On some smooth approximations on thick periodic multi-structures

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Summary. In this paper we study the approximation properties of measurable and square-integrable functions. In particular we show that any \( L^2 \)-bounded functions can be approximated in \( L^2 \)-norm by smooth functions defined on a highly oscillating boundary of thick multi-structures in \( \mathbb{R}^n \). We derive the norm estimates for the approximating functions and study their asymptotic behavior.

Key words: thick multi-structure, approximating properties, singular measures, convergence in variable spaces

1 Introduction

The main object of our consideration in this paper is a class of functions defined on a domain \( \Omega_\varepsilon \subset \mathbb{R}^n \), whose boundary \( \partial \Omega_\varepsilon \) contains the very highly oscillating part with respect to a small parameter \( \varepsilon \), as \( \varepsilon \to 0 \).

We say that \( \Omega_\varepsilon \) is a thick multi-structure in \( \mathbb{R}^n \), if \( \Omega_\varepsilon \) consists of some fixed domain \( \Omega^+ \) and a large number of cylinders with axes parallel to \( Ox_n \) and \( \varepsilon \)-periodically distributed along some manifold \( \Sigma \) on the boundary of \( \Omega^+ \) (see Fig. 1). This manifold is called the joint zone and the domain \( \Omega^+ \) is called the junction’s body. Here \( \varepsilon \) is a small positive parameter, which characterizes the distance between the neighboring cylinders and their thickness. So, each attached cylinder has a small cross section of the size \( \varepsilon \) and its limiting dimension (as \( \varepsilon \to 0 \)) is equal to 1. In view of this, such cylinders will be called thin domains.

Thick junctions have a special character of the connectedness: there are points in a thick junction which are at a short distance of order \( O(\varepsilon) \), but the length of all curves, which connect these points in the junction, is order \( O(1) \). As a result, there are no extension operators \( P : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}_{loc}(\mathbb{R}^n) \) that would be uniformly bounded in \( \varepsilon \). At the same time thick multi-structures (or thick junctions) are prototypes of widely used engineering constructions as
well as many other physical and biological systems with very distinct characteristic scales (microscopic radiators, ferrite-filled rod radiators, and others). As a rule, the computational calculation of the solutions of any problems on $\Omega_\varepsilon$ is very complicated due to singularities of the thick junctions. As a result, asymptotic analysis is one of the main approaches to study majority of problems in such domains.

In view of this we consider the approximation problem of $L^2$-bounded functions by sequences of smooth functions defined on lateral side of thin cylinders and their bases which form a quickly oscillating counterpart of thick junctions. We derive the norm estimates for the approximating functions and study their asymptotic behavior.

2 Notation and Preliminaries

Let $a > 0$, $d > 0$, and $c > 0$ be given positive constants. Let $B = (0, a)^{n-1}$ and $C$ be bounded open smooth domain in $\mathbb{R}^{n-1}$ ($n \geq 2$). Moreover we assume that $C \subset (0,1)^{n-1}$, that is the set $C$ belongs to the $n-1$-dimensional cube $(0,1)^{n-1}$ together with its boundary $\partial C$. Throughout this paper the parameter $\varepsilon$ varies in a strictly decreasing sequence of positive numbers which converges to zero, i.e. we can suppose that $\varepsilon = a/N$, where $N$ is a large positive integer. When we write $\varepsilon > 0$, we consider only the elements of this sequence.

Let us denote the elements of $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$ and introduce the following sets: $\theta_\varepsilon = \{ k = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{N}^{n-1} : \varepsilon C + \varepsilon k \subset B \}$,
\[ \Omega = B \times (-d, c), \quad G_{\varepsilon}^{k} = \{(x', x_n) : x' \in \varepsilon C + \varepsilon k, \ -d < x_n \leq 0\}, \]
\[ \Sigma = B \times \{0\}, \quad \Omega^{+} = B \times (0, c), \quad \Omega^{-} = B \times (-d, 0), \]
\[ \Gamma_0 = B \times \{-d\}, \quad \Omega_{\varepsilon} = \Omega \cap \Omega_{\varepsilon}. \]

Then in view of our previous description, the set

\[ \Omega_{\varepsilon} = (B \times (0, c)) \cup \left( \bigcup_{k \in \theta_{\varepsilon}} (\varepsilon C + \varepsilon k) \times (-d, 0) \right) = \Omega^{+} \cup \left( \bigcup_{k \in \theta_{\varepsilon}} G_{\varepsilon}^{k} \right), \]

is a thick multi-structure in \( \mathbb{R}^n \), which consists of the junction’s body \( \Omega^{+} \) and a large number \( N^{n-1} \) of the thin cylinders \( G_{\varepsilon}^{k} \) with axis \( O x_n \) and \( \varepsilon \)-periodically distributed on the basis \( \Sigma \) of \( \Omega^{+} \) (see Fig. 1 for 3-d example). Here, each cylinder \( G_{\varepsilon}^{k} \) is obtained with \( \varepsilon \)-homothety in the first \( (n-1) \) variables. It is easy to see that \( \Omega_{\varepsilon} = \Omega^{+} \cup \left( \bigcup_{k \in \theta_{\varepsilon}} G_{\varepsilon}^{k} \right) \).

