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ON EXISTENCE OF WEAK SOLUTIONS TO A CAUCHY PROBLEM FOR ONE CLASS OF CONSERVATION LAWS

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We discuss the existence of weak solutions to the Cauchy problem for one class of hyperbolic conservation laws that models a highly re-entrant production system. The output of the factory is described as a function of the work in progress and the position of the so-called push-pull point (PPP) where we separate the beginning of the factory employing a push policy from the end of the factory, which uses a pull policy. The main question we discuss in this paper is about the optimal choice of the input in-flux, push and pull constituents, and the position of PPP.

Key words: conservation laws, clearing function, existence result, re-entrant systems

1. Introduction

The aim of this article is to analyze the existence of weak solutions for a highly re-entrant production system which is described by a scalar nonlinear conservation law. Typically, in high-technological semi-conductor manufacturing, many machines are repeatedly used for similar processing operations. In such production lines, semi-conductor wafers return to the same set of machines many times. So, the product flow has a re-entrant character. Typically, the semi-conductor systems are characterized by the very high volume (number of parts manufactured per unit time) and the very large number of consecutive production steps. This fact motivates to consider the scalar nonlinear conservation laws for the simulation of such processes. Partial differential equations, which are related with nonlinear conservation laws, are rather popular due to their superior analytic properties and availability of efficient numerical tools for simulation. For more detailed discussions of these models we refer to [1, 2, 4, 8, 12–23].

From the optimization point of view, in manufacturing systems the natural control input is the in-flux. However, the output of a factory can be changed via dispatch policies. Specifically, re-entrant production creates the opportunity to set priority rules for the various stages of production competing for capacity at the same machines. This dispatch policy, as it was indicated in [3], typically allows for two models of operations — the so-called push and pull policies. A puch policy, also known as first buffer first step, is typically assigned to the front of the factory. A pull policy gives priority to later or fixed production steps over the earlier production steps. The step where push policy switches to pull policy is called the push-pull point (PPP). Moving the PPP leads not only to a change in dispatch rules, but also it has significant effect on the total output. This fact motivates to consider the PPP as a control variable.

A modern introduction to the study of hyperbolic conservation laws can be found in [6]. Fundamental are questions of well-posedness, regularity properties of solutions, existence, and their uniquenesss. Existence of solutions, regularity and well-posedness of nonlinear conservation laws have been widely studied under diverse sets of hypotheses, see e.g. [4,5,7]. Further results can be found in [9,11,23].

The characteristic feature of the re-entrant, we deal with in this article, is the fact that we consider a Cauchy problem for nonlinear hyperbolic conservation law for the part density $\rho(t,x)$ including a PPP at position x^*

$$\partial_t\rho(t,x) + \partial_x\left(V\rho(t,x)\right) = 0 \quad \text{in} \quad Q = (0,T)\times(0,1),$$

where the multiplier

$$V = H(x-x^*)V_2\left(\int_x^1 \rho(t,y-x+x^*)\,dy\right) + H(x^*-x)V_1\left(\int_0^x \rho(t,y)\,dy\right)$$

depends explicitly on the specific PPP $x^*\in[0,1]$ and via the functional dependence V_1 and V_2 on the push and pull regimes

$$\int_0^x \rho(t,y)\,dy \quad \text{and} \quad \int_x^1 \rho(t,y-x+x^*)\,dy,$$

respectively. As for the right choice of functions V_1 and V_2, this question is definitely open. Typically, they use functions like (see [3,11,24])

$$V(W) = \frac{1}{1+W}, \quad V(W) = \exp\left(-\beta W\right), \quad \text{or} \quad V(W) = \frac{C_1}{C_2 - \log(\alpha - W)}.$$

This fact motivates us to consider the functions V_1 and V_2 as arbitrary given. As a result, we deal with the Cauchy problem for the nonlinear conservation law with a nonlocal character of the velocity and with three different control actions — the in-flux, the PPP, and the so-called clearing functions V_1 and V_2.

The paper is organized as follows. In Section 2 we give the precise statement of the Cauchy problem for a highly re-entrant production system. The aim of Section 3 is to give some preliminaries and auxiliary results that we make use for our further analysis. In Section 4 we prove the existence of a unique weak solution to the Cauchy problem associated with the re-entrant system under given control functions when the initial and boundary conditions we consider in $L^1(0,T)$ and $L^1(0,1)$ sense, respectively.

2. Statement of the Problem

Let $\alpha_2 > \alpha_1 > 0$ and $\alpha_3 > 0$ be given constants. Let $\mathfrak{A}_{ad}$ be the following subset of $C^1([0,\infty))$

$$\mathfrak{A}_{ad} = \left\{ V \in C^1([0,\infty)) \;\middle|\; \begin{array}{c} 0 \le \alpha_1 \le V(x) \le \alpha_2 \;\forall\, x \in [0,\infty), \\ \|V'\|_{C^0([0,\infty))} \le \alpha_3. \end{array} \right\} \tag{2.1}$$

Following the concept of the continuous flow model, describing the flow of products through a factory like a fluid flow, we denote $\rho(t,x)$ the product density at the stage $x \in [0,1]$ and time $t \in [0,T]$. Here, $x = 0$ refers to the point of raw material and $x = 1$ to the finished product.

Definition 2.1. We say that a mapping $F : [0,T] \times [0,1] \mapsto [0,\infty)$ is the clearing function if there exists a point $x^* \in [0,1]$ and functions $V_1, V_2 \in \mathfrak{A}_{ad}$ such that

$$F(t,x) := \rho(t,x) \left[H(x^* - x) V_1 \left(W^{push}(t,x) \right) + H(x - x^*) V_2 \left(W^{pull}(t,x) \right) \right], \tag{2.2}$$

where $H(x)$ stands for the Heaviside function and

$$W^{pull}(t,x) = \int_x^1 \rho(t, y - x + x^*)\, dy, \quad W^{push}(t,x) = \int_0^x \rho(t,y)\, dy. \tag{2.3}$$

As follows from this definition $F(t,x)$ can be associated with the flux (production rate) at the time $t \in [0,T]$ and stage $x \in [0,1]$ in the factory, whereas $x^* \in [0,1]$ is the PPP where the push policy switches to pull policy.

Taking into account that in the manufacturing systems the natural control input is the in-flux, we arrive at the following Cauchy problem:

$$\partial_t \rho(t,x) + \partial_x \left(V \rho(t,x) \right) = 0 \quad \text{in} \;\; Q := (0,T) \times (0,1), \tag{2.4}$$

$$V = H(x - x^*) V_2 \left(\int_x^1 \rho(t, y - x + x^*)\, dy \right) + H(x^* - x) V_1 \left(\int_0^x \rho(t,y)\, dy \right), \tag{2.5}$$

$$\rho(0,x) = \rho_0(x) \quad \text{for} \;\; x \in [0,1], \quad \rho(t,0) V_1(0) = u, \quad \text{for} \;\; t \in [0,T], \tag{2.6}$$

$$V_1, V_2 \in \mathfrak{A}_{ad}, \quad x^* \in [0,1], \tag{2.7}$$

$$u \in \mathfrak{U}_{ad} := \left\{ w \in L^2(0,T) \mid \|w\|_{L^2(0,T)} \le \alpha_4, \; w(x) \ge 0 \;\text{ a.e. on } (0,T) \right\}, \tag{2.8}$$

where $\rho_0 \in L^2(0,1)$ is a given nonnegative function.

Hereinafter, a tuple

$$(u, V_1, V_2, x^*) \in L^2(0,T) \times C^1([0,a_1]) \times C^1([0,a_2]) \times [0,1] \tag{2.9}$$

with properties (2.7)–(2.8) we call an admissible control.

3. Preliminaries and Auxiliary Results

It is easy to see that, for each admissible control (u, V_1, V_2, x^*), the Cauchy problem (2.4)–(2.6) can be represented in the form of coupled system

$$\partial_t \rho_1(t,x) + \partial_x \left(V_1 \left(\int_0^x \rho_1(t,y)\, dy \right) \rho_1(t,x) \right) = 0 \quad \text{in } \ Q_1, \tag{3.1}$$

$$\rho_1(0,x) = \rho_0(x) \quad \text{for } \ x \in [0,x^*], \quad \rho_1(t,0)V_1(0) = u(t), \quad \text{for } \ t \in [0,T], \tag{3.2}$$

$$\partial_t \rho_2(t,x) + \partial_x \left(V_2 \left(\int_x^1 \rho_2(t, y - x + x^*)\, dy \right) \rho_2(t,x) \right) = 0 \quad \text{in } \ Q_2, \tag{3.3}$$

$$\rho_2(0,x) = \rho_0(x) \quad \text{for } \ x \in [x^*,1], \tag{3.4}$$

$$\rho_2(t,x^*) V_2 \left(\int_{x^*}^1 \rho_2(t,y)\, dy \right) = \rho_1(t,x^*) V_1 \left(\int_0^{x^*} \rho_1(t,y)\, dy \right), \ \forall\, t \in [0,T], \tag{3.5}$$

where $Q_1 := (0,T) \times (0,x^*)$, $Q_2 = (0,T) \times (x^*,1)$ the compatibility condition (3.5) means that the output flux at $x = x^*$ of the push region must be considered as the in-flux for the pull region.

Remark 3.1. It is easy to note that the following representation for the solutions to the Cauchy problem (2.4)–(2.6)

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if } \ t \in [0,T], \ x \in [0,x^*), \\ \rho_2(t,x), & \text{if } \ t \in [0,T], \ x \in (x^*,1]), \end{cases} \tag{3.6}$$

holds, where $x = x^*$ is the discontinuity point for the work in progress (wip) profile.

Following [10], we adopt the following definition of a weak solution to the problem (3.1)–(3.5).

Definition 3.1. Let $T > 0$, $\rho_0 \in L^1(0,1)$, $u \in L^1(0,T)$, $x^* \in [0,1]$, and $V_1, V_2 \in \mathfrak{A}_{ad}$ be given. We say that a pair $(\rho_1, \rho_2) \in C^0([0,T]; L^1(0,x^*) \times L^1(x^*,1))$ is a weak solution to the Cauchy problem (3.1)–(3.5) if for every $\tau \in [0,T]$ and every test functions $(\varphi_1, \varphi_2) \in C^1([0,T] \times [0,x^*]) \times C^1([0,T] \times [x^*,1])$ such that

$$\begin{aligned} &\varphi_1(\tau,x) = 0, \quad \forall\, x \in [0,x^*], \quad \varphi_1(t,x^*) = 0, \quad \forall\, t \in [0,\tau], \\ &\varphi_2(\tau,x) = 0, \quad \forall\, x \in [x^*,1], \quad \varphi_2(t,1) = 0, \quad \forall\, t \in [0,\tau], \end{aligned} \tag{3.7}$$

the following integral identities hold true

$$\begin{aligned} &\int_0^\tau \int_0^{x^*} \rho_1(t,x) \left[\partial_t \varphi_1(t,x) + V_1 \left(\int_0^x \rho_1(t,y)\, dy \right) \partial_x \varphi_1(t,x) \right] dxdt \\ &+ \int_0^\tau u(t) \varphi_1(t,0)\, dt + \int_0^{x^*} \rho_0(x) \varphi_1(0,x)\, dx = 0, \end{aligned} \tag{3.8}$$

$$\int_0^\tau \int_{x^*}^1 \rho_2(t,x) \left[\partial_t \varphi_2(t,x) + V_2\left(\int_x^1 \rho_2(t,y-x+x^*)\,dy\right)\partial_x\varphi_2(t,x)\right] dxdt$$
$$+\int_0^\tau \rho_1(t,x^*)V_1\left(\int_0^{x^*}\rho_1(t,y)\,dy\right)\varphi_2(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_2(0,x)\,dx = 0. \tag{3.9}$$

For our further analysis, we make use of a couple of auxiliary results.

