tag{4.1}$$

holds true for any $p \in (1;2)$.

Proof. We are aware about the following inequality

$$\left(|\nabla u_1|^{p-2}\,\nabla u_1 - |\nabla u_2|^{p-2}\,\nabla u_2, \nabla u_1 - \nabla u_2\right)_{\mathbb{R}^N} \ge \frac{|\nabla u_1 - \nabla u_2|^2_{\mathbb{R}^N}}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} \tag{4.2}$$

which is valid for any $p \in [1;2]$.

Now we observe that

$$\begin{aligned} \|u_1 - u_2\|^p_{W_0^{1,p}(\Omega)} &= \int\limits_\Omega |\nabla(u_1 - u_2)|^p\,dx \\ &= \int_\Omega |\nabla(u_1 - u_2)|^p \left(\frac{1}{|\nabla u_1| + |\nabla u_2|}\right)^{\frac{p(2-p)}{2}} \left(|\nabla u_1| + |\nabla u_2|\right)^{\frac{p(2-p)}{2}} dx. \end{aligned}$$

Then, by Hölder inequality with $p' = \dfrac{2}{p}$ and $q' = \dfrac{2}{2-p}$, we have

$$
\begin{aligned}
&\int_\Omega |\nabla (u_1 - u_2)|^p \, dx \\
&\leq \left(\int\limits_\Omega \left(\frac{|\nabla (u_1 - u_2)|^p}{(|\nabla u_1| + |\nabla u_2|)^{\frac{p(2-p)}{2}}} \right)^{p'} dx \right)^{\frac{1}{p'}} \left(\int\limits_\Omega (|\nabla u_1| + |\nabla u_2|)^{\frac{p(2-p)}{2} \cdot q'} \, dx \right)^{\frac{1}{q'}} \\
&= \left(\int_\Omega \frac{|\nabla (u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_1| + |\nabla u_2|)^p \, dx \right)^{\frac{2-p}{2}} . \qquad (4.3)
\end{aligned}
$$

Hence, in view of (4.2), we get

$$
\begin{aligned}
&\left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \right)_{\mathbb{R}^N} \geq \frac{|\nabla u_1 - \nabla u_2|^2_{\mathbb{R}^N}}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} \\
&\overset{by (4.3)}{\geq} \left(\int_\Omega |\nabla (u_1 - u_2)|^p \, dx \right)^{\frac{2}{p}} \left(\int_\Omega (|\nabla u_1| + |\nabla u_2|)^p \, dx \right)^{\frac{p-2}{p}} \\
&= \|u_1 - u_2\|^2_{W_0^{1,p}(\Omega)} \left(\int\limits_\Omega (|\nabla u_1| + |\nabla u_2|)^p \, dx \right)^{\frac{p-2}{p}} \\
&\geq \|u_1 - u_2\|^2_{W_0^{1,p}(\Omega)} \left(\int\limits_\Omega |\nabla u_1|^p \, dx + \int_\Omega |\nabla u_2|^p \, dx \right)^{\frac{p-2}{p}} \\
&= \|u_1 - u_2\|^2_{W_0^{1,p}(\Omega)} \left(\|\nabla u_1\|^p_{W_0^{1,p}(\Omega)} + \|\nabla u_2\|^p_{W_0^{1,p}(\Omega)} \right)^{\frac{p-2}{p}} .
\end{aligned}
$$

Taking into account the above results, we arrive at the following inequality

$$
\begin{aligned}
&\int_\Omega \sigma \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla (u_1 - u_2) \right) \, dx \\
&\qquad \geq \sigma_0 \|u_1 - u_2\|^2_{W_0^{1,p}(\Omega)} \left(\|\nabla u_1\|^p_{W_0^{1,p}(\Omega)} + \|\nabla u_2\|^p_{W_0^{1,p}(\Omega)} \right)^{\frac{p-2}{p}} .
\end{aligned}
$$