Denote by \( \Gamma_{\varepsilon} \) the union of the lower bases

\[ \Gamma_{\varepsilon}^{k} = \{(x', x_n) : x' \in \varepsilon \cdot C + \varepsilon k, \ x_n = -d\} \]

of the thin cylinders \( G_{\varepsilon}^{k} \) when \( k \in \theta_{\varepsilon} \) (i.e. \( \Gamma_{\varepsilon} = \Gamma_0 \cap \partial \Omega_{\varepsilon} \)), and by \( S_{\varepsilon} \) the union of their boundaries along the axis \( O x_n \):

\[ S_{\varepsilon}^{k} = \{(x', x_n) : x' \in \varepsilon \cdot \partial C + \varepsilon k, \ -d < x_n < 0\}, \]

We make also use of the idea of classical smoothing. Let \( K(x) \) be a positive compactly supported smooth function (\( K \in C_0^\infty \)) such that \( \int_{\mathbb{R}^n} K(x) \, dx = 1 \) and \( K(x) = K(-x) \) for all \( x \in \mathbb{R}^n \). For an arbitrary measurable function \( \varphi \in L_0^2(\mathbb{R}^n) \) we define an ordinary smoothing operator by the equality

\[ (\varphi)_h(x) = h^{-n} \int_{\mathbb{R}^n} K\left(\frac{x - y}{h}\right) \varphi(y) \, dy = \int_{\mathbb{R}^n} K\left(y\right) \varphi(x + hy) \, dy. \]

3 On smooth \( \Gamma_{\varepsilon} \)-approximation of \( L^2(\Gamma_0) \)-functions

Let \( u^0 \in L^2(\Gamma_0) \) be a given function such that \( \|u^0\|_{L^2(\Gamma_0)} \leq C_u \) with some constant \( C_u > 0 \). Our aim is to construct a sequence of smooth functions \( \{u_{\varepsilon} \in C_0^\infty(\Gamma_\varepsilon)\}_{\varepsilon > 0} \) satisfying the following conditions:

\[ \|u_{\varepsilon}\|_{L^2(\Gamma_\varepsilon)} \leq \left(\sqrt{|C|}\right)^{-1} C_u, \quad \forall \varepsilon > 0, \quad (3) \]
\[ \bar{u}_\varepsilon \to u^0 \text{ in } L^2(\Gamma_0) \text{ as } \varepsilon \to 0. \quad (4) \]

Here \( |C| \) is the \( (n-1) \)-dimensional Lebesgue measure of the set \( C \), and \( \bar{u}_\varepsilon \) is the trivial extension by zero of a function \( u_{\varepsilon} \in L^2(\Gamma_{\varepsilon}) \) to the set \( \Gamma_0 \). It is clear that in this case \( \bar{u}_\varepsilon \in L^2(\Gamma_0) \).
Remark 1. Note that for every $\varepsilon > 0$ the restriction $u^0|_{T^\varepsilon}$ is an element of $L^2(T^\varepsilon)$ whereas the functions $u_\varepsilon$ in (3)–(4) must be elements of $C^\infty(T^\varepsilon)$. In view of this we call the sequence $\{u_\varepsilon \in C^\infty(T^\varepsilon)\}_{\varepsilon > 0}$ with properties (3)–(4) a smooth $T^\varepsilon$-approximation of $u^0 \in L^2(T_0)$.

In order to construct the sequence $\{u_\varepsilon \in C^\infty(T^\varepsilon)\}_{\varepsilon > 0}$, we make use the idea of classical smoothing. Let $K_u \in C^\infty_0(\mathbb{R}^{n-1})$ be nonnegative function such that

$$\int_{\mathbb{R}^{n-1}} K_u(x') \, dx' = 1 \text{ and } K_u(x') = K_u(-x') \quad \forall x' \in \mathbb{R}^{n-1}.$$

Let $\tilde{u}^0$ be the trivial extension by zero of function $u^0 \in L^2(T_0)$ to the set $O = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n = -\beta_0\}$. For every fixed $\varepsilon > 0$ we define the following smoothing operator

$$(u^0)_\varepsilon = (K_\varepsilon u^0)(x') = \varepsilon^{-n+1} \int_O K_u \left( \frac{x' - y'}{\varepsilon} \right) \tilde{u}^0(y', y_n) \, dH^{n-1}$$

$$= \varepsilon^{-n+1} \int_{\mathbb{R}^{n-1}} K_u \left( \frac{x' - y'}{\varepsilon} \right) \tilde{u}^0(y') \, dy'.$$  \hspace{1cm} (5)

We begin with the following result:

**Proposition 1.** Let $u^0 \in L^2(T_0)$ be a given function such that $\|u^0\|_{L^2(T_0)} \leq C_u$. Then for every $\delta > 0$ there exists $\varepsilon(\delta) = a/N$, where $N \in \mathbb{N}$, such that

$$\|u_\varepsilon\|_{L^2(T^\varepsilon)} \leq \left( \sqrt{|C|} \right)^{-1} C_u$$

provided

$$u_\varepsilon := |C|^{-1} (u^0)_\delta|_{T^\varepsilon}. \hspace{1cm} (6)$$

**Proof.** First of all we note that $u_\varepsilon \in C^\infty(T^\varepsilon)$ by the properties of smoothing operator (5). Following the definition of the smoothing operator and using the Cauchy-Bunyakovskii inequality, we have
\[ \left\| (u^0)_\delta \right\|_{L^2(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} \left( \delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u \left( \frac{x' - y'}{\delta} \right) \bar{u}^0(y') \, dy' \right)^2 \, dx' \]
\[ \leq \int_{\mathbb{R}^{n-1}} \left( \delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u \left( \frac{x' - y'}{\delta} \right) \, dy' \right) \times \left( \delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u \left( \frac{x' - y'}{\delta} \right) \bar{u}^0(y')^2 \, dy' \right) \, dx' \]
\[ = \int_{\mathbb{R}^{n-1}} \left( \delta^{-n+1} \int_{\mathbb{R}^{n-1}} K_u \left( \frac{x' - y'}{\delta} \right) \bar{u}^0(y')^2 \, dy' \right) \, dx' \]
\[ \leq \int_{\mathbb{R}^{n-1}} \left( \bar{u}^0(y')^2 \right) \, dy' = \int_{\Gamma_0} (u^0(y'))^2 \, dy' = \| u^0 \|^2_{L^2(\Gamma_0)} \quad (8) \]

Due to this estimate, we come to the inequality
\[ \left\| (u^0)_\delta \right\|_{L^2(\Gamma_\varepsilon)} = \left\| \chi_{\Gamma_0} (u^0)_\delta \right\|_{L^2(\Gamma_\varepsilon)} \leq \left\| \chi_{\Gamma_0} (u^0)_\delta \right\|_{L^2(\mathbb{R}^{n-1})} \leq \| u^0 \|_{L^2(\Gamma_0)} \leq C_u \quad (9) \]

which holds true for every \( \varepsilon > 0 \). Here \( \chi_{\Gamma_0} \in L^\infty(\Gamma_0) \) is the characteristic function of the zone \( \Gamma_\varepsilon \), i.e.,
\[ \chi_{\Gamma_\varepsilon}(x') = \begin{cases} 1, & x' \in \Gamma_\varepsilon, \\ 0, & \text{otherwise} \end{cases} \quad \forall x' \in \Gamma_0. \]

Note that
\[ \chi_{\Gamma_\varepsilon} \xrightarrow{\varepsilon \to 0} |C| \quad \text{in} \quad L^\infty(\Gamma_0) \quad \text{as} \quad \varepsilon \to 0. \quad (10) \]

Indeed, in view of \( \varepsilon [0,1]^{n-1} \)-periodicity of this characteristic function and the mean value property, we have
\[ \lim_{\varepsilon \to 0} \int_{\Gamma_0} \varphi(x') \chi_{\Gamma_\varepsilon}(x') \, d\mathcal{H}^{n-1} = \lim_{\varepsilon \to 0} \int_B \varphi(x') \chi_{\Gamma_\varepsilon \in \Theta_{\varepsilon}(\varepsilon \mathbb{C} + \mathbb{C} \kappa)}(x') \, dx' \]
\[ = \int_B \varphi(x') \, dx' \left( \int_{(0,1)^{n-1}} \chi_{\Gamma_\varepsilon}(x') \, dx' \right) \]
\[ = |C| \int_B \varphi(x') \, dx' = |C| \int_{\Gamma_0} \varphi(x') \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_0^\infty(\Gamma_0). \]

Thus, the property (10) holds true.

Further for a given \( \delta > 0 \) we define a parameter \( \varepsilon = a/N \) so that the following inequality would be valid
\[ \frac{1}{|\Gamma_\varepsilon|} \int_{\Gamma_\varepsilon} (u^0)^2 \, dx' \leq \frac{1}{|\Gamma_0|} \int_{\Gamma_0} (u^0)^2 \, dx'. \]
Then from (9) we deduce

\[ \| |C|^{-1} (u^0)_{\delta} \|^2_{L^2(\Gamma_\varepsilon)} \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} \| |C|^{-1} (u^0)_{\delta} \|^2_{L^2(\Gamma_0)} \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} \|C\|^{-2} C u^2. \] (11)

Since \(|\Gamma_0| = |B| = a^{n-1}\) and

\[ |\Gamma_\varepsilon| = \sum_{\mathbf{k} \in \Theta} \int_{\mathbb{R}^n} dx = \sum_{\mathbf{k} \in \Theta} \varepsilon^{n-1} |C| = \left( \frac{a}{\varepsilon} \right)^{n-1} \varepsilon^{n-1} |C| = a^{n-1} |C|, \]

it follows from (11) that

\[ \| u_\varepsilon \|_{L^2(\Gamma_\varepsilon)} \equiv \| |C|^{-1} (u^0)_{\delta} \|_{L^2(\Gamma_\varepsilon)} \leq \left( \sqrt{|C|} \right)^{-1} C u. \] (12)

Thus for the chosen value of \(\varepsilon\) inequality (6) is valid. This concludes the proof. \(\square\)

The next step to prove the property of strong convergence (4).