Lemma 3.1. *Let $\rho_0 \in L^1(0,1)$ and $u, v \in L^1(0,T)$ be nonnegative functions. Let $x^* \in [0,1]$, $x \in [0,x^*]$, $z \in [x^*,1]$, and $V_1, V_2 \in \mathfrak{A}_{ad}$ be given and such that $V_i(s) = V_i(0)$ for all $s < 0$. Then there exists $\delta \in [0,T]$ independent of x and y such that the Cauchy problem*

$$\begin{cases} \dfrac{d\xi(t)}{dt} = V_1\left(\displaystyle\int_0^t u(\sigma)\,d\sigma + \int_0^{x-\xi(t)}\rho_0(y)\,dy\right), & t\in[0,\delta], \qquad \xi(0)=0,\\ \dfrac{d\zeta(t)}{dt} = V_2\left(\displaystyle\int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta(t)+x^*-z}\rho_0(y)\,dy\right), & t\in[0,\delta], \quad \zeta(0)=0,\end{cases} \tag{3.10}$$

has a unique solution $(\xi_x, \zeta_z) \in \left[C^1([0,\delta])\right]^2$.

Proof. We associate with the Cauchy problem (3.10) the mapping $(\xi,\zeta)\mapsto F(\xi,\zeta) : \Omega_\delta\times\Omega_\delta \to \left[C^0([0,\delta])\right]^2$ such that

$$F(\xi,\zeta)(t) = \begin{bmatrix} \displaystyle\int_0^t V_1\left(\int_0^s u(\sigma)\,d\sigma + \int_0^{x-\xi(s)}\rho_0(y)\,dy\right) ds \\ \displaystyle\int_0^t V_2\left(\int_0^s v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta(s)+x^*-z}\rho_0(y)\,dy\right) ds \end{bmatrix}, \ \forall\, t\in[0,\delta] \tag{3.11}$$

and

$$\Omega_\delta = \left\{\xi\in C^0([0,\delta]) \mid \xi(0)=0,\ \alpha_1 \le \frac{\xi(s)-\xi(t)}{s-t} \le \alpha_2,\ \forall\, s,t\in[0,\delta],\ s>t\right\} \tag{3.12}$$

where the constants α_1 and α_2 are defined as in (2.1). It is clear that Ω_δ consists of monotonically increasing functions on $[0,\delta]$.

Let us show that there exists a constant $\kappa\in(0,1)$ such that

$$\|F(\xi_1,\zeta_1) - F(\xi_2,\zeta_2)\|_{[C^0([0,\delta])]^2} \le \kappa\left[\|\zeta_1-\zeta_2\|_{C^0([0,\delta])} + \|\xi_1-\xi_2\|_{C^0([0,\delta])}\right] \tag{3.13}$$

for all $\xi_i,\zeta_i\in\Omega_\delta$ and $\delta>0$ small enough. Since F maps into Ω_δ provided $\delta<\alpha_2^{-1}$, it follows from (3.13) that $F(\xi,\zeta):\Omega_\delta\times\Omega_\delta\to\Omega_\delta\times\Omega_\delta$ is a contraction mapping. Then, by the Banach fixed point theorem, there exists a unique pair (ξ_x,ζ_z) such that $F(\xi_x,\zeta_z)=(\xi_x,\zeta_z)$, i.e. (ξ_x,ζ_z) is the unique solution to the Cauchy problem (3.10). Moreover, as follows from definition of the set $\mathfrak{A}_{ad}$ and the fact that $V_1,V_2\in\mathfrak{A}_{ad}$, the unique fixed pair (ξ_x,ζ_z) for F is in $\left[C^1([0,\delta])\right]^2$.

Let ξ_i, ζ_i $(i = 1,2)$ be arbitrary elements of Ω_δ. Then (3.11) implies the estimate

$$\begin{aligned} |F(\xi_1,\zeta_1) - F(\xi_2,\zeta_2)|_1 &\le \alpha_3 \int_0^t \left| \int_{x-\xi_1(s)}^{x-\xi_2(s)} \rho_0(y)\,dy \right| ds \\ &\quad + \alpha_3 \int_0^t \left| \int_{1-\zeta_1(s)+x^*-z}^{1-\zeta_2(s)+x^*-z} \rho_0(y)\,dy \right| ds \\ &= \alpha_3 \left[J_1(\xi_1,\xi_2) + J_2(\zeta_1,\zeta_2) \right]. \end{aligned} \tag{3.14}$$

We define $\widehat{\xi}, \underline{\xi} \in C^0([0,\delta])$ by

$$\widehat{\xi}(t) := \max\{\xi_1(t),\xi_2(t)\} \quad \text{and} \quad \underline{\xi}(t) := \min\{\xi_1(t),\xi_2(t)\}.$$

Since ξ_i are monotonically increasing functions, it follows that the inverse functions $\widehat{\xi}^{-1}$ and $\underline{\xi}^{-1}$ are well defined. Then, changing the order of integrations in (3.14) as it is shown at Figure 1(left) and following in many aspects [11], we obtain

$$\begin{aligned} J_1(\xi_1,\xi_2) &= \int_0^t \left| \int_{x-\xi_1(s)}^{x-\xi_2(s)} \rho_0(y)\,dy \right| ds \\ &= \int_{x-\widehat{\xi}(t)}^{x-\underline{\xi}(t)} \rho_0(y) \left(t - \widehat{\xi}^{-1}(x-y) \right) dy \\ &+ \int_{x-\underline{\xi}(t)}^{x} \rho_0(y) \left(\underline{\xi}^{-1}(x-y) - \widehat{\xi}^{-1}(x-y) \right) dy \\ &\le \int_{x-\widehat{\xi}(t)}^{x-\underline{\xi}(t)} \rho_0(y) \left(\underline{\xi}^{-1}\left(\underline{\xi}(t)\right) - \widehat{\xi}^{-1}\left(\underline{\xi}(t)\right) \right) dy \\ &\quad + \int_{x-\underline{\xi}(t)}^{x} \rho_0(y) \left(\underline{\xi}^{-1}(x-y) - \widehat{\xi}^{-1}(x-y) \right) dy \\ &\le \int_{x-\widehat{\xi}(t)}^{x} \rho_0(y)\,dy \sup_{0\le y\le \underline{\xi}(t)} \left[\underline{\xi}^{-1}(y) - \widehat{\xi}^{-1}(y) \right], \end{aligned} \tag{3.15}$$

where for the term $\underline{\xi}^{-1}(y) - \widehat{\xi}^{-1}(y)$ we have the following estimate for each $y \in [0, \underline{\xi}(t)]$ (for details we refer to [11] and Figure 1(right))

$$\begin{aligned} 0 &\le \underline{\xi}^{-1}(y) - \widehat{\xi}^{-1}(y) \\ &= \left(\underline{\xi}^{-1}(y) - \frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} \right) + \left(\frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} - \widehat{\xi}^{-1}(y) \right) \\ &\le \frac{1}{\alpha_1} \left[y - \underline{\xi}\left(\frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} \right) \right] + \frac{1}{\alpha_1} \left[\widehat{\xi}\left(\frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} \right) - y \right] \\ &= \frac{1}{\alpha_1} \left[\widehat{\xi}\left(\frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} \right) - \underline{\xi}\left(\frac{\underline{\xi}^{-1}(y) + \widehat{\xi}^{-1}(y)}{2} \right) \right] \le \frac{1}{\alpha_1} \|\xi_1 - \xi_2\|_{C^0([0,\delta])}. \end{aligned} \tag{3.16}$$

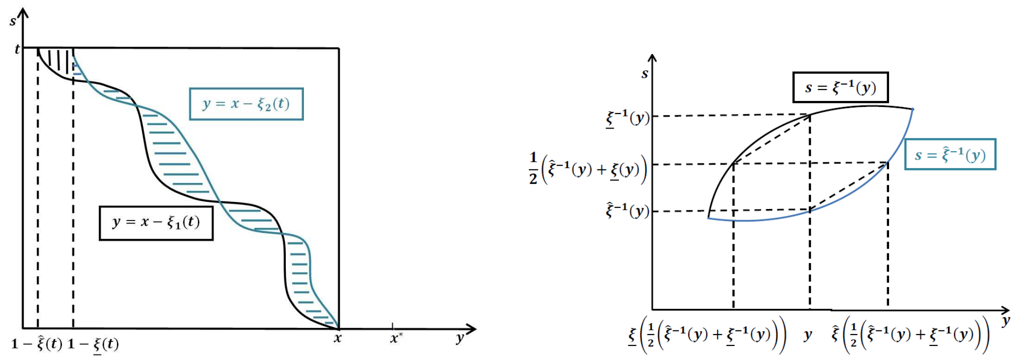

Fig. 1. Left: Change the order of integration in (3.15). Right: Explanation to formula (3.16)

Combining the above results, we finally get

$$J_1(\xi_1,\xi_2) \le \frac{1}{\alpha_1}\|\xi_1-\xi_2\|_{C^0([0,\delta])}\int_{x-\widehat{\xi}(t)}^{x}\rho_0(y)\,dy \le \frac{1}{\alpha_1}\|\xi_1-\xi_2\|_{C^0([0,\delta])}\int_{x-\delta\alpha_2}^{x}\rho_0(y)\,dy.$$

By analogy, it can be shown that

$$J_2(\zeta_1,\zeta_2) \le \frac{1}{\alpha_1}\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\int_{1-\widehat{\zeta}(t)}^{1}\rho_0(y)\,dy \le \frac{1}{\alpha_1}\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\int_{1-\delta\alpha_2}^{1}\rho_0(y)\,dy.$$

As a result, the inequality (3.14) implies

$$\begin{aligned}|F(\xi_1,\zeta_1)(y)-F(\xi_2,\zeta_2)(y)|_1 \le \frac{\alpha_3}{\alpha_1}&\left[\int_{x-\delta\alpha_2}^{x}\rho_0(y)\,dy+\int_{1-\delta\alpha_2}^{1}\rho_0(y)\,dy\right]\\ &\times\left[\|\xi_1-\xi_2\|_{C^0([0,\delta])}+\|\zeta_1-\zeta_2\|_{C^0([0,\delta])}\right]. \qquad (3.17)\end{aligned}$$

Since $\rho_0 \in L^1(0,1)$, it follows that there exists $\delta \in (0,T)$ small enough such that

$$\int_{x-\delta\alpha_2}^{x}\rho_0(y)\,dy+\int_{1-\delta\alpha_2}^{1}\rho_0(y)\,dy < \frac{\alpha_1}{2\alpha_3}. \qquad (3.18)$$

In view of estimate (3.17), this immediately leads us to inequality (3.13). □

Our next intention is to study the properties of the mappings $x \mapsto \xi_x(t)$ and $z \mapsto \zeta_z(t)$.

Lemma 3.2. *Assume that $\rho_0 \in L^\infty(0,1)$. Then, for given $u, v \in L^1(0,T)$, $V_1, V_2 \in \mathfrak{A}_{ad}$, and $t \in [0,\delta]$, the mappings*

$$x \mapsto \xi_x(t) : [0,x^*] \to \mathbb{R}_+ \quad \textit{and} \quad z \mapsto \zeta_z(t) : [x^*,1] \to \mathbb{R}_+ \qquad (3.19)$$

are continuous.

