On some smooth approximations on thick periodic multi-structures

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Summary. In this paper we study the approximation properties of measurable and square-integrable functions. In particular we show that any L^2-bounded functions can be approximated in L^2-norm by smooth functions defined on a highly oscillating boundary of thick multi-structures in $\mathbb{R}^n$. We derive the norm estimates for the approximating functions and study their asymptotic behavior.

Key words: thick multi-structure, approximating properties, singular measures, convergence in variable spaces

1 Introduction

The main object of our consideration in this paper is a class of functions defined on a domain $\Omega_\varepsilon \subset \mathbb{R}^n$, whose boundary $\partial\Omega_\varepsilon$ contains the very highly oscillating part with respect to a small parameter ε, as $\varepsilon \to 0$.

Wa say that Ω_ε is a thick multi-structure in $\mathbb{R}^n$, if Ω_ε consists of some fixed domain Ω^+ and a large number of cylinders with axes parallel to Ox_n and ε-periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 1). This manifold is called the joint zone and the domain Ω^+ is called the junction's body. Here ε is a small positive parameter, which characterizes the distance between the neighboring cylinders and their thickness. So, each attached cylinder has a small cross section of the size ε and its limiting dimension (as $\varepsilon \to 0$) is equal to 1. In view of this, such cylinders will be called thin domains.

Thick junctions have a special character of the connectedness: there are points in a thick junction which are at a short distance of order $\mathcal{O}(\varepsilon)$, but the length of all curves, which connect these points in the junction, is order $\mathcal{O}(1)$. As a result, there are no extension operators $P : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}_{loc}(\mathbb{R}^n)$ that would be uniformly bounded in ε. At the same time thick multi-structures (or thick junctions) are prototypes of widely used engineering constructions as well as many other physical and biological systems with very distinct characteristic scales (microscopic radiators, ferrite-filled rod radiators, and others). As a rule, the computational calculation of the solutions of any problems on Ω_ε is very complicated due to singularities of the thick junctions. As a result, asymptotic analysis is one of the main approaches to study majority of problems in such domains.

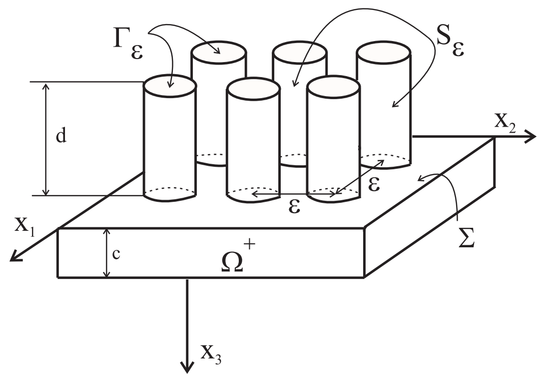

Fig. 1. Thick multi-structure Ω_ε.

In view of this we consider the approximation problem of L^2-bounded functions by sequences of smooth functions defined on lateral side of thin cylinders and their bases which form a quickly oscillating counterpart of thick junctions. We derive the norm estimates for the approximating functions and study their asymptotic behavior.

2 Notation and Preliminaries

Let $a >$, $d > 0$, and $c > 0$ be given positive constants. Let $B = (0, a)^{n-1}$ and C be bounded open smooth domain in $\mathbb{R}^{n-1}$ $(n \geq 2)$. Moreover we assume that $C \subset\subset (0,1)^{n-1}$, that is the set C belongs to the $n-1$-dimensional cube $(0,1)^{n-1}$ together with its boundary ∂C. Throughout this paper the parameter ε varies in a strictly decreasing sequence of positive numbers which converges to zero, i.e. we can suppose that $\varepsilon = a/N$, where N is a large positive integer. When we write $\varepsilon > 0$, we consider only the elements of this sequence.