**Proposition 2.** Assume the functions \(u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\) are defined by (7). Then the property (4) holds true.

**Proof.** To begin with, we note that

\[ (u^0)_{\delta} \rightarrow u^0 \text{ strongly in } L^2(\Gamma_0) \text{ as } \delta \rightarrow 0 \] (13)

by properties of the classical smoothing. Hence in view of (10), (13), and the fact that the value of \(\varepsilon\) depends on a given parameter \(\delta\), we have

\[ \tilde{u}_{\varepsilon(\delta)} = |C|^{-1} \chi_{\Gamma_\varepsilon(\delta)} (u^0)_{\delta} \rightarrow |C|^{-1} |C| u^0 = u^0 \text{ as } \delta \rightarrow 0 \]

as a limit of the product of the weak and strong convergent sequences in \(L^2(\Gamma_0)\). This implies the required property (4). The proof is complete. \(\square\)

### 4 Some auxiliary results

We begin this section with the description of the \((n-1)\)-dimensional surface \(S_\varepsilon\) in the terms of singular measures in \(\mathbb{R}^n\) which do not satisfy the regularity property with respect to the corresponding Lebesgue measure.

Let \(\mu_0\) be a periodic finite positive Borel measure in \(\mathbb{R}^{n-1}\). Let \(\square = [0,1)^{n-1}\) be the cell or torus of periodicity for \(\mu_0\). We assume that Borel measure \(\mu_0\) is the probability measure in \(\mathbb{R}^{n-1}\), concentrated and uniformly distributed on the set \(\partial C\), so \(\int_{\square} d\mu_0 = 1\).
Remark 2. By definition we have $\mu_0(\square \setminus \partial C) = 0$. Therefore any functions, taking the same values on the set $\partial C$, coincide as elements of $L^2(\square, d\mu_0)$. Here the Lebesgue space $L^2(\square, d\mu_0)$ with respect to the measure $\mu_0$ is defined in a usual way with the corresponding norm

$$
\|f\|_{L^2(\square, d\mu_0)}^2 = \int_\square |f(x)|^2 \, d\mu_0
$$

(we adopt the standard notation $L^2(\square)$ when $\mu_0$ is the Lebesgue measure).

We set $\square_n = \square \times [0,1) = [0,1]^n$ and consider the following measure $d\mu = d\mu_0 \times dx_n$ on $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1)$, and for any smooth function $g$ we have

$$
\int_{\square_n} g \, d\mu = \int_0^1 \int_{\square} \, dx_n \, d\mu_0 = \left[ \mathcal{H}^{n-1}(\partial C \times [0,1]) \right]^{-1} \int_{\partial C \times [0,1)} g \, d\mathcal{H}^{n-1}.
$$

However, as follows from the properties of the Hausdorff measure, we have $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$ (see [1]). In what follows, we use the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$ for $(n-2)$-dimensional Hausdorff measure of the set $\partial C$. Then the previous relation can be rewritten in the form

$$
\int_{\square_n} g \, d\mu = \int_0^1 \int_{\square} \, dx_n \, d\mu_0 = \frac{1}{|\partial C|_H} \int_{\partial C \times [0,1)} g \, d\mathcal{H}^{n-1}.
$$

In order to clarify relation (14), we consider the planar thick multi-structure $\Omega_\varepsilon \subset \mathbb{R}$. Then $n = 2$ and the set $C$ is some part of the segment $(0,1)$. For instance, let $C = \{x_1 \in (0,1) : |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed value. In this case $|\partial C|_H = 2$ and the 1-periodic measure $\mu_0$ in $\mathbb{R}^1$ can be defined as

$$
\mu_0(B) = \frac{1}{|\partial C|_H} (\delta_{M_1} + \delta_{M_2})(B) = \frac{1}{2} (\delta_{M_1} + \delta_{M_2})(B)
$$

for any Borel set $B \subseteq [0,1)$, where $M_i = \frac{i}{2} + \left( \frac{i}{2} \right) h$, $i = 1, 2$. Here $\delta_{M_i}$ is the Dirac measures located at the points $M_i$. Thus, the multiplier $|\partial C|_H^{-1}$ in (14) is equal to $1/2$.

Let $A$ be any Borel set of $\mathbb{R}^n$. We introduce the scaling measure $\mu_\varepsilon$ by the rule $\mu_\varepsilon(A) = \varepsilon^n \mu(\varepsilon^{-1} A)$. This measure has a period $\varepsilon$, and moreover, since $\mu(\varepsilon \square_n) = \varepsilon \mu_0(\varepsilon \square)$, it follows that

$$
\mu_\varepsilon(\varepsilon \square_n) = \varepsilon^n \int_{\square} \int_{\varepsilon \square} \, dx_n \, d\mu_0(x') = \varepsilon^n \int_{\square} \, dx_n = \varepsilon^n.
$$

As a result, the measure $\mu_\varepsilon$ weakly converges to Lebesgue measure in $\mathbb{R}^n$ as $\varepsilon \to 0$ (in symbols $d\mu_\varepsilon \rightharpoonup dx$), that is,