Proof. Let $x, y \in [0, x^*]$ be arbitrary points. Then, in view of definition of the class $\mathfrak{A}_{ad}$, we can derive from the first equation of (3.10) the estimate

$$\begin{aligned}|\xi_x(t)-\xi_y(t)| &\le \alpha_3 \int_0^t \left| \int_{y-\xi_y(s)}^{x-\xi_x(s)} \rho_0(\sigma)\, d\sigma \right| ds \\ &\le \alpha_3 \|\rho_0\|_{L^\infty(0,1)} \int_0^t \left(|x-y| + |\xi_x(s)-\xi_y(s)|\right) ds \\ &\le \alpha_3 \|\rho_0\|_{L^\infty(0,1)} \delta |x-y| + \alpha_3 \|\rho_0\|_{L^\infty(0,1)} \int_0^t |\xi_x(s)-\xi_y(s)|\, ds.\end{aligned}$$

As a result, by Gronwall-Bellman inequality, we see that

$$|\xi_x(t)-\xi_y(t)| \le \alpha_3 \|\rho_0\|_{L^\infty(0,1)} \delta |x-y| \exp\left(\alpha_3 \|\rho_0\|_{L^\infty(0,1)} t\right) \le C|x-y|, \quad (3.20)$$

that is, $x \mapsto \xi_x(t) : [0, x^*] \to \mathbb{R}_+$ is a continuous mapping. The continuity of $z \mapsto \zeta_z(t)$ can be established in a similar manner. $\square$

Let $x \in [0, x^*]$ and $z \in [x^*, 1]$ be fixed. Let $(\xi_x, \zeta_z) \in \left[C^1([0,\delta])\right]^2$ be the corresponding solution of the system (3.10) on some small time interval $[0, \delta]$. For given $\rho_0 \in L^1(0,1)$, $u, v \in L^1(0,T)$, and $x^* \in [0,1]$, we introduce the following couple of functions

$$\widetilde{\rho}_{1,x}(t,y) = \begin{cases} \dfrac{u\left(\xi_x^{-1}\left(\xi_x(t)-y\right)\right)}{\xi_x'\left(\xi_x^{-1}\left(\xi_x(t)-y\right)\right)}, & 0 \le y \le \xi_x(t), \\ \rho_0(y-\xi_x(t)), & \xi_x(t) \le y \le x^*, \end{cases} \quad \forall\, t \in [0,\delta], \quad (3.21)$$

$$\widetilde{\rho}_{2,z}(t,y) = \begin{cases} \dfrac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}, & x^* \le y \le x^*+\zeta_z(t), \\ \rho_0(y-\zeta_z(t)), & x^*+\zeta_z(t) \le y \le 1. \end{cases} \quad \forall\, t \in [0,\delta]. \quad (3.22)$$

Lemma 3.3. *For given $\rho_0 \in L^1(0,1)$, $u, v \in L^1(0,T)$, $x^* \in [0,1]$, $x \in [0,x^*]$, $z \in [x^*, 1]$, and $(\xi_x, \zeta_z) \in \left[C^1([0,\delta])\right]^2$, the functions $\widetilde{\rho}_{1,x}$ and $\widetilde{\rho}_{2,z}$, defined by* (3.21)–(3.22) *are such that*

$$\widetilde{\rho}_{1,x} \in C([0,\delta]; L^1(0,x^*)), \qquad \widetilde{\rho}_{2,z} \in C([0,\delta]; L^1(x^*,1)). \quad (3.23)$$

Proof. We only prove the inclusion $\rho_2 \in C([0,\delta]; L^1(x^*,1))$, since the second one in (3.23) can be established by analogy. Let $\varepsilon > 0$ be an arbitrary value. Our aim it to show that there exists $\theta = \theta(\varepsilon) > 0$ such that, for arbitrary points $s, t \in [0, \delta]$, we have

$$\|\widetilde{\rho}_{2,z}(s,\cdot) - \widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)} < \varepsilon, \quad \text{provided} \quad |s-t| < \theta.$$

Indeed, having assumed for simplicity that $s > t$, we have

$$\begin{aligned}
&\int_{x^*}^{1} |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \\
&\quad \le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \\
&\quad + \int_{x^*+\zeta_z(s)}^{1} |\rho_0(y-\zeta_z(s)) - \rho_0(y-\zeta_z(t))|\, dy = J_1 + J_2 + J_3. \qquad (3.24)
\end{aligned}$$

Since

$$\begin{aligned}
J_2 &:= \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} |\widetilde{\rho}_{2,z}(s,y) - \widetilde{\rho}_{2,z}(t,y)|\, dy \\
&\le \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \widetilde{\rho}_{2,z}(s,y)\, dy + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \widetilde{\rho}_{2,z}(t,y)\, dy \\
&\overset{\text{by (3.22)}}{=} \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}\, dy + \int_{x^*+\zeta_z(t)}^{x^*+\zeta_z(s)} \rho_0(y-\zeta_z(t))\, dy \\
&= \int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))} v(\sigma)\, d\sigma + \int_{x^*}^{x^*+\zeta_z(s)-\zeta_z(t)} \rho_0(\gamma)\, d\gamma
\end{aligned}$$

and $\rho_0 \in L^1(0,1)$ and $v \in L^1(0,T)$, it is easy to conclude from monotonicity property of $\zeta_z \in C^1([0,\delta])$ and condition $\zeta_z(0) = 0$ that there exists a value $\theta_2(\varepsilon) > 0$ such that $J_2 < \varepsilon/3$.

Now we show that the same conclusion can be obtained with respect to the term J_3. Indeed, let $\{\rho_0^k\}_{k\in\mathbb{N}} \subset C^1([x^*,1])$ be an arbitrary sequence such that $\rho_0^k \to \rho_0$ in $L^1(x^*,1)$ as $k \to \infty$. Then

$$\begin{aligned}
J_3 &:= \int_{x^*+\zeta_z(s)}^{1} |\rho_0(y-\zeta_z(s)) - \rho_0(y-\zeta_z(t))|\, dy \\
&\le \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0(y-\zeta_z(s)) - \rho_0^k(y-\zeta_z(s))\right| dy \\
&\quad + \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0^k(y-\zeta_z(s)) - \rho_0^k(y-\zeta_z(t))\right| dy \\
&\quad + \int_{x^*+\zeta_z(s)}^{1} \left|\rho_0^k(y-\zeta_z(t)) - \rho_0(y-\zeta_z(t))\right| dy \\
&\le \int_{x^*}^{1-\zeta_z(s)} \left|\rho_0(y) - \rho_0^k(y)\right| dy + \int_{x^*+\zeta_z(s)-\zeta_z(t)}^{1-\zeta_z(t)} \left|\rho_0^k(y) - \rho_0(y)\right| dy \\
&\quad + C(k)|\zeta_z(s)-\zeta_z(t)| \le 2\int_{x^*}^{1} \left|\rho_0(y) - \rho_0^k(y)\right| dy + C(k)|\zeta_z(s)-\zeta_z(t)|,
\end{aligned}$$

where the constant $C(k)$ depends on $k \in \mathbb{N}$ but does not depend on t and s. Hence, in view of the strong convergence $\rho_0^k \to \rho_0$ in $L^1(x^*,1)$ and monotonicity of $\zeta_z \in C^1([0,\delta])$, there exists a value $\theta_3(\varepsilon) > 0$ such that $J_3 < \varepsilon/3$.

It remains to estimate the first term in the right hand side of (3.24). Let $\{v_k\}_{k\in\mathbb{N}} \subset C^1([0,T])$ be a strongly convergent sequence to v in $L^1(0,T)$. Then

$$\begin{aligned}
J_1 &:= \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} - \frac{v\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

Since $A_1 = \displaystyle\int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{s} |v(\sigma) - v_k(\sigma)|\, d\sigma \le \|v - v_k\|_{L^1(0,T)}$,

$A_3 = \displaystyle\int_0^t |v(\sigma) - v_k(\sigma)|\, d\sigma \le \|v - v_k\|_{L^1(0,T)}$, and

$$\begin{aligned}
A_2 &:= \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\le \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right) - v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} \right| dy \\
&\quad + \int_{x^*}^{x^*+\zeta_z(t)} \left| \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right)} - \frac{v_k\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)}{\zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right)} \right| dy \\
&\quad \le C(k)|\zeta_z(s) - \zeta_z(t)| \\
&\quad + \widehat{C}(k) \int_{x^*}^{x^*+\zeta_z(t)} \left| \zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(t)+x^*-y\right)\right) - \zeta_z'\left(\zeta_z^{-1}\left(\zeta_z(s)+x^*-y\right)\right) \right| dy
\end{aligned}$$

it follows from definition of function ζ_z (see the Cauchy problem (3.10)) that

$$\begin{aligned}
A_2 &\le C(k)|\zeta_z(s) - \zeta_z(t)| \\
&\quad + \widehat{C}(k) \int_{x^*}^{x^*+\zeta_z(t)} \left| V_2(\cdot)\big|_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)} - V_2(\cdot)\big|_{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} \right| dy \\
&\quad \le C(k)|\zeta_z(s) - \zeta_z(t)| + \widehat{C}(k)\alpha_3 \int_{x^*}^{x^*+\zeta_z(t)} \int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\, d\sigma dy \\
&\quad + \widehat{C}(k)\alpha_3 \int_{x^*}^{x^*+\zeta_z(t)} \int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y} \rho_0(\gamma)\, d\gamma dy. \qquad (3.25)
\end{aligned}$$

To estimate the right hand side in (3.25), we change the order of integration. As a result, we obtain (for the details, see Figure 2(left))

$$\begin{aligned}
&\int_{x^*}^{x^*+\zeta_z(t)}\int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\,d\sigma dy = \int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}\int_{\zeta_z(t)-\zeta_z(\sigma)+x^*}^{x^*+\zeta_z(t)} v(\sigma)\,dyd\sigma\\
&\quad+\int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{t}\int_{\zeta_z(t)-\zeta_z(\sigma)+x^*}^{\zeta_z(s)-\zeta_z(\sigma)+x^*} v(\sigma)\,dyd\sigma+\int_t^s\int_{x^*}^{\zeta_z(s)-\zeta_z(\sigma)+x^*} v(\sigma)\,dyd\sigma\\
&=\int_0^{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}\zeta_z(\sigma)v(\sigma)\,d\sigma+\int_{\zeta_z^{-1}(\zeta_z(s)-\zeta_z(t))}^{t}(\zeta_z(s)-\zeta_z(t))v(\sigma)\,d\sigma\\
&\quad+\int_t^s(\zeta_z(s)-\zeta_z(\sigma))v(\sigma)\,d\sigma. \qquad (3.26)
\end{aligned}$$

Taking into account that

$$\begin{aligned}
0\le\zeta_z(\sigma)\le\zeta_z(s)-\zeta_z(t), &\qquad \text{if } 0\le\sigma\le\zeta_z^{-1}\left(\zeta_z(s)-\zeta_z(t)\right),\\
0\le\zeta_z(s)-\zeta_z(\sigma)\le\zeta_z(s)-\zeta_z(t), &\qquad \text{if } t\le\sigma\le s,
\end{aligned}$$

we can conclude from (3.26) the following estimate

$$\begin{aligned}
\int_{x^*}^{x^*+\zeta_z(t)}\int_{\zeta_z^{-1}(\zeta_z(t)+x^*-y)}^{\zeta_z^{-1}(\zeta_z(s)+x^*-y)} v(\sigma)\,d\sigma dy &\le |\zeta_z(s)-\zeta_z(t)|\int_0^s v(\sigma)\,d\sigma\\
&\le \|v\|_{L^1(0,T)}|\zeta_z(s)-\zeta_z(t)|. \qquad (3.27)
\end{aligned}$$