Let us denote the elements of $\mathbb{R}^n$ by $x = (x_1, x_2, \dots, x_n) = (x', x_n)$ and introduce the following sets: $\theta_\varepsilon = \{\mathbf{k} = (k_1, k_2, \dots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ \varepsilon C + \varepsilon \mathbf{k} \subset\subset B\}$,

$$\begin{gathered} \Omega = B \times (-d,c),\ G_\varepsilon^{\mathbf{k}} = \{(x',x_n)\ :\ x' \in \varepsilon C + \varepsilon \mathbf{k},\ -d < x_n \le 0\}, \\ \Sigma = B \times \{0\}, \quad \Omega^+ = B \times (0,c), \quad \Omega^- = B \times (-d,0), \\ \Gamma_0 = B \times \{-d\}, \quad \Omega_\varepsilon^- = \Omega_\varepsilon \cap \Omega^-. \end{gathered} \tag{1}$$

Then in view of our previous description, the set

$$\Omega_\varepsilon = (B \times (0,c)) \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} (\varepsilon C + \varepsilon \mathbf{k}) \times (-d,0] \right) = \Omega^+ \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} G_\varepsilon^{\mathbf{k}} \right),$$

is a thick multi-structure in $\mathbb{R}^n$, which consists of the junction's body Ω^+ and a large number N^{n-1} of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ with axis Ox_n and ε-periodically distributed on the basis Σ of Ω^+ (see Fig. 1 for 3-d example). Here, each cylinder $G_\varepsilon^{\mathbf{k}}$ is obtained with ε-homothety in the first $(n-1)$ variables. It is easy to see that $\Omega_\varepsilon = \Omega^+ \cup \left(\bigcup_{\mathbf{k}\in\theta_\varepsilon} G_\varepsilon^{\mathbf{k}}\right)$.

Denote by Γ_ε the union of the lower bases

$$\Gamma_\varepsilon^{\mathbf{k}} = \{(x',x_n)\ :\ x' \in \varepsilon \cdot C + \varepsilon \mathbf{k},\ x_n = -d\}$$

of the thin cylinders $G_\varepsilon^{\mathbf{k}}$ when $\mathbf{k} \in \theta_\varepsilon$ (i.e. $\Gamma_\varepsilon = \Gamma_0 \cap \partial\Omega_\varepsilon$), and by S_ε the union of their boundaries along the axis Ox_n:

$$S_\varepsilon^{\mathbf{k}} = \{(x',x_n)\ :\ x' \in \varepsilon \cdot \partial C + \varepsilon \mathbf{k},\ -d < x_n < 0\},$$

We make also use of the idea of classical smoothing. Let $K(x)$ be a positive compactly supported smooth function ($K \in C_0^\infty$) such that $\int_{\mathbb{R}^n} K(x)\,\mathrm{d}x = 1$ and $K(x) = K(-x)$ for all $x \in \mathbb{R}^n$. For an arbitrary measurable function $\varphi \in L^1_{loc}(\mathbb{R}^n)$ we define an ordinary smoothing operator by the equality

$$(\varphi)_h(x) = h^{-n} \int_{\mathbb{R}^n} K\left(\frac{x-y}{h}\right) \varphi(y)\,dy = \int_{\mathbb{R}^n} K(y)\,\varphi(x+hy)\,dy. \tag{2}$$

3 On smooth Γ_ε-approximation of $L^2(\Gamma_0)$-functions

Let $u^0 \in L^2(\Gamma_0)$ be a given function such that $\|u^0\|_{L^2(\Gamma_0)} \le \mathbf{C_u}$ with some constant $\mathbf{C_u} > 0$. Our aim is to construct a sequence of smooth functions $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon>0}$ satisfying the following conditions:

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \le \left(\sqrt{|C|}\right)^{-1} \mathbf{C_u}, \quad \forall\, \varepsilon > 0, \tag{3}$$

$$\widetilde{u}_\varepsilon \rightharpoonup u^0 \ \text{ in } \ L^2(\Gamma_0) \ \text{ as } \ \varepsilon \to 0. \tag{4}$$

Here $|C|$ is the $(n-1)$–dimensional Lebesgue measure of the set C, and $\widetilde{u}_\varepsilon$ is the trivial extension by zero of a function $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ to the set Γ_0. It is clear that in this case $\widetilde{u}_\varepsilon \in L^2(\Gamma_0)$.