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi \, d\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi \, dx \text{ for all functions } \varphi \in C_0^\infty(\mathbb{R}^n).
$$
Now, we give the convergence formalism of the sequence of boundary controls \( \{ \hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon) \}_{\varepsilon > 0} \). First of all we recall that the sequence of Borel measures \( \{ d\mu_\varepsilon \} \) weakly converges to Lebesgue measure \( dx \). Let \( \{ \hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon) \}_{\varepsilon > 0} \) be any bounded sequence, i.e.

\[
\limsup_{\varepsilon \to 0} \int_{\Omega^-} \hat{p}_\varepsilon^2 \, d\mu_\varepsilon < +\infty.
\]

**Definition 1.** [3]. We say that a bounded sequence \( \{ \hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon) \}_{\varepsilon > 0} \) is weakly convergent if there exists an element \( p^* \in L^2(\Omega^-) \) such that

\[
\lim_{\varepsilon \to 0} \int_{\Omega^-} \varphi \hat{p}_\varepsilon \, d\mu_\varepsilon = \int_{\Omega^-} \varphi p^* \, dx \quad \text{for any function } \varphi \in C_0^\infty(\mathbb{R}^n).
\]

(in symbols we will use the following notation \( \hat{p}_\varepsilon \rightharpoonup p^* \) in \( L^2(\Omega^-, d\mu_\varepsilon) \)).

Further we make use the observation

\[
\varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} = \varepsilon \sum_{j=1}^{N_{n-1}} \int_{\varepsilon(\partial C + k_j)} \int_{-\beta(x')} p_\varepsilon \varphi \, d\mathcal{H}^{n-2} \, dx_n
\]

\[
= \varepsilon |\partial C|_H \sum_{j=1}^{N_{n-1}} \int_{\varepsilon(\partial C + k_j)} \int_{-\beta(x')} \hat{p}_\varepsilon \varphi \, \varepsilon^{-n-2} \, d\mu_0(x'/\varepsilon) \, dx_n
\]

\[
= |\partial C|_H \sum_{j=1}^{N_{n-1}} \int_{\varepsilon(\partial + k_j)} \int_{-\beta(x')} \hat{p}_\varepsilon \varphi \, \varepsilon^n \, d\mu_0(x'/\varepsilon) \, d(x_n/\varepsilon)
\]

\[
= |\partial C|_H \sum_{j=1}^{N_{n-1}} \int_{\{ (x',x_n) \mid x' \in \varepsilon(\partial + k_j), -\beta(x') < x_n < 0 \}} \hat{p}_\varepsilon \varphi \, d\mu_\varepsilon
\]

\[
= |\partial C|_H \int_{\Omega^-} \hat{p}_\varepsilon \varphi \, d\mu_\varepsilon \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).
\]

(15)

Here a function \( \hat{p}_\varepsilon \) is defined as follows: \( \hat{p}_\varepsilon = p_\varepsilon \) on \( S_\varepsilon \equiv \text{supp}(\mu_\varepsilon) \), and \( \hat{p}_\varepsilon = 0 \) in \( \Omega^- \setminus S_\varepsilon \). Hence, \( \hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon) \). It is clear that any functions, taking the same values on the set \( S_\varepsilon \), coincide as elements of \( L^2(\Omega^-, d\mu_\varepsilon) \). In view of this, any function \( \hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon) \) with the property

\[
|\partial C|_H \int_{\Omega^-} \hat{p}_\varepsilon \varphi \, d\mu_\varepsilon = \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)
\]

will be called a prototype of the function \( p_\varepsilon \in L^2(S_\varepsilon) \). We note also that the integral \( \int_{\Omega^-} \hat{p}_\varepsilon \varphi \, d\mu_\varepsilon \) is well defined for every function \( \varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon) \).

Indeed, since the set \( \Omega^- \) is bounded, and \( \hat{p}_\varepsilon \, d\mu_\varepsilon \) is a Radon measure, it follows that \( \int_{\Omega^-} \hat{p}_\varepsilon \varphi \, d\mu_\varepsilon \) is a linear continuous functional on \( C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon) \).

In a similar manner to (15), we can derive the following relation
\[ \varepsilon \| p_\varepsilon \|^2_{L^2(S_\varepsilon)} = \varepsilon \int_{S_\varepsilon} p_\varepsilon^2 \, d\mathcal{H}^{n-1} \]
\[ = |\partial C| H \int_{\Omega^-} \bar{p}_\varepsilon^2 \, d\mu_\varepsilon = |\partial C| H \| \bar{p}_\varepsilon \|^2_{L^2(\Omega^-, d\mu_\varepsilon)}. \quad (16) \]

5 On smooth \( S_\varepsilon \)-approximation of \( L^2(\Omega^-) \)-functions

Let \( p^0 \in L^2(\Omega^-) \) be a given function such that \( \| p^0 \|_{L^2(\Omega^-)} \leq C_p \) with some constant \( C_p > 0 \). Our aim is to construct a sequence of smooth functions \( \{ p_\varepsilon \in C_\infty(S_\varepsilon) \} \) satisfying the following conditions:

\[ \| p_\varepsilon \|^2_{L^2(S_\varepsilon)} \leq \sqrt{|\partial C| H / \varepsilon} C_p, \quad \forall \varepsilon > 0, \quad (17) \]
\[ \bar{p}_\varepsilon \rightharpoonup p^0 \text{ in } L^2(\Omega^-, d\mu_\varepsilon) \text{ as } \varepsilon \to 0. \quad (18) \]