It remains to estimate the last term in (3.25). Following in the similar manner (see Figure 2(right) for the details), we change the order of integration. As a result, we obtain

$$\begin{aligned}
&\int_{x^*}^{x^*+\zeta_z(t)}\int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y}\rho_0(\gamma)\,d\gamma dy=\int_{1-\zeta_z(s)}^{1-\zeta_z(t)}\int_{x^*}^{\gamma+\zeta_z(s)+x^*-1}\rho_0(\gamma)\,dyd\gamma\\
&\quad+\int_{1-\zeta_z(t)}^{1-\zeta_z(s)+\zeta_z(t)}\int_{\gamma+\zeta_z(t)+x^*-1}^{\gamma+\zeta_z(s)+x^*-1}\rho_0(\gamma)\,dyd\gamma\\
&\quad+\int_{1-\zeta_z(s)+\zeta_z(t)}^{1}\int_{\gamma+\zeta_z(t)+x^*-1}^{x^*+\zeta_z(t)}\rho_0(\gamma)\,dyd\gamma\\
&=\int_{1-\zeta_z(s)}^{1-\zeta_z(t)}(\gamma+\zeta_z(s)-1)\rho_0(\gamma)\,d\gamma+\int_{1-\zeta_z(t)}^{1-\zeta_z(s)+\zeta_z(t)}(\zeta_z(s)-\zeta_z(t))\rho_0(\gamma)\,d\gamma\\
&\quad+\int_{1-\zeta_z(s)+\zeta_z(t)}^{1}(1-\gamma)\rho_0(\gamma)\,d\gamma. \qquad (3.28)
\end{aligned}$$

Since

$$\begin{aligned}
0\le\gamma+\zeta_z(s)-1\le\zeta_z(s)-\zeta_z(t), &\qquad \text{provided } 1-\zeta_z(s)\le\gamma\le1-\zeta_z(t),\\
0\le1-\gamma\le\zeta_z(s)-\zeta_z(t), &\qquad \text{provided } 1-\zeta_z(s)+\zeta_z(t)\le\gamma\le1,
\end{aligned}$$

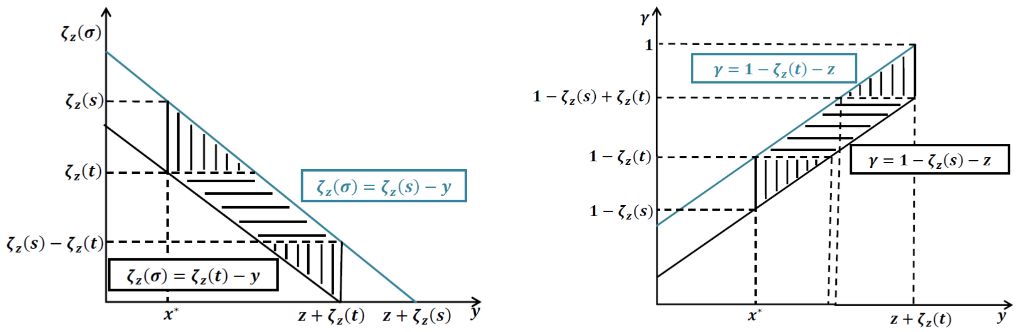

Fig. 2. Change the order of integration in (3.26) (on the left) and in (3.28) (on the right)

we deduce from (3.28) that

$$\int_{x^*}^{x^*+\zeta_z(t)}\int_{1-\zeta_z(s)-x^*+y}^{1-\zeta_z(t)-x^*+y}\rho_0(\gamma)\,d\gamma dy \le |\zeta_z(s)-\zeta_z(t)|\int_{1-\zeta_z(s)}^{1}\rho_0(\gamma)\,d\gamma$$
$$\le \|\rho_0\|_{L^1(0,1)}|\zeta_z(s)-\zeta_z(t)|. \tag{3.29}$$

Thus, combining the estimates (3.25), (3.27), and (3.29), we get

$$A_3 \le \left[C(k)+\widehat{C}(k)\alpha_3\left(\|v\|_{L^1(0,T)}+\|\rho_0\|_{L^1(0,1)}\right)\right]|\zeta_z(s)-\zeta_z(t)|$$
$$= D(k)|\zeta_z(s)-\zeta_z(t)||\zeta_z(s)-\zeta_z(t)|,$$

and, hence,

$$J_1 \le A_1+A_2+A_3 \le 2\|v-v_k\|_{L^1(0,T)}+D(k)|\zeta_z(s)-\zeta_z(t)||\zeta_z(s)-\zeta_z(t)|, \tag{3.30}$$

where the constant $D(k)$ depends on $k\in\mathbb{N}$ but does not depend on t and s. As follows from (3.30), for $k\in\mathbb{N}$ large enough there exists a value $\theta_1(\varepsilon)>0$ such that $J_1<\varepsilon/3$. As a result, we arrive at the following conclusion: for a given $\varepsilon>0$ and all $t,s\in[0,\delta]$ such that $|s-t|<\theta=\min\{\theta_1(\varepsilon),\theta_2(\varepsilon),\theta_3(\varepsilon)\}$, the estimate

$$\|\widetilde{\rho}_{2,z}(s,\cdot)-\widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)} \le J_1+J_2+J_3<\varepsilon$$

holds true. □

As a consequence of Lemma 3.2, we have the following important property.

Corollary 3.1. *If, in additional to the assumptions of Lemma 3.3, $\rho_0\in L^\infty(0,1)$, then the mappings*

$$x\mapsto \|\widetilde{\rho}_{1,x}(t,\cdot)\|_{L^1(0,x^*)}:[0,x^*]\to\mathbb{R}_+ \quad \textit{and} \quad z\mapsto \|\widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)}:[x^*,1]\to\mathbb{R}_+ \tag{3.31}$$

are continuous for each $t\in[0,\delta]$.

Proof. It is easy to check that the following relations

$$\int_0^{x^*} \widetilde{\rho}_{1,x}(t,y)\,dy = \int_0^t u(\sigma)\,d\sigma + \int_0^{x^*-\xi_x(t)} \rho_0(y)\,dy, \tag{3.32}$$

$$\int_{x^*}^1 \widetilde{\rho}_{2,z}(t,y)\,dy = \int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta_z(t)} \rho_0(y)\,dy \tag{3.33}$$

hold true for each $x \in [0,x^*]$, $z \in [x^*,1]$. As a result, for any $x, y \in [0,x^*]$, we have

$$\begin{aligned}
&\Big|\|\widetilde{\rho}_{1,x}(t,\cdot)\|_{L^1(0,x^*)} - \|\widetilde{\rho}_{1,y}(t,\cdot)\|_{L^1(0,x^*)}\Big| \\
&= \left|\int_0^{x^*} \widetilde{\rho}_{1,x}(t,\sigma)\,d\sigma - \int_0^{x^*} \widetilde{\rho}_{1,y}(t,\sigma)\,d\sigma\right| \\
&\overset{\text{by (3.32)}}{=} \left|\int_{x^*-\xi_x(t)}^{x^*-\xi_y(t)} \rho_0(\sigma)\,d\sigma\right| \le \|\rho_0\|_{L^\infty(0,1)}\Big[|\xi_x(t)-\xi_y(t)|\Big] \\
&\overset{\text{by (3.20)}}{\le} \|\rho_0\|_{L^\infty(0,1)} C|x-y|.
\end{aligned}$$

The continuity of the mapping $z \mapsto \|\widetilde{\rho}_{2,z}(t,\cdot)\|_{L^1(x^*,1)}$ can be shown in a similar way. □

By Lemma 3.2, the following limits

$$\begin{aligned}
\lim_{y\to x} \int_0^y \widetilde{\rho}_{1,y}(t,\gamma)\,d\gamma &\overset{\text{by (3.21)}}{=} \lim_{y\to x}\left[\int_0^t u(\sigma)\,d\sigma + \int_0^{x-\xi_y(t)} \rho_0(y)\,dy\right], \\
\lim_{z\to x} \int_z^1 \widetilde{\rho}_{2,z}(t,\gamma+x^*-z)\,d\gamma &\overset{\text{by (3.22)}}{=} \lim_{z\to x}\left[\int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta_z(t)-z+x^*} \rho_0(y)\,dy\right]
\end{aligned} \tag{3.34}$$

are well defined provided $t \in [0, \xi_x^{-1}(x)$ in $(3.34)_1$ and $t \in [0, \zeta_x^{-1}(1-x+x^*)]$ in $(3.34)_2$. In view of this, we make use of the following notations

$$\begin{aligned}
\int_0^x \rho_1(t,\gamma)\,d\gamma &:= \lim_{y\to x}\int_0^y \widetilde{\rho}_{1,y}(t,\gamma)\,d\gamma, \\
\int_x^1 \rho_2(t,\gamma+x^*-x)\,d\gamma &:= \lim_{z\to x}\int_z^1 \widetilde{\rho}_{2,z}(t,\gamma+x^*-z)\,d\gamma.
\end{aligned} \tag{3.35}$$

Then relations (3.32)–(3.33) and Lemma 3.2 imply the following representation for the limit functions ρ_1 and ρ_2

$$\int_0^x \rho_1(t,\gamma)\,d\gamma = \lim_{y\to x}\left[\int_0^t u(\sigma)\,d\sigma + \int_0^{y-\xi_y(t)} \rho_0(\gamma)\,d\gamma\right]$$
$$= \int_0^t u(\sigma)\,d\sigma + \int_0^{x-\xi_x(t)} \rho_0(\gamma)\,d\gamma, \qquad \forall\, t \in [0, \min\{\delta, \xi_x^{-1}(x)\}] \quad (3.36)$$
$$\int_x^1 \rho_2(t,\gamma + x^* - x)\,d\gamma$$
$$= \lim_{z\to x}\left[\int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta_z(t)-z+x^*} \rho_0(\gamma)\,d\gamma\right] = \int_0^t v(\sigma)\,d\sigma$$
$$+ \int_{x^*}^{1-\zeta_x(t)-x+x^*} \rho_0(\gamma)\,d\gamma,\ \forall\, t \in [0, \min\{\delta, \zeta_x^{-1}(1-x+x^*)\}]. \quad (3.37)$$

4. Existence of Weak Solutions to the Cauchy Problem (2.4)–(2.6)

We begin this section with the following result.

Theorem 4.1. *For given $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1, V_2 \in \mathfrak{A}_{ad}$, and $x^* \in [0,1]$, let $\rho_1 = \rho_1(t,x)$ be defined by* (3.36)*, and let $\rho_2 = \rho_2(t,x)$ be defined by* (3.37) *with*

$$v(t) := \rho_1(t,x^*) V_1\left(\int_0^{x^*} \rho_1(t,y)\,dy\right). \quad (4.1)$$

Then $(\rho_1,\rho_2) \in C([0,\delta]; L^1(0,x^*)) \times C([0,\delta]; L^1(x^*,1))$ and

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if } t\in[0,\delta],\ x \in [0,x^*), \\ \rho_2(t,x), & \text{if } t\in[0,\delta],\ x \in (x^*,1]) \end{cases} \quad (4.2)$$

is a weak solution to the Cauchy problem (2.4)–(2.6) in the strip

$$\Pi_\delta := \{(t,x)\ :\ t\in[0,\delta],\ x\in[0,1]\}. \quad (4.3)$$

Proof. In view of Remark 3.1, a weak solution to the Cauchy problem (2.4)–(2.6) in the strip (4.3) can be defined in the sense of Definition 3.1. Following this definition, we fix an arbitrary $\tau \in [0,\delta]$ and a couple of test functions $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*]) \times C^1([0,\tau]\times[x^*,1])$ such that

$$\begin{aligned} &\varphi_1(\tau,x) = 0, \quad \forall\, x\in[0,x^*], \quad \varphi_1(t,x^*) = 0, \quad \forall\, t\in[0,\tau], \\ &\varphi_2(\tau,x) = 0, \quad \forall\, x\in[x^*,1], \quad \varphi_2(t,1) = 0, \quad \forall\, t\in[0,\tau]. \end{aligned} \quad (4.4)$$