Remark 1. Note that for every $\varepsilon > 0$ the restriction $u^0\big|_{\Gamma_\varepsilon}$ is an element of $L^2(\Gamma_\varepsilon)$ whereas the functions u_ε in (3)–(4) must be elements of $C^\infty(\Gamma_\varepsilon)$. In view of this we call the sequence $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon>0}$ with properties (3)–(4) a smooth Γ_ε-approximation of $u^0 \in L^2(\Gamma_0)$.

In order to construct the sequence $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon>0}$, we make use the idea of classical smoothing. Let $K_u \in C_0^\infty(\mathbb{R}^{n-1})$ be nonnegative function such that

$$\int_{\mathbb{R}^{n-1}} K_u(x')\,dx' = 1 \quad \text{and} \quad K_u(x') = K_u(-x')\ \forall\, x' \in \mathbb{R}^{n-1}.$$

Let $\widetilde{u}^{\,0}$ be the trivial extension by zero of function $u^0 \in L^2(\Gamma_0)$ to the set $\mathcal{O} = \big\{(x', x_n) \in \mathbb{R}^n \ :\ x' \in \mathbb{R}^{n-1},\ x_n = -\beta_0\big\}$. For every fixed $\varepsilon > 0$ we define the following smoothing operator

$$\begin{aligned}\left(u^0\right)_\varepsilon = \left(\mathcal{K}_u^\varepsilon u^0\right)(x') &= \varepsilon^{-n+1}\int_{\mathcal{O}} K_u\left(\frac{x'-y'}{\varepsilon}\right)\widetilde{u}^{\,0}(y', y_n)\,d\mathcal{H}^{n-1}\\ &= \varepsilon^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\varepsilon}\right)\widetilde{u}^{\,0}(y')\,dy'.\end{aligned} \tag{5}$$

We begin with the following result:

Proposition 1. *Let $u^0 \in L^2(\Gamma_0)$ be a given function such that $\left\|u^0\right\|_{L^2(\Gamma_0)} \le \mathbf{C_u}$. Then for every $\delta > 0$ there exists $\varepsilon(\delta) = a/N$, where $N \in \mathbb{N}$, such that*

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \le \left(\sqrt{|C|}\right)^{-1}\mathbf{C_u} \tag{6}$$

provided

$$u_\varepsilon := |C|^{-1}\left(u^0\right)_\delta\Big|_{\Gamma_\varepsilon}. \tag{7}$$

Proof. First of all we note that $u_\varepsilon \in C^\infty(\Gamma_\varepsilon)$ by the properties of smoothing operator (5). Following the definition of the smoothing operator and using the Cauchy-Bunyakovskiĭ inequality, we have

$$\begin{aligned}
\left\|\left(u^0\right)_\delta\right\|^2_{L^2(\mathbb{R}^{n-1})} &= \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right)\widetilde{u}^{\,0}(y')\,\mathrm{d}y'\right)^2 \mathrm{d}x' \\
&\leq \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right)\mathrm{d}y'\right) \\
&\times \left(\delta^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right)\left(\widetilde{u}^{\,0}(y')\right)^2\,\mathrm{d}y'\right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right)\left(\widetilde{u}^{\,0}(y')\right)^2\,\mathrm{d}y'\right) dx' \\
&\int_{\mathbb{R}^{n-1}} \left(\delta^{-n+1}\int_{\mathbb{R}^{n-1}} K_u\left(\frac{x'-y'}{\delta}\right) dx'\right)\left(\widetilde{u}^{\,0}(y')\right)^2\,dy' \\
&= \int_{\mathbb{R}^{n-1}} \left(\widetilde{u}^{\,0}(y')\right)^2\,dy' = \int_{\Gamma_0} \left(u^{\,0}(y')\right)^2\,dy' = \|u^{\,0}\|^2_{L^2(\Gamma_0)}. \qquad (8)
\end{aligned}$$