Remark 3. Note that we can not use the restriction of this function to the construction of the sequence \( \{ p_\varepsilon \} \) because a trace of an \( L^2 \)-function on the \((n-1)\)-dimensional surface \( S_\varepsilon \) is not defined. In view of this we call the sequence \( \{ p_\varepsilon \in C_\infty(S_\varepsilon) \} \) with properties (17)–(18) a smooth \( S_\varepsilon \)-approximation of the function \( p^0 \in L^2(I_0) \).

Let \( K_p \in C_\infty_0(\mathbb{R}^n) \) be nonnegative function such that
\[ \int_{\mathbb{R}^n} K_p(x) \, dx = 1, \quad \text{and} \quad K_p(x) = K_p(-x) \quad \forall \, x \in \mathbb{R}^n. \]

Let \( \bar{p}^0 \) be the trivial extension by zero of the function \( p^0 \in L^2(\Omega^-) \) to \( \mathbb{R}^n \). For every fixed \( \varepsilon > 0 \) we define the following smoothing operator
\[ (p^0)_\varepsilon = (K_{\varepsilon}^\varepsilon p^0)(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} K_p \left( \frac{x-y}{\varepsilon} \right) \bar{p}^0(y) \, dy. \]

We begin with the following result:

**Proposition 3.** Let \( p^0 \in L^2(\Omega^-) \) be a given function such that \( \| p^0 \|_{L^2(\Omega^-)} \leq C_p \). Then there exists \( \varepsilon_0 > 0 \) such that the functions
\[ p_\varepsilon = (p^0)_\varepsilon |_{S_\varepsilon} \quad (19) \]
satisfy the estimate (17) for all \( \varepsilon \in (0, \varepsilon_0) \).

**Proof.** Since \( (p^0)_\varepsilon \in C(\overline{\Omega^-}) \) it is clear that \( p_\varepsilon^{\text{sub}} = (p^0)_\varepsilon |_{S_\varepsilon} \in L^2(S_\varepsilon, \mathcal{H}^{n-1}). \)

By analogy with (8) it is easy to show the validity of the following inequality
\[ \| (p^0)_\varepsilon \|^2_{L^2(\Omega^-)} \leq \int_{\Omega^-} (p^0)^2 \, dx = \| p^0 \|^2_{L^2(\Omega^-)} \leq C_p^2. \quad (20) \]
Thus, taking into account (20), we have to prove that the estimate (17) is valid at least for \( \varepsilon \) small enough. In order to verify this fact we will show that the value

\[
\sigma = \left| \varepsilon |\partial C|^{-1}_H \left\| (p^0)_{\varepsilon} \right\|_{L^2(S_\varepsilon; \mathcal{H}^{n-1})}^2 - \left\| (p^0)_{\varepsilon} \right\|_{L^2(\Omega^-)}^2 \right|
\]

can be done as small as is wished for \( \varepsilon \) small enough.

To begin with, we note that

\[
\| (p^0)_{\varepsilon} \|_{L^2(\Omega^-)}^2 = \sum_{j=1}^{N^{n-1}} \int \left\{ x = (x',x_n) : x' \in \varepsilon [0,1]^{n-1}, \right. \delta_{\varepsilon} < x_n < 0 \} \left. (p^0)_{\varepsilon} \right|^2 \, \mathrm{d}x.
\]

Here \( \square = [0,1]^{n-1} \). Further we make use the following notation:

\[
L^k_{\varepsilon} = \{(x',x_n) : x' \in \varepsilon \square + \varepsilon k, -\beta_0 < x_n \leq 0 \},
\]
\[
G^k_{\varepsilon} = \{(x',x_n) : x' \in \varepsilon C + \varepsilon k, -\beta_0 < x_n \leq 0 \},
\]
\[
\Delta_i S^k_{\varepsilon} = \{(x',x_n) : x' \in \varepsilon C + \varepsilon k, -\beta_0 + (i-1) \Delta < x_n \leq -\beta_0 + i \Delta \}.
\]

where \( \Delta = \beta_0 / J \).