Then, direct computations show that

$$
\begin{aligned}
A &:= \int_0^\tau \int_0^{x^*} \rho_1(t,x) \left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^x \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\right] dxdt \\
&\overset{\text{by (3.35)}}{=} \lim_{y\to x}\int_0^\tau \int_0^{x^*} \widetilde{\rho}_{1,y}(t,x) \left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\right] dxdt \\
&\overset{\text{by Lemma 3.2 and Corollary 3.1}}{=} \lim_{y\to x}\int_0^\tau \int_0^{\xi_y(t)} \frac{u\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}{\xi_y'\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}\,\partial_t\varphi_1(t,x)\,dxdt \\
&+ \lim_{y\to x}\int_0^\tau \int_0^{\xi_y(t)} \frac{u\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}{\xi_y'\left(\xi_y^{-1}\left(\xi_y(t)-x\right)\right)}\,V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\,dxdt \\
&\quad + \lim_{y\to x}\int_0^\tau \int_{\xi_y(t)}^{x^*} \rho_0(x-\xi_y(t))\partial_t\varphi_1(t,x)\,dxdt \\
&\quad + \lim_{y\to x}\int_0^\tau \int_{\xi_y(t)}^{x^*} \rho_0(x-\xi_y(t))V_1\left(\int_0^y \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,x)\,dxdt \\
&= \lim_{y\to x}\int_0^\tau \int_0^t u(\sigma)\partial_t\varphi_1(t,\xi_y(t)-\xi_y(\sigma))\,d\sigma dt \\
&\quad + \lim_{y\to x}\int_0^\tau \int_0^t u(\sigma)V_1\left(\int_0^{\xi_y(t)-\xi_y(\sigma)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\xi_y(t)-\xi_y(\sigma))\,d\sigma dt \\
&\quad + \lim_{y\to x}\int_0^\tau \int_0^{x^*-\xi_y(t)} \rho_0(\sigma)\partial_t\varphi_1(t,\sigma+\xi_y(t))\,d\sigma dt \\
&\quad + \lim_{y\to x}\int_0^\tau \int_0^{x^*-\xi_y(t)} \rho_0(\sigma)V_1\left(\int_0^{\sigma+\xi_y(t)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\sigma+\xi_y(t))\,d\sigma dt.
\end{aligned}
$$

Using again Lemma 3.2 and Corollary 3.1, we can pass to the limit as $y \to x-0$. Therefore,

$$
\begin{aligned}
A = &\int_0^\tau \int_0^t u(\sigma)\partial_t\varphi_1(t,\xi_x(t)-\xi_x(\sigma))\,d\sigma dt \\
&+ \int_0^\tau \int_0^t u(\sigma)V_1\left(\int_0^{\xi_x(t)-\xi_x(\sigma)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\xi_x(t)-\xi_x(\sigma))\,d\sigma dt \\
&+ \int_0^\tau \int_0^{x^*-\xi_x(t)} \rho_0(\sigma)\partial_t\varphi_1(t,\sigma+\xi_x(t))\,d\sigma dt \\
&+ \int_0^\tau \int_0^{x^*-\xi_x(t)} \rho_0(\sigma)V_1\left(\int_0^{\sigma+\xi_x(t)} \rho_1(t,s)\,ds\right)\partial_x\varphi_1(t,\sigma+\xi_x(t))\,d\sigma dt.
\end{aligned}
$$

As a result, making use of relations (3.10) and (3.32), we arrive at the following relation

$$
\begin{aligned}
A &= \int_0^\tau \int_0^t u(\sigma) \frac{d\varphi_1(t, \xi_x(t) - \xi_x(\sigma))}{dt}\, d\sigma dt \\
&\quad + \int_0^\tau \int_0^{x^* - \xi_x(t)} \rho_0(\sigma) \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, d\sigma dt \\
&\overset{\text{changing the order of integration}}{=} \int_0^\tau u(\sigma) \left(\int_\sigma^\tau \frac{d\varphi_1(t, \xi_x(t) - \xi_x(\sigma))}{dt}\, dt \right) d\sigma \\
&\quad + \int_0^{x^* - \xi_x(\tau)} \rho_0(\sigma) \left(\int_0^\tau \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, dt \right) d\sigma \\
&\quad + \int_{x^* - \xi_x(\tau)}^{x^*} \rho_0(\sigma) \left(\int_0^{\xi_x^{-1}(x^* - \sigma)} \frac{d\varphi_1(t, \sigma + \xi_x(t))}{dt}\, dt \right) d\sigma \\
&\overset{\text{by (4.4)}}{=} -\int_0^\tau u(\sigma) \varphi_1(\sigma, 0)\, d\sigma - \int_0^{x^* - \xi_x(\tau)} \rho_0(\sigma) \varphi_1(0, \sigma)\, d\sigma \\
&\quad - \int_{x^* - \xi_x(\tau)}^{x^*} \rho_0(\sigma) \varphi_1(0, \sigma)\, d\sigma \\
&= -\int_0^\tau u(t) \varphi_1(t, 0)\, dt - \int_0^{x^*} \rho_0(x) \varphi_1(0, x)\, dx,
\end{aligned}
$$

which immediately yields the integral identity (3.8).

Following the similar scheme, it can be shown that

$$
\begin{aligned}
&\int_0^\tau \int_{x^*}^1 \rho_2(t, x) \left[\partial_t \varphi_2(t, x) + V_2 \left(\int_x^1 \rho_2(t, y + x^* - x)\, dy \right) \partial_x \varphi_2(t, x) \right] dx dt \\
&\overset{(3.35)}{=} \lim_{z \to x} \int_0^\tau \int_{x^*}^1 \rho_2(t, x) \left[\partial_t \varphi_2(t, x) + V_2 \left(\int_z^1 \rho_2(t, y)\, dy \right) \partial_x \varphi_2(t, x) \right] dx dt \\
&= \int_0^\tau \int_0^t v(\sigma) \frac{d\varphi_2(t, \zeta_x(t) - \zeta_x(\sigma) + x^*)}{dt}\, d\sigma dt \\
&\quad + \int_0^\tau \int_{x^*}^{1 - \zeta_x(t)} \rho_0(\sigma) \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, d\sigma dt \\
&= \int_0^\tau v(\sigma) \left(\int_\sigma^\tau \frac{d\varphi_2(t, \zeta_x(t) - \zeta_x(\sigma) + x^*)}{dt}\, dt \right) d\sigma \\
&\quad + \int_{x^*}^{1 - \zeta_x(\tau)} \rho_0(\sigma) \left(\int_0^\tau \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, dt \right) d\sigma \\
&\quad + \int_{1 - \zeta_x(\tau)}^1 \rho_0(\sigma) \left(\int_0^{\zeta_x^{-1}(1 - \sigma)} \frac{d\varphi_2(t, \sigma + \zeta_x(t))}{dt}\, dt \right) d\sigma \\
&= -\int_0^\tau v(t) \varphi_2(t, x^*)\, dt + \int_{x^*}^1 \rho_0(x) \varphi_2(0, x)\, dx
\end{aligned}
$$

for all $\varphi_2 \in C([0, \tau] \times [x^*, 1])$ with properties (4.4), where the input-flux $v(t)$ is defined by (4.1). Since the inclusion $(\rho_1, \rho_2) \in C([0, \delta]; L^1(0, x^*)) \times C([0, \delta]; L^1(x^*, 1))$

is a consequence of Lemma 3.3 and representations (3.36)–(3.37), the existence result to the Cauchy problem (2.4)–(2.6) in the strip (4.3) is established. □

Theorem 4.2. *Under assumptions of Theorem 4.1, a weak solution to the Cauchy problem (2.4)–(2.6) in the strip* (4.3) *is unique.*

Proof. In order to show that the distribution (4.2) defined in Theorem 4.1 is the unique solution to this problem, we make use of some ideas from [11, Theorem 3.2]. Let us assume, by contraposition, that there exists another distribution

$$\widehat{\rho}(t,x) = \begin{cases} \widehat{\rho}_1(t,x), & \text{if } t \in [0,\delta],\ x \in [0,x^*), \\ \widehat{\rho}_2(t,x), & \text{if } t \in [0,\delta],\ x \in (x^*,1]) \end{cases} \tag{4.5}$$

such that $\rho(t,x) \neq \widehat{\rho}(t,x)$. It is worth to emphasize that, in general, it is unknown whether this function can be represented in the form like (3.36)–(3.37), because in this case Lemma 3.1 immediately leads to the conclusion

$$\int_0^x (\rho_1(t,\gamma) - \widehat{\rho}_1(t,\gamma))\,d\gamma = 0,\ \int_y^1 (\rho_2(t,\gamma + x^* - y) - \widehat{\rho}_2(t,\gamma + x^* - y))\,d\gamma = 0,$$

for all $t \in [0,T]$, almost all $x \in [0,x^*]$ and $y \in [x^*,1]$, and, therefore, $\rho_1(t,\gamma) = \widehat{\rho}_1(t,x)$ and $\rho_2(t,\gamma) = \widehat{\rho}_2(t,x)$ almost everywhere in the corresponding domains.

In view of this, we assume that $\widehat{\rho}(t,x)$ is merely a weak solution to the Cauchy problem (2.4)–(2.6) in the sense of Definition 3.1. For each $\tau \in [0,\delta]$, $\varepsilon \in (0,\tau)$, and a test function $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*])\times C^1([0,\tau]\times[x^*,1])$ with properties (see for comparison (4.4))

$$\varphi_1(t,x^*) = 0 \quad \text{and} \quad \varphi_2(t,1) = 0, \quad \forall\, t \in [0,\tau], \tag{4.6}$$

we set $\varphi_{1,\varepsilon}(t,x) := \eta_\varepsilon(t)\varphi_1(t,x)$ and $\varphi_{2,\varepsilon}(t,x) := \eta_\varepsilon(t)\varphi_2(t,x)$, where

$$\eta_\varepsilon(\tau) = 0 \ \text{ and } \ \eta_\varepsilon(t) = 1, \quad \forall\, t \in [0,\tau-\varepsilon] \ \text{ and } \ \eta'_\varepsilon(t) \le 0,\ \forall\, t \in [0,\tau]. \tag{4.7}$$

It is clear that, in this case, the new test function $(\varphi_{1,\varepsilon},\varphi_{2,\varepsilon})$ satisfies properties (4.4). Hence, by Definition 3.1, we have the equalities

$$\begin{aligned} &\int_0^\tau \int_0^{x^*} \widehat{\rho}_1(t,x)\left[\partial_t \varphi_{1,\varepsilon}(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x \varphi_{1,\varepsilon}(t,x)\right] dxdt \\ &\qquad + \int_0^\tau u(t)\varphi_{1,\varepsilon}(t,0)\,dt + \int_0^{x^*} \rho_0(x)\varphi_{1,\varepsilon}(0,x)\,dx = 0, \\ &\int_0^\tau \int_{x^*}^1 \widehat{\rho}_2(t,x)\left[\partial_t \varphi_{2,\varepsilon}(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x \varphi_{2,\varepsilon}(t,x)\right] dxdt \\ &\qquad + \int_0^\tau \widehat{v}(t)\varphi_{2,\varepsilon}(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_{2,\varepsilon}(0,x)\,dx = 0, \end{aligned}$$

where

$$\widehat{v}(t) := \widehat{\rho}_1(t,x^*)V_1\left(\int_0^{x^*} \widehat{\rho}_1(t,y)\,dy\right). \tag{4.8}$$