As follows from the proof of Lemma 3.1, we have the estimate

$$
\begin{aligned}
&\int_\Omega \sigma_1 \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla (u_1 - u_2) \right) \, dx \\
&\qquad = \int_\Omega (\sigma_2 - \sigma_1) \left(|\nabla u_2|^{p-2} \nabla u_2, \nabla (u_1 - u_2) \right) dx \\
&\qquad \leqslant \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)} \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} .
\end{aligned}
$$

Combining the last two inequalities, we can deduce that

$$\begin{aligned}&\sigma_0 \|u_1 - u_2\|^2_{W_0^{1,p}(\Omega)} \left(\|\nabla u_1\|^p_{W_0^{1,p}(\Omega)} + \|\nabla u_2\|^p_{W_0^{1,p}(\Omega)} \right)^{\frac{p-2}{p}} \\ &\le \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \ \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)} \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} .\end{aligned}$$

As a result, the last inequality implies the expected estimate (4.1) that can be represented as follows

$$\begin{aligned}&\|R_{\sigma_1}(f) - R_{\sigma_2}(f)\|_{W_0^{1,p}(\Omega)} \\ &\le \|\sigma_2 - \sigma_1\|_{L^\infty(\Omega)} \ \sigma_0^{-2} \|f\|_{W^{-1,q}(\Omega)} \left(\|R_{\sigma_1}(f)\|^p_{W_0^{1,p}(\Omega)} + \|R_{\sigma_2}(f)\|^p_{W_0^{1,p}(\Omega)} \right)^{\frac{2-p}{p}} . \qquad (4.4)\end{aligned}$$

□

In order to complete the proof of Theorem 1.3, we make use of the following auxiliary proposition.

Proposition 4.1. Let u, v be given elements of $W_0^{1,p}(\Omega)$. Let $\sigma \in \Sigma$ be a fixed diffusivity coefficient. Then the relation

$$\begin{aligned}&\|A_\sigma(u) - A_\sigma(v)\|_{W^{-1,q}(\Omega)} \\ &\le \|\sigma\|_{L^\infty(\Omega)} \left(\min\left\{ \|u\|_{W_0^{1,p}(\Omega)}, \|v\|_{W_0^{1,p}(\Omega)} \right\} \right)^{p-2} \|u - v\|_{W_0^{1,p}(\Omega)} \qquad (4.5)\end{aligned}$$

is valid for any $p \in (1; 2)$.

Proof. Indeed, following in a similar manner to the proof of the Proposition 3.1, we have

$$\begin{aligned}&\langle A_\sigma(u) - A_\sigma(v), \varphi \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\ &\le \|\sigma\|_{L^\infty(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega \left| |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\overset{by (2.18)}{\le} \|\sigma\|_{L^\infty(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega \left((\min\{|\nabla u|, |\nabla v|\})^{p-2} |\nabla u - \nabla v| \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &= \|\sigma\|_{L^\infty(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega (\min\{|\nabla u|, |\nabla v|\})^{\frac{p(p-2)}{p-1}} |\nabla(u - v)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\le \|\sigma\|_{L^\infty(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \left(\int_\Omega (\min\{|\nabla u|, |\nabla v|\})^p \, dx \right)^{\frac{p-2}{p}} \left(\int_\Omega |\nabla(u - v)|^p \, dx \right)^{\frac{1}{p}} \\ &= \|\sigma\|_{L^\infty(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \left(\min\left\{ \|u\|_{W_0^{1,p}(\Omega)}, \|v\|_{W_0^{1,p}(\Omega)} \right\} \right)^{p-2} \|u - v\|_{W_0^{1,p}(\Omega)}\end{aligned}$$

for any $\varphi \in W_0^{1,p}(\Omega)$. □

We are now in position to derive the second part of inequality (1.9).

Lemma 4.2. *Let $f \in W^{-1,q}(\Omega)$ be a given distribution. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be two arbitrary coefficients of diffusivity and let $u_1 = R_{\sigma_1}(f)$ and $u_2 = R_{\sigma_2}(f)$ be the corresponding generalized solutions to the Dirichlet problem (1.1)–(1.2). Then the following estimate*

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \le \sigma_1^* \left(\min\left\{1; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)}\right\}\right)^{\frac{p-2}{p-1}} \times \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)} \tag{4.6}$$

holds true for any $p \in (1;2)$.