Since \( (p^0)_{\varepsilon} \in C^0_0(\mathbb{R}^n) \) it means that a wild oscillation of this function is excluded on the cross sections of thin cylinders \( L^k_{\varepsilon} \) for \( \varepsilon \) small enough. Hence for a given \( \eta > 0 \) there exist an \( \varepsilon_0 > 0 \), an integer \( J \in \mathbb{N} \), and a collection of points \( \{x^k_{\varepsilon, i} \in \Delta_i S^k_{\varepsilon}\} \) such that

\[
\left| \int_{L^k_{\varepsilon}} (p^0)_{\varepsilon}^2 \, \mathrm{d}x - \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \mathcal{L}^n(\varepsilon \square) \Delta \right| < \frac{\eta}{2N^{n-1}} \forall \varepsilon \in (0, \varepsilon_0).
\]

Taking into account the chain of transformations

\[
\sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \mathcal{L}^n(\varepsilon \square) \Delta
\]

\[
= \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \varepsilon^n \Delta = \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \varepsilon^n \frac{\mathcal{H}^{n-1}(\Delta_i S^k_{\varepsilon})}{\mathcal{H}^{n-2}(\varepsilon \partial C)}
\]

\[
= \frac{\varepsilon}{\mathcal{H}^{n-2}(\partial C)} \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \mathcal{H}^{n-1}(\Delta_i S^k_{\varepsilon})
\]

\[
= \varepsilon |\partial C|^{-1}_H \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \mathcal{H}^{n-1}(\Delta_i S^k_{\varepsilon})
\]

and the fact that

\[
\left| \sum_{i=1}^{J} (p^0)_{\varepsilon}^2 (x^k_{\varepsilon, i}) \mathcal{H}^{n-1}(\Delta_i S^k_{\varepsilon}) - \int_{S^k_{\varepsilon}} (p^0)_{\varepsilon}^2 \, \mathrm{d}\mathcal{H}^{n-1} \right| < \frac{\eta |\partial C|_H}{2 \varepsilon N^{n-1}}
\]
for $J$ large enough, we deduce: for a given $\eta > 0$ there exists an $\varepsilon_\eta > 0$ such that

$$\left| \int_{L^2_\varepsilon} (p^0)^2 \, dx - \varepsilon|\partial C|^{-1}_H \int_{S^k_\varepsilon} (p^0)^2 \, dH^{n-1} \right|$$

$$\leq \left| \int_{L^2_\varepsilon} (p^0)^2 \, dx - \varepsilon|\partial C|^{-1}_H \sum_{i=1}^J (p^0)_\varepsilon^2 (x^k_i) H^{n-1}(\Delta_i S^k_\varepsilon) \right|$$

$$+ \varepsilon|\partial C|^{-1}_H \sum_{i=1}^J (p^0)_\varepsilon^2 (x^k_i) H^{n-1}(\Delta_i S^k_\varepsilon)$$

$$- \int_{S^k_\varepsilon} (p^0)^2 \, dH^{n-1} \right| < \frac{\eta}{N^N - 1} \quad (22)$$

for all $\varepsilon \in (0, \varepsilon_\eta)$.

As a result, combining this estimate with the representation (21), we come to the conclusion:

$$\left| \| (p^0)^{\varepsilon}_\{\Omega\}^2 - \varepsilon|\partial C|^{-1}_H \| (p^0)^{\varepsilon}_\{\Omega\}^2 \right|$$

$$= \sum_{j=1}^{N^N - 1} \left( \int_{L^2_\varepsilon} (p^0)^2 \, dx - \varepsilon|\partial C|^{-1}_H \int_{S^k_\varepsilon} (p^0)^2 \, dH^{n-1} \right) < \eta. \quad (23)$$

Since relation (20) turns into the equality only in the case when $p^0$ is a constant, it follows that choosing an appropriate value $\eta > 0$ we can find an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta)$ the inequality

$$\varepsilon|\partial C|^{-1}_H \| (p^0)^{\varepsilon}_\{\Omega\}^2 \leq \| p^0 \|^2_{L^2(\Omega^-)} \leq C_p^2$$

holds true. Hence the estimate (17) is valid for $\varepsilon > 0$ small enough. \(\Box\)

It remains to substantiate the convergence property (18).

**Proposition 4.** Assume the functions $p_\varepsilon \in C^\infty(S_\varepsilon)$ are defined by (19). Then the property (18) holds true.

**Proof.** Let $\{\zeta_\varepsilon\}$ be any sequence of smooth functions such that $\zeta_\varepsilon \to p$ strongly in $L^2(\Omega^-)$ as $\varepsilon \to 0$. Let us show that

$$\lim_{\varepsilon \to 0} \int_{\Omega^-} \zeta_\varepsilon \varphi \, d\mu_\varepsilon = \int_{\Omega^-} p \varphi \, dx \quad \forall \varphi \in C^\infty_0(\mathbb{R}^n). \quad (24)$$

Indeed, let us partition $\Omega^-$ into cubes with edges $\varepsilon$ and denote these cubes by the symbols $\varepsilon \square_\varepsilon$. Then
\[ \int_{\Omega^-} \zeta \varphi \, d\mu_\varepsilon = \sum_j \int_{\varepsilon \Box_{\varepsilon}} \zeta(x_j) \varphi(x) \, d\mu_\varepsilon \]

\[ + \sum \int_{\Omega^- \cap \varepsilon \Box_{\varepsilon}} \zeta(x) \varphi(x) \, d\mu_\varepsilon = \sum_j \zeta(x_j) \varphi(x_j) \int_{\varepsilon \Box_{\varepsilon}} d\mu_\varepsilon \]

\[ + \sum \int_{\Omega^- \cap \varepsilon \Box_{\varepsilon}} \zeta(x) \varphi(x) \, d\mu_\varepsilon, \]

where \( x_j \) is a Lebesgue point of \( p \) in the cube \( \varepsilon \Box_{\varepsilon} \) and the second sum is calculated over the set of boundary cubes. By definition of the measure \( \mu_\varepsilon \) we have

\[ \int_{\varepsilon \Box_{\varepsilon}} d\mu_\varepsilon = \varepsilon^n \int_{\Box_n} d\mu = \varepsilon^n. \]