In view of (4.7), these relations can be rewritten as follows

$$\begin{aligned}
&\int_0^\tau \int_0^{x^*} \widehat{\rho}_1(t,x) \left[\partial_t \varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right] dxdt \\
&\qquad + \int_0^\tau u(t)\varphi_1(t,0)\,dt + \int_0^{x^*} \rho_0(x)\varphi_1(0,x)\,dx \\
&= \int_{\tau-\varepsilon}^\tau \int_0^{x^*} (1-\eta_\varepsilon)\widehat{\rho}_1(t,x) \left[\partial_t \varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right] dxdt, \\
&\qquad + \int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)u(t)\varphi_1(t,0)\,dt - \int_{\tau-\varepsilon}^\tau \eta_\varepsilon'(t)\left(\int_0^{x^*} \widehat{\rho}_1(t,x)\varphi_1(t,x)\,dx\right) dt,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\int_0^\tau \int_{x^*}^1 \widehat{\rho}_2(t,x) \left[\partial_t \varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right] dxdt \\
&\qquad + \int_0^\tau \widehat{v}(t)\varphi_2(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_2(0,x)\,dx \\
&= \int_{\tau-\varepsilon}^\tau \int_{x^*}^1 (1-\eta_\varepsilon)\widehat{\rho}_2(t,x) \\
&\qquad \times \left[\partial_t \varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right] dxdt \\
&\qquad + \int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)\widehat{v}(t)\varphi_2(t,x^*)\,dt - \int_{\tau-\varepsilon}^\tau \eta_\varepsilon'(t)\left(\int_{x^*}^1 \widehat{\rho}_2(t,x)\varphi_2(t,x)\,dx\right) dt.
\end{aligned} \tag{4.10}$$

Since $\widehat{\rho} \in C([0,\delta];L^1(0,x^*)) \times C([0,\delta];L^1(x^*,1))$, $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*]) \times C^1([0,\tau]\times[x^*,1])$, and $V_1, V_2 \in \mathfrak{A}_{ad} \subset C^1([0,\infty))$, it follows that there exists a constant D independent of ε such that

$$\begin{aligned}
&\Bigg|\int_{\tau-\varepsilon}^\tau \int_0^{x^*} (1-\eta_\varepsilon)\widehat{\rho}_1(t,x) \\
&\qquad\qquad \times \left[\partial_t \varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right] dxdt\Bigg| \le D\varepsilon,
\end{aligned}$$

$$\begin{aligned}
&\Bigg|\int_{\tau-\varepsilon}^\tau \int_{x^*}^1 (1-\eta_\varepsilon)\widehat{\rho}_2(t,x) \\
&\qquad\qquad \times \left[\partial_t \varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy\right)\partial_x\varphi_2(t,x)\right] dxdt\Bigg| \le D\varepsilon,
\end{aligned}$$

$$\left|\int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)u(t)\varphi_1(t,0)\,dt\right| \le D\varepsilon, \quad \left|\int_{\tau-\varepsilon}^\tau (1-\eta_\varepsilon)\widehat{v}(t)\varphi_2(t,x^*)\,dt\right| \le D\varepsilon.$$

At the same time, the last terms in (4.9)–(4.10) possess the following properties

$$\int_{\tau-\varepsilon}^{\tau} \eta_\varepsilon'(t) \left(\int_0^{x^*} \widehat{\rho}_1(t,x)\varphi_1(t,x)\,dx \right) dt \overset{\varepsilon\to 0}{\to} -\int_0^{x^*} \widehat{\rho}_1(\tau,x)\varphi_1(\tau,x)\,dx,$$
$$\int_{\tau-\varepsilon}^{\tau} \eta_\varepsilon'(t) \left(\int_{x^*}^{1} \widehat{\rho}_2(t,x)\varphi_2(t,x)\,dx \right) dt \overset{\varepsilon\to 0}{\to} -\int_{x^*}^{1} \widehat{\rho}_2(\tau,x)\varphi_2(\tau,x)\,dx.$$

Thus, passing to the limit in (4.9)–(4.10), we arrive at the extended integral identities for the weak solution $\widehat{\rho} \in C([0,\delta];L^1(0,x^*)) \times C([0,\delta];L^1(x^*,1))$:

$$\begin{aligned}
&\int_0^\tau \int_0^{x^*} \widehat{\rho}_1(t,x) \left[\partial_t\varphi_1(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,y)\,dy \right) \partial_x\varphi_1(t,x) \right] dxdt \\
&\quad + \int_0^\tau u(t)\varphi_1(t,0)\,dt + \int_0^{x^*} \rho_0(x)\varphi_1(0,x)\,dx \\
&- \int_0^{x^*} \widehat{\rho}_1(\tau,x)\varphi_1(\tau,x)\,dx = 0, \qquad (4.11)\\
&\int_0^\tau \int_{x^*}^{1} \widehat{\rho}_2(t,x) \left[\partial_t\varphi_2(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,y+x^*-x)\,dy \right) \partial_x\varphi_2(t,x) \right] dxdt \\
&\quad + \int_0^\tau \widehat{v}(t)\varphi_2(t,x^*)\,dt + \int_{x^*}^{1} \rho_0(x)\varphi_2(0,x)\,dx \\
&- \int_{x^*}^{1} \widehat{\rho}_2(\tau,x)\varphi_2(\tau,x)\,dx = 0. \qquad (4.12)
\end{aligned}$$

We are now in a position to specify the choice of test functions $(\varphi_1,\varphi_2) \in C^1([0,\tau]\times[0,x^*])\times C^1([0,\tau]\times[x^*,1])$ in (4.11)–(4.12) with properties (4.6). With that in mind, for given $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1,V_2 \in \mathfrak{A}_{ad}$, $x^* \in [0,1]$, and arbitrary $y \in [0,x^*]$ and $z \in [x^*,1]$, we define functions $(\widehat{\xi}_y(t),\widehat{\zeta}_z(t))$ by the rule

$$\begin{aligned}
\widehat{\xi}_y(t) &:= \int_0^t V_1\left(\int_0^y \widehat{\rho}_1(s,\sigma)\,d\sigma \right) ds,\\
\widehat{\zeta}_z(t) &:= \int_0^t V_2\left(\int_z^1 \widehat{\rho}_2(s,\sigma+x^*-z)\,d\sigma \right) ds.
\end{aligned} \qquad (4.13)$$

It is clear that these functions are monotonically increasing, $(\widehat{\xi}_y,\widehat{\zeta}_z) \in \left[C^1([0,\delta])\right]^2$, and the mappings

$$y \mapsto \widehat{\xi}_y(t) : [0,x^*] \to \mathbb{R}_+ \quad \text{and} \quad z \mapsto \widehat{\zeta}_z(t) : [x^*,1] \to \mathbb{R}_+ \qquad (4.14)$$

are continuous. Moreover, direct computations show that

$$\begin{aligned}
\frac{\partial}{\partial y}\widehat{\xi}_y(t) &:= \int_0^t V_1'\left(\int_0^y \widehat{\rho}_1(s,\sigma)\,d\sigma \right) \widehat{\rho}_1(s,y)\,ds,\\
\frac{\partial}{\partial z}\widehat{\zeta}_z(t) &:= -\int_0^t V_2'\left(\int_z^1 \widehat{\rho}_2(s,\sigma+x^*-z)\,d\sigma \right) \widehat{\rho}_2(s,x^*)\,ds
\end{aligned}$$

and, therefore, the mappings

$$y \mapsto \frac{\partial}{\partial y}\widehat{\xi}_y(t) : [0, x^*] \to \mathbb{R}_+ \quad \text{and} \quad z \mapsto \frac{\partial}{\partial z}\widehat{\zeta}_z(t) : [x^*, 1] \to \mathbb{R}_+ \tag{4.15}$$

are measurable and integrable. Thus, the mappings (4.14) are absolutely continuous.

Let (ψ_1, ψ_2) be compactly supported functions in $C_0^1([0, x^*]) \times C_0^1([x^*, 1])$. As a result, we define the test functions (φ_1, φ_2) for (4.11)–(4.12) as follows

$$\varphi_1^y(t, x) := \begin{cases} \psi_1\left(\widehat{\xi}_y(\tau) - \widehat{\xi}_y(t) + x\right), & 0 \le x \le \widehat{\xi}_y(t) - \widehat{\xi}_y(\tau) + x^*, \\ 0, & \widehat{\xi}_y(t) - \widehat{\xi}_y(\tau) + x^* \le x \le x^*, \end{cases} \quad t \in [0, \tau], \tag{4.16}$$

$$\varphi_2^z(t, x) := \begin{cases} \psi_2\left(\widehat{\zeta}_z(\tau) - \widehat{\zeta}_z(t) + x\right), & x^* \le x \le 1 - \widehat{\zeta}_z(\tau) + \widehat{\zeta}_z(t), \\ 0, & 1 - \widehat{\zeta}_z(\tau) + \widehat{\zeta}_z(t) \le x \le 1, \end{cases} \quad t \in [0, \tau]. \tag{4.17}$$

It is clear now that $(\varphi_1^y, \varphi_2^z) \in C^1([0, \delta] \times [0, x^*]) \times C^1([0, \delta] \times [x^*, 1])$ and for each $y \in [0, x^*]$ and $z \in [x^*, 1]$ these functions satisfy the Cauchy problems

$$\begin{cases} \partial_t \varphi_1^y(t, x) + V_1\left(\int_0^y \widehat{\rho}_1(t, \sigma)\, d\sigma\right) \partial_x \varphi_1^y(t, x) = 0, & (t, x) \in (0, \delta) \times (0, x^*), \\ \varphi_1^y(\tau, x) = \psi_1(x), & x \in [0, x^*], \\ \varphi_1^y(t, x^*) = 0, & t \in [0, \delta], \end{cases} \tag{4.18}$$

and

$$\begin{cases} \partial_t \varphi_2^z(t, x) + V_2\left(\int_z^1 \widehat{\rho}_2(t, \sigma + x^* - z)\, d\sigma\right) \partial_x \varphi_2^z(t, x) = 0, & (t, x) \in (0, \delta) \\ & \times (x^*, 1), \\ \varphi_2^z(\tau, x) = \psi_2(x), & x \in [x^*, 1], \\ \varphi_2^z(t, 1) = 0, & t \in [0, \delta], \end{cases} \tag{4.19}$$

respectively.

As immediately follows from (4.16)–(4.17) and properties (4.14), the mapping

$$\begin{gathered} y \mapsto \varphi_1^y(t, x) : [0, x^*] \to \mathbb{R} \quad \text{and} \quad z \mapsto \varphi_2^z(t, x) : [x^*, 1] \to \mathbb{R}, \\ y \mapsto \partial_t \varphi_1^y(t, x) : [0, x^*] \to \mathbb{R} \quad \text{and} \quad z \mapsto \partial_t \varphi_2^z(t, x) : [x^*, 1] \to \mathbb{R}, \\ y \mapsto V_1\left(\int_0^y \widehat{\rho}_1(t, \sigma)\, d\sigma\right) \partial_x \varphi_1^y(t, x) : [0, x^*] \to \mathbb{R} \quad \text{and} \\ z \mapsto V_2\left(\int_z^1 \widehat{\rho}_2(t, \sigma + x^* - z)\, d\sigma\right) \partial_x \varphi_2^z(t, x) : [x^*, 1] \to \mathbb{R} \end{gathered}$$

are continuous. Hence, the limit passage in (4.18)–(4.19) as $y \to x$ and $z \to x$ yields

$$\begin{cases} \partial_t \varphi_1^x(t,x) + V_1\left(\int_0^x \widehat{\rho}_1(t,\sigma)\,d\sigma\right)\partial_x\varphi_1^x(t,x) = 0, & (t,x)\in(0,\delta)\times(0,x^*),\\ \varphi_1^x(\tau,x) = \psi_1(x), & x\in[0,x^*],\\ \varphi_1^x(t,x^*) = 0, & t\in[0,\delta], \end{cases} \tag{4.20}$$

$$\begin{cases} \partial_t \varphi_2^x(t,x) + V_2\left(\int_x^1 \widehat{\rho}_2(t,\sigma+x^*-x)\,d\sigma\right)\partial_x\varphi_2^x(t,x) & = 0,\ t\in(0,\delta),\\ & \quad x\in(x^*,1),\\ \varphi_2^x(\tau,x) = \psi_2(x+\widehat{z}-x^*), & x\in[x^*,1+x^*-\widehat{z}],\\ \varphi_2^x(\tau,x) = 0, & x\in[1+x^*-\widehat{z},1],\\ \varphi_2^x(t,1) = 0, & t\in[0,\delta]. \end{cases} \tag{4.21}$$