Proof. By analogy with the proof of the Lemma 3.2, we introduce the notation

$$I_1(u,v) = \langle A_{\sigma_1}(u) - A_{\sigma_2}(u), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)},$$
$$I_2(u,v) = \int_\Omega (\sigma_1 - \sigma_2) |\nabla u|^{p-2} (\nabla u, \nabla v)\, dx.$$

It is worth to note that

$$I_1(u,v) = I_2(u,v), \quad \forall v \in W_0^{1,p}(\Omega).$$

Then we have

$$I_1(u,u) \overset{\text{by (4.4)}}{\le} \|\sigma_2\|_{L^\infty(\Omega)} \left(\min\left\{\|R_{\sigma_2}(A_{\sigma_1}(u))\|_{W_0^{1,p}(\Omega)}, \|R_{\sigma_1}(A_{\sigma_1}(u))\|_{W_0^{1,p}(\Omega)}\right\}\right)^{p-2} \times \|R_{\sigma_2}(A_{\sigma_1}(u)) - R_{\sigma_1}(A_{\sigma_1}(u))\|_{W_0^{1,p}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)}$$

and

$$I_2(u,u) = \int_\Omega (\sigma_1 - \sigma_2) |\nabla u|^p\, dx \le \sigma_1^* \left(\min\left\{1; \sigma_0^{-1} \|f\|_{W^{-1,q}(\Omega)}\right\}\right)^{\frac{p-2}{p-1}} \times \|R_{\sigma_2}(f) - R_{\sigma_1}(f)\|_{W_0^{1,p}(\Omega)} \tag{4.7}$$

As a result, Lemma 2.1 and inequality (4.7) imply the estimate (4.6). □

References

1. R.A. Adams, *Sobolev Spaces*, *Academic Press*, New York, 1975.
2. P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk, *Convergence rates for greedy algorithms in reduced basis methods*, *SIAM J. Math. Anal.*, **43** (3) (2011), 1457–1472.
3. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
4. A. Buffa, Y. Maday, A. T. Patera, C. Prudhomme, G. Turinici, *A Priori convergence of the greedy algorithm for the parameterized reduced basis*, *Math. Model. Numer. Anal.*, **46** (2012), 595–603.

5. P.G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, SIAM, Philadelfia, 2013.
6. M. Choulli, E. Zuazua, *Lipschitz dependence of the coefficients on the resolvent and greedy approximation for scalar elliptic problems*, Comptes Rendus Mathématique, Elsevier Masson, **354** (2016), 1174–1187.
7. R. DeVore, G. Petrova, P. Wojtaszczyk, *Greedy algorithms for reduced bases in Banach spaces*, Constr. Approx., **37** (2013), 455–466.
8. P. Drabek, Y. Milota, *Methods of Nonlinear Analysis. Applications to Differential Equations*, Birkhäuser, Berlin, 2007.
9. P. Drabek, A. Kufner, F. Nicolosi, *Nonlinear Elliptic Equations, Singular and Degenerate Cases*, University of West Bohemia, 1996.
10. L.C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
11. R. Griesse, *Stability and Sensitivity Analysis in Optimal Control of Partial Differential Equations*, Karl-Franzens University Graz, Habilitation Thesis, 2007.
12. P.I. Kogut, O.P. Kupenko, G. Leugering, *Optimal control problems in coefficients for nonlinear Dirichlet problems of monotone type: Optimality conditions. Part I*, Journal of Analysis and its Applications (ZAA), **34**, Issue 1, 2015, 85–108.
13. P.I. Kogut, G. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains: Approximation and Asymptotic Analysis*, Birkhäuser, Boston, 2011.
14. S. Salsa, *Partial Differential Equations in Action. From Modelling to Theory*, Springer, Milan, 2008.
15. T. Roubiček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Boston, 2005.

Надійшла до редколегії 12.11.2017