Moreover,

\[ \left| \sum \int_{\Omega^- \cap \varepsilon \Box_{\varepsilon}} \zeta(x) \varphi(x) \, d\mu_\varepsilon \right| \leq \sup_{x \in \Omega^-} |\zeta(x) \varphi(x)| \varepsilon^n \cdot D(\varepsilon), \]

where \( D(\varepsilon) \) is a quantity of boundary cubes, and \( \varepsilon^n D(\varepsilon) \to 0 \) by Jordan’s measurability of the set \( \partial \Omega^- \). Thus, summarizing the above cited facts, we have

\[ \lim_{\varepsilon \to 0} \int_{\Omega^-} \zeta \varphi \, d\mu_\varepsilon = \lim_{\varepsilon \to 0} \left[ \sum_j \zeta(x_j) \varphi(x_j) \varepsilon^n - \int_{\Omega^-} \zeta \varphi \, dx \right] \]

\[ + \lim_{\varepsilon \to 0} \int_{\Omega^-} \zeta \varphi \, dx = \lim_{\varepsilon \to 0} \sum_j \left[ \zeta(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \Box_{\varepsilon}} \zeta \varphi \, dx \right] \varepsilon^n \]

\[ + \lim_{\varepsilon \to 0} \sum \int_{\Omega^- \cap \varepsilon \Box_{\varepsilon}} \zeta(x) \varphi(x) \, dx + \int_{\Omega^-} p \varphi \, dx \]

\[ = \lim_{\varepsilon \to 0} \sum_j \left[ \zeta(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \Box_{\varepsilon}} \zeta \varphi \, dx \right] \varepsilon^n + \int_{\Omega^-} p \varphi \, dx. \]

Let us suppose that

\[ \lim_{\varepsilon \to 0} \sum_j \left| \zeta(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \Box_{\varepsilon}} \zeta \varphi \, dx \right| \varepsilon^n > 0. \]

Then there exist a constant \( C^* > 0 \) and a value \( \varepsilon^* > 0 \) such that

\[ \left| \zeta(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \Box_{\varepsilon}} \zeta \varphi \, dx \right| \geq C^* \forall \varepsilon \in (0, \varepsilon^*) \]

(possibly except, for a finite number of indices \( j \) for every fixed \( \varepsilon \)). Hence extremely wild oscillations are present in the sequence \( \{ \zeta \varphi \} \). However (see
On some smooth approximations on thick periodic multi-structures, if we have very rapid fluctuations in the functions $\{\zeta_\epsilon \varphi\}$, then the convergence $\zeta_\epsilon \varphi \to p \varphi$ almost everywhere in $\Omega^-$ is excluded. This fact immediately reflects the failure of the strong convergence $\zeta_\epsilon \varphi \to p \varphi$ in $L^2(\Omega^-)$ as $\epsilon \to 0$ (see Valadier’s Theorem [2]). So, our supposition was wrong and we get

$$\lim_{\epsilon \to 0} \sum_j \left| \zeta_\epsilon(x_j) \varphi(x_j) - \epsilon^{-n} \int_{\epsilon \Box^n} \zeta_\epsilon \varphi \, dx \right| \epsilon^n = 0.$$ 

As a result, we come to the relation (24).

To conclude the proof it remains to use the main property of classical smoothing $(p^0)_{\epsilon} \to p^0$ strongly in $L^2(\Omega^-)$ as $\epsilon \to 0$ and apply the relation (24). □

As the direct consequence of Proposition 3, which we feel to be interesting per se, we present one observation concerning $L^2$-functions defined on thick junctions.

**Corollary 1.** Let $\Omega_\epsilon$ be a thick multi-structure in $\mathbb{R}^n$, which consists of some domain $\Omega^+$ and a large number of thin cylinders $G^k_\epsilon$ with a small cross section of the size $\epsilon C$ and $\epsilon$-periodically distributed along some manifold $\Sigma$ on the boundary of $\Omega^+$ (see Fig. 1 for 3-d example). Let $\Omega^- = \Sigma \times (a, b)$ be a domain which is filled up by the thin cylinders in the limit passage as $\epsilon \to 0$. Let $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ a given function. Then

$$\|f\|^2_{L^2(\Omega^-)} \geq \frac{\epsilon}{\mathcal{H}^{n-2}(\partial C)} \|(f)_{\epsilon}\|^2_{L^2(S; \mathcal{H}^{n-1})}$$

for $\epsilon$ small enough, where $(f)_{\epsilon}$ denotes the direct smoothing of the function $\chi_{\Omega^+} f$, i.e.

$$(f)_{\epsilon} = \epsilon^{-n} \int_{\mathbb{R}^n} K \left( \frac{x - y}{\epsilon} \right) \chi_{\Omega^-} f(y) \, dy.$$ 

**References**