As a result, we deduce from (4.11)–(4.12) that

$$\begin{aligned} 0 &= \lim_{y\to x}\int_0^\tau\int_0^{x^*}\widehat{\rho}_1(t,x)\left[\partial_t\varphi_1^y(t,x) + V_1\left(\int_0^x\widehat{\rho}_1(t,y)\,dy\right)\partial_x\varphi_1^y(t,x)\right]dxdt\\ &\quad + \lim_{y\to x}\int_0^\tau u(t)\varphi_1^y(t,0)\,dt + \lim_{y\to x}\int_0^{x^*}\rho_0(x)\varphi_1^y(0,x)\,dx\\ &\quad - \lim_{y\to x}\int_0^{x^*}\widehat{\rho}_1(\tau,x)\varphi_1^y(\tau,x)\,dx \overset{\text{by } (4.20),(4.16)}{=} \lim_{y\to x}\int_0^\tau u(t)\psi_1\left(\widehat{\xi}_y(\tau)-\widehat{\xi}_y(t)\right)dt\\ &\quad + \lim_{y\to x}\int_0^{x^*-\widehat{\xi}_y(\tau)}\rho_0(x)\psi_1\left(\widehat{\xi}_y(\tau)+x\right)dx - \int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx\\ &= \lim_{y\to x}\left[\int_0^{\widehat{\xi}_y(\tau)}\frac{u\left(\widehat{\xi}_y^{-1}(\widehat{\xi}_y(\tau)-\sigma)\right)}{\widehat{\xi}_y'\left(\widehat{\xi}_y^{-1}(\widehat{\xi}_y(\tau)-\sigma)\right)}\psi_1(\sigma)\,d\sigma + \int_{\widehat{\xi}_y(\tau)}^{x^*}\rho_0(\sigma-\widehat{\xi}_y(\tau))\psi_1(\sigma)\,d\sigma\right]\\ &\quad - \int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx = -\int_0^{x^*}\widehat{\rho}_1(\tau,x)\psi_1(x)\,dx\\ &\quad + \int_0^{\widehat{\xi}_x(\tau)}\frac{u\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(\tau)-\sigma)\right)}{\widehat{\xi}_x'\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(\tau)-\sigma)\right)}\psi_1(\sigma)\,d\sigma + \int_{\widehat{\xi}_x(\tau)}^{x^*}\rho_0(\sigma-\widehat{\xi}_x(\tau))\psi_1(\sigma)\,d\sigma, \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} 0 = \lim_{z\to x}&\int_0^\tau\int_{x^*}^1\widehat{\rho}_2(t,x)\\ &\times\left[\partial_t\varphi_2^z(t,x) + V_2\left(\int_z^1\widehat{\rho}_2(t,y+x^*-z)\,dy\right)\partial_x\varphi_2^z(t,x)\right]dxdt \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&+\lim_{z\to x}\int_0^\tau \widehat{v}(t)\varphi_2^z(t,x^*)\,dt+\lim_{z\to x}\int_{x^*}^1 \rho_0(x)\varphi_2^z(0,x)\,dx\\
&\quad-\lim_{z\to x}\int_{x^*}^1 \widehat{\rho}_2(\tau,x)\varphi_2^z(\tau,x)\,dx\\
&\overset{\text{by }(4.21),(4.17)}{=}\lim_{z\to x}\int_0^\tau \widehat{v}(t)\psi_2\left(\widehat{\zeta}_z(\tau)-\widehat{\zeta}_z(t)+x^*\right)\,dt-\int_{x^*}^1 \widehat{\rho}_2(\tau,x)\psi_2(x)\,dx\\
&+\lim_{z\to x}\int_{x^*}^{1-\widehat{\zeta}_z(\tau)} \rho_0(x)\psi_2\left(\widehat{\zeta}_z(\tau)+x\right)\,dx\\
&=\lim_{z\to x}\int_{x^*}^{x^*+\widehat{\zeta}_z(\tau)} \frac{\widehat{v}\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(\tau)+x^*-x\right)\right)}{\widehat{\zeta}_z'\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(\tau)+x^*-x\right)\right)}\psi_2(x)\,dx\\
&+\lim_{z\to x}\int_{x^*+\widehat{\zeta}_z(\tau)}^1 \rho_0(x-\widehat{\zeta}_z(\tau))\psi_2(x)\,dx-\int_{x^*}^1 \widehat{\rho}_2(\tau,x)\psi_2(x)\,dx. \qquad (4.24)
\end{aligned}
$$

Taking into account the continuity result (3.19) and the fact that functions $(\psi_1,\psi_2)\in C_0^1([0,\widehat{x}])\times C_0^1([\widehat{z},1])$ and parameter $\tau\in[0,\delta]$ were arbitrary, after localization, we can conclude from (4.22)–(4.24) the following relations

$$
\begin{aligned}
\int_0^x \widehat{\rho}_1(t,\sigma)\,d\sigma &:= \int_0^{\widehat{\xi}_x(t)} \frac{u\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(t)-x)\right)}{\widehat{\xi}_x'\left(\widehat{\xi}_x^{-1}(\widehat{\xi}_x(t)-x)\right)}\,dx+\int_{\widehat{\xi}_x(t)}^x \rho_0(\sigma-\widehat{\xi}_x(t))\,d\sigma\\
&=\int_0^t u(s)\,ds+\int_0^{x-\widehat{\xi}_x(t)} \rho_0(\sigma)\,d\sigma, \qquad (4.25)\\
\int_{x^*}^1 \widehat{\rho}_2(t,\sigma)\,d\sigma &:= \lim_{z\to x}\int_{x^*}^{x^*+\widehat{\zeta}_z(t)} \frac{\widehat{v}\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(t)+x^*-x\right)\right)}{\widehat{\zeta}_z'\left(\widehat{\zeta}_z^{-1}\left(\widehat{\zeta}_z(t)+x^*-x\right)\right)}\,dx\\
&+\lim_{z\to x}\int_{x^*+\widehat{\zeta}_z(t)}^1 \rho_0(\sigma-\widehat{\zeta}_z(t))\,d\sigma=\int_0^t \widehat{v}(s)\,ds+\int_{x^*}^{1-\widehat{\zeta}_x(t)} \rho_0(\sigma)\,d\sigma \qquad (4.26)
\end{aligned}
$$

which evidently hold true in $C([0,\delta];L^1(0,x^*))$ and $C([0,\delta];L^1(x^*,1))$, respectively. Moreover, as immediately follows from (4.26), we have

$$
\int_x^1 \widehat{\rho}_2(t,\gamma+x^*-x)\,d\gamma=\int_0^t \widehat{v}(\sigma)\,d\sigma+\int_{x^*}^{1-\widehat{\zeta}_x(t)-x+x^*} \rho_0(\gamma)\,d\gamma. \qquad (4.27)
$$

Then, combining relations (4.13), (4.27), and (4.25), we see that the functions $(\widehat{\xi}_y(t),\widehat{\zeta}_z(t))$ satisfy the Cauchy problem (3.10). Since, by Lemma 3.1, this problem has a unique solution, it follows that $\widehat{\xi}_y(t)\equiv\xi_y(t)$ and $\widehat{\zeta}_z(t)\equiv\zeta_z$ as elements of $C^1([0,\delta])$. Hence, $\rho=\widehat{\rho}$ by comparing (4.27) and (4.25) with (3.36) and (3.37). Thus, a weak solution to the Cauchy problem (2.4)–(2.6) is unique for small time. □

As a consequence of Theorem 4.1, we have the following hidden regularity property of the weak solutions.

Corollary 4.1. *Let $\rho = (\rho_1, \rho_2) \in C([0,\tau]; L^1(0,x^*)) \times C([0,\tau]; L^1(x^*,1))$ be a weak solution to the Cauchy problem (2.4)–(2.6) for some $\tau \in (0,T]$. Then for given $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1, V_2 \in \mathfrak{A}_{ad}$, and $x^* \in [0,1]$, we have*

$$(\rho_1, \rho_2) \in C([0,x^*]; L^1(0,\tau)) \times C([x^*,1]; L^1(0,\tau)). \tag{4.28}$$

Proof. Let $x \in (0,x^*)$ be an arbitrary point. Then, by the first mean value theorem for integration, we get

$$\begin{aligned} A(x) &:= \int_0^\tau \rho_1(t,x)\,dt = \lim_{\Delta\to 0} \frac{1}{2\Delta} \int_0^\tau \left(\int_{x-\Delta}^{x+\Delta} \rho_1(t,y)\,dy \right) dt = \{\text{by } (3.35)\} \\ &= \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \int_0^\tau \left(\int_{x-\Delta}^{x+\Delta} \widetilde{\rho}_{1,z}(t,y)\,dy \right) dt. \end{aligned} \tag{4.29}$$

As follows from (3.21), we have the following representation

$$\begin{aligned} &\int_{x-\Delta}^{x+\Delta} \widetilde{\rho}_{1,z}(t,y)\,dy \\ &= \begin{cases} \displaystyle\int_{x-\Delta}^{x+\Delta} \rho_0(y - \xi_z(t))\,dy, & 0 < t < \xi_z^{-1}(x-\Delta), \\ \displaystyle\int_{x-\Delta}^{\xi_z(t)} \frac{u\left(\xi_z^{-1}\left(\xi_z(t) - y\right)\right)}{\xi_z'\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}\,dy \\ \qquad + \displaystyle\int_{\xi_z(t)}^{x+\Delta} \rho_0(y-\xi_z(t))\,dy, & \xi_z^{-1}(x-\Delta) < t < \xi_z^{-1}(x+\Delta), \\ \displaystyle\int_{x-\Delta}^{x+\Delta} \frac{u\left(\xi_z^{-1}\left(\xi_z(t) - y\right)\right)}{\xi_z'\left(\xi_z^{-1}\left(\xi_z(t)-y\right)\right)}\,dy, & \xi_z^{-1}(x+\Delta) < t < \tau, \end{cases} \\ &= \begin{cases} \displaystyle\int_{x-\Delta-\xi_z(t)}^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma, & 0 < t < \xi_z^{-1}(x-\Delta), \\ \displaystyle\int_0^{\xi_z^{-1}(\xi_z(t)-x+\Delta)} u(s)\,ds \\ \qquad + \displaystyle\int_0^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma, & \xi_z^{-1}(x-\Delta) < t < \xi_z^{-1}(x+\Delta), \\ \displaystyle\int_{\xi_z^{-1}(\xi_z(t)-x+\Delta)}^{\xi_z^{-1}(\xi_z(t)-x-\Delta)} u(s)\,ds, & \xi_z^{-1}(x+\Delta) < t < \tau. \end{cases} \end{aligned} \tag{4.30}$$

In view of (4.29), we can conclude from (4.30) that

$$\begin{aligned} A(x) = \lim_{\Delta\to 0} \lim_{z\to x} \frac{1}{2\Delta} \Bigg[&\int_0^{\xi_z^{-1}(x-\Delta)} \int_{x-\Delta-\xi_z(t)}^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\,d\sigma dt \\ &+ \int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta)} \int_0^{\xi_z^{-1}(\xi_z(t)-x+\Delta)} u(s)\,ds dt \end{aligned}$$

$$
\begin{aligned}
&+\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta)} \int_0^{x+\Delta-\xi_z(t)} \rho_0(\sigma)\, d\sigma dt \\
&+\int_{\xi_z^{-1}(x+\Delta)}^{\tau} \int_{\xi_z^{-1}(\xi_z(t)-x+\Delta)}^{\xi_z^{-1}(\xi_z(t)-x-\Delta)} u(s)\, ds\, dt\Bigg] \\
&= A_1(x)+A_2(x)+A_3(x)+A_4(x). \qquad (4.31)
\end{aligned}
$$

Changing the order of integration in each terms of (4.31), we arrive at the following relations

$$
\begin{aligned}
A_1(x) &= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\left[\int_0^{2\Delta}\left(\int_{\xi_z^{-1}(x-\Delta-\sigma)}^{\xi_z^{-1}(x-\Delta)} dt\right)\rho_0(\sigma)\, d\sigma\right. \\
&+\int_{2\Delta}^{x-\Delta}\left(\int_{\xi_z^{-1}(x-\Delta-\sigma)}^{\xi_z^{-1}(x+\Delta-\sigma)} dt\right)\rho_0(\sigma)\, d\sigma \\
&+\left.\int_{x-\Delta}^{x+\Delta}\left(\int_0^{\xi_z^{-1}(x+\Delta-\sigma)} dt\right)\rho_0(\sigma)\, d\sigma\right] \\
&= \lim_{\Delta\to 0}\lim_{z\to x}\left[\int_0^{2\Delta}\frac{\xi_z^{-1}(x-\Delta)-\xi_z^{-1}(x-\Delta-\sigma)}{2\Delta}\rho_0(\sigma)\, d\sigma\right. \\
&\qquad +\int_{2\Delta}^{x-\Delta}\frac{\xi_z^{-1}(x+\Delta-\sigma)-\xi_z^{-1}(x-\Delta-\sigma)}{2\Delta}\rho_0(\sigma)\, d\sigma \\
&\qquad +\left.\int_{x-\Delta}^{x+\Delta}\frac{\xi_z^{-1}(x+\Delta-\sigma)}{2\Delta}\rho_0(\sigma)\, d\sigma\right] \overset{\text{by Lemmas 3.1 and 3.2}}{=} \\
&= \lim_{\Delta\to 0}\int_{2\Delta}^{x-\Delta}\frac{\xi_z^{-1}(x+\Delta-\sigma)-\xi_z^{-1}(x-\Delta-\sigma)}{2\Delta}\rho_0(\sigma)\, d\sigma \\
&= \int_0^x \left.\frac{d\xi_x^{-1}(y)}{dy}\right|_{y=x-\sigma}\rho_0(\sigma)\, d\sigma, \qquad (4.32)
\end{aligned}
$$

$$
\begin{aligned}
A_2(x) &= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\int_0^{2\Delta}\left(\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(\xi_z(s)+x-\Delta)} dt\right)u(s)\, ds \\
&= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\int_0^{2\Delta}\left(\xi_z^{-1}(\xi_z(s)+x-\Delta)-\xi_z^{-1}(x-\Delta)\right)u(s)\, ds = 0, \qquad (4.33)
\end{aligned}
$$

$$
\begin{aligned}
A_3(x) &= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\int_0^{2\Delta}\left(\int_{\xi_z^{-1}(x-\Delta)}^{\xi_z^{-1}(x+\Delta-\sigma)} dt\right)\rho_0(\sigma)\, d\sigma \\
&= \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\int_0^{2\Delta}\left(\xi_z^{-1}(x+\Delta-\sigma)-\xi_z^{-1}(x-\Delta)\right)\rho_0(\sigma)\, d\sigma = 0, \qquad (4.34)
\end{aligned}
$$

$$
A_3(x) = \lim_{\Delta\to 0}\lim_{z\to x}\frac{1}{2\Delta}\left[\int_0^{\xi_z^{-1}(2\Delta)}\left(\int_{\xi_z^{-1}(\xi_z(s)+x-\Delta)}^{\xi_z^{-1}(x+\Delta)} dt\right)u(s)\, ds\right.
$$

$$+ \int_{\xi_z^{-1}(2\Delta)}^{\xi_z^{-1}(\xi_z(\tau)-x-\Delta)} \left(\int_{\xi_z^{-1}(\xi_z(s)+x-\Delta)}^{\xi_z^{-1}(\xi_z(s)+x+\Delta)} dt \right) u(s)\, ds \Bigg]$$
$$= \lim_{\Delta\to 0} \frac{1}{2\Delta} \left[\int_{\xi_x^{-1}(2\Delta)}^{\xi_x^{-1}(\xi_x(\tau)-x-\Delta)} \left(\int_{\xi_x^{-1}(\xi_x(s)+x-\Delta)}^{\xi_x^{-1}(\xi_x(s)+x+\Delta)} dt \right) u(s)\, ds \right]$$
$$= \int_0^{\xi_z^{-1}(\xi_z(\tau)-x)} \left.\frac{d\xi_x^{-1}(y)}{dy}\right|_{y=\xi_x(s)+x} u(s)\, ds. \qquad (4.35)$$

Combining results (4.31)–(4.35), we finally get

$$\int_0^\tau \rho_1(t,x)\, dt = \int_0^x \left.\frac{d\xi_x^{-1}(y)}{dy}\right|_{y=x-\sigma} \rho_0(\sigma)\, d\sigma + \int_0^{\xi_z^{-1}(\xi_z(\tau)-x)} \left.\frac{d\xi_x^{-1}(y)}{dy}\right|_{y=\xi_x(s)+x} u(s)\, ds, \quad x \in [0,x^*]. \qquad (4.36)$$

By analogy, it can be shown that

$$\int_0^\tau \rho_2(t,x)\, dt = \int_{x^*}^x \left.\frac{d\zeta_x^{-1}(y)}{dy}\right|_{y=x-x^*-\sigma} \rho_0(\sigma)\, d\sigma + \int_0^{\zeta_z^{-1}(\zeta_z(\tau)-x+x^*)} \left.\frac{d\zeta_x^{-1}(y)}{dy}\right|_{y=\zeta_x(s)+x-x^*} v(s)\, ds, \quad x \in [x^*,1]. \qquad (4.37)$$

It is worth to note that $\xi_x^{-1} \in C^1([0,\xi_x(\tau)])$ and $\zeta_x^{-1} \in C^1([0,\zeta_x(\tau)])$ because $(\xi_x,\zeta_z) \in \left[C^1([0,\delta])\right]^2$ are monotonically increasing functions. Hence, to conclude the proof, it remains to apply the arguments of Lemma 3.3 to relations (4.36)–(4.37). □

Remark 4.1. Taking into account the fact that

$$\left(\xi_x^{-1}(y)\right)' = \frac{1}{\xi_x'\left(\xi_x^{-1}(y)\right)}, \quad \left(\zeta_x^{-1}(y)\right)' = \frac{1}{\zeta_x'\left(\zeta_x^{-1}(y)\right)}, \qquad (4.38)$$

and by Lemma 3.1 and relations (3.36)-(3.37)

$$\xi_x'\left(\xi_x^{-1}(y)\right) = V_1\left(\int_0^x \rho_1(\xi_x^{-1}(y),\gamma)\, d\gamma\right), \qquad (4.39)$$
$$\zeta_x'\left(\zeta_x^{-1}(y)\right) = V_2\left(\int_x^1 \rho_2(\zeta_x^{-1}(y),\gamma+x^*-x)\, d\gamma\right),$$

it is easy to deduce from definition of the set $\mathfrak{A}_{ad}$ and representations (4.36)–(4.37) the following estimates

$$\|\rho_1(\cdot,x)\|_{L^1(0,\tau)} := \int_0^\tau \rho_1(t,x)\,dt \le \alpha_1^{-1}\left[\|u\|_{L^1(0,\tau)} + \|\rho_0\|_{L^1(0,x^*)}\right] \quad \forall\, x\in[0,x^*], \tag{4.40}$$

$$\|\rho_2(\cdot,x)\|_{L^1(0,\tau)} := \int_0^\tau \rho_2(t,x)\,dt \le \alpha_1^{-1}\left[\|v\|_{L^1(0,\tau)} + \|\rho_0\|_{L^1(x^*,1)}\right] \quad \forall\, x\in[x^*,1]. \tag{4.41}$$

We are now in a position to prove the main result of this section.

Theorem 4.3. *Let $\rho_0\in L^\infty(0,1)$, $u\in L^1(0,T)$, $V_1,V_2\in\mathfrak{A}_{ad}$, and $x^*\in[0,1]$ be given. Then the Cauchy problem (2.4)–(2.6) admits a unique global solution*

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & \text{if } t\in[0,T],\ x\in[0,x^*),\\ \rho_2(t,x), & \text{if } t\in[0,T],\ x\in(x^*,1]) \end{cases} \tag{4.42}$$

such that

$$\begin{aligned}(\rho_1,\rho_2)&\in C([0,T];L^1(0,x^*))\times C([0,T];L^1(x^*,1)),\\ (\rho_1,\rho_2)&\in C([0,x^*];L^1(0,T))\times C([x^*,1];L^1(0,T)).\end{aligned} \tag{4.43}$$

Proof. As follows from Theorem 4.1, there exists a value $\delta\in(0,T]$ such that the Cauchy problem (2.4)–(2.6) is uniquely solvable in the strip $(t,x)\in[0,\delta]\times(0,1)$. Moreover, in view of representation (3.36)–(3.37), we have the following estimates for the weak solution $(\rho_1,\rho_2)\in C([0,\delta];L^1(0,x^*))\times C([0,\delta];L^1(x^*,1))$

$$\begin{aligned}0\le \int_0^{x^*}\rho_1(t,\gamma)\,d\gamma &= \int_0^t u(\sigma)\,d\sigma + \int_0^{x^*-\xi_{x^*}(t)}\rho_0(\gamma)\,d\gamma\\ &\le \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}\end{aligned} \tag{4.44}$$

$$\begin{aligned}0\le \int_{x^*}^{1}\rho_2(t,\gamma)\,d\gamma &= \int_0^t v(\sigma)\,d\sigma + \int_{x^*}^{1-\zeta_{x^*}(t)}\rho_0(\gamma)\,d\gamma\\ &\le \|v\|_{L^1(0,\delta)} + \|\rho_0\|_{L^\infty(0,1)}\end{aligned} \tag{4.45}$$

for all $t\in[0,\delta]$. In order to estimate the term

$$\|v\|_{L^1(0,\delta)} := \int_0^\delta \rho_1(t,x^*)V_1\left(\int_0^{x^*}\rho_1(t,y)\,dy\right)dt, \tag{4.46}$$

we apply the inequality (4.40). Then

$$\|v\|_{L^1(0,\delta)} \le \alpha_2\int_0^\delta \rho_1(t,x^*)\,dt \le \frac{\alpha_2}{\alpha_1}\left[\|u\|_{L^1(0,T)} + \|\rho_0\|_{L^1(0,x^*)}\right].$$

Since $\|\rho_0\|_{L^1(0,x^*)}\le\|\rho_0\|_{L^\infty(0,1)}$, we finally get

$$\|\rho_1\|_{C([0,\delta];L^1(0,x^*)}\le \|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}, \tag{4.47}$$

$$\|\rho_2\|_{C([0,\delta];L^1(x^*,1)}\le \frac{2\alpha_2}{\alpha_1}\left[\|u\|_{L^1(0,T)} + \|\rho_0\|_{L^\infty(0,1)}\right]. \tag{4.48}$$

Since both the a priori estimates for weak solution (ρ_1, ρ_2) and the choice rule (3.18) do not depend on δ, it follows that the weak solution

$$(\rho_1, \rho_2) \in C([0, \delta]; L^1(0, x^*)) \times C([0, \delta]; L^1(x^*, 1))$$

can be extended to the next time interval $[\delta, 2\delta] \cap [0, T]$. Hence, following this iterative procedure, we finally find a unique global solution

$$(\rho_1, \rho_2) \in C([0, T]; L^1(0, x^*)) \times C([0, T]; L^1(x^*, 1)).$$

It remains to note that inclusion $(4.43)_2$ is a direct consequence of Corollary 4.1. □

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