Due to this estimate, we come to the inequality

$$\begin{aligned}
\left\|\left(u^0\right)_\delta\right\|_{L^2(\Gamma_\varepsilon)} &= \left\|\chi_{\Gamma_\varepsilon}\left(u^0\right)_\delta\right\|_{L^2(\Gamma_\varepsilon)} \\
&\leq \left\|\chi_{\Gamma_0}\left(u^0\right)_\delta\right\|_{L^2(\mathbb{R}^{n-1})} \leq \left\|u^{\,0}\right\|_{L^2(\Gamma_0)} \leq \mathbf{C_u} \qquad (9)
\end{aligned}$$

which holds true for every $\varepsilon > 0$. Here $\chi_{\Gamma_\varepsilon} \in L^\infty(\Gamma_0)$ is the characteristic function of the zone Γ_ε, i.e.,

$$\chi_{\Gamma_\varepsilon}(x') = \begin{cases} 1, \, x' \in \Gamma_\varepsilon, \\ 0, \text{ otherwise} \end{cases} \qquad \forall\, x' \in \Gamma_0.$$

Note that

$$\chi_{\Gamma_\varepsilon} \overset{*}{\rightharpoonup} |C| \;\text{ in }\; L^\infty(\Gamma_0) \;\text{ as }\; \varepsilon \to 0. \qquad (10)$$

Indeed, in view of $\varepsilon[0,1)^{n-1}$-periodicity of this characteristic function and the mean value property, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Gamma_0} \varphi(x')\chi_{\Gamma_\varepsilon}(x')\,d\mathcal{H}^{n-1} &= \lim_{\varepsilon\to 0}\int_B \varphi(x')\chi_{\cup_{\mathbf{k}\in\theta_\varepsilon}(\varepsilon C+\varepsilon\mathbf{k})}(x')\,\mathrm{d}x' \\
&= \int_B \varphi(x')\,dx'\left(\int_{(0,1)^{n-1}} \chi_C(x')\,dx'\right) \\
&= |C|\int_B \varphi(x')\,dx' = |C|\int_{\Gamma_0} \varphi(x')\,d\mathcal{H}^{n-1} \quad \forall\,\varphi \in C_0^\infty(\Gamma_0).
\end{aligned}$$

Thus, the property (10) holds true.

Further for a given $\delta > 0$ we define a parameter $\varepsilon = a/N$ so that the following inequality would be valid

$$\frac{1}{|\Gamma_\varepsilon|}\int_{\Gamma_\varepsilon} \left(u^0\right)^2_\delta\,dx' \leq \frac{1}{|\Gamma_0|}\int_{\Gamma_0} \left(u^0\right)^2_\delta\,dx'.$$

Then from (9) we deduce

$$\left\| |C|^{-1} \left(u^0\right)_\delta \right\|^2_{L^2(\Gamma_\varepsilon)} \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} \left\| |C|^{-1} \left(u^0\right)_\delta \right\|^2_{L^2(\Gamma_0)} \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} |C|^{-2} \mathbf{C_u}^2. \tag{11}$$

Since $|\Gamma_0| = |B| = a^{n-1}$ and

$$|\Gamma_\varepsilon| = \sum_{\mathbf{k}\in\theta_\varepsilon} \int_{\Gamma_\varepsilon^{\mathbf{k}}} dx' = \sum_{\mathbf{k}\in\theta_\varepsilon} \varepsilon^{n-1}|C| = \left(\frac{a}{\varepsilon}\right)^{n-1} \varepsilon^{n-1}|C| = a^{n-1}|C|,$$

it follows from (11) that

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \equiv \left\| |C|^{-1} \left(u^0\right)_\delta \right\|_{L^2(\Gamma_\varepsilon)} \leq \left(\sqrt{|C|}\right)^{-1} \mathbf{C_u}. \tag{12}$$

Thus for the chosen value of ε inequality (6) is valid. This concludes the proof. □

The next step to prove the property of strong convergence (4).

Proposition 2. *Assume the functions $u_\varepsilon \in C^\infty(\Gamma_\varepsilon)$ are defined by* (7)*. Then the property* (4) *holds true.*

Proof. To begin with, we note that

$$\left(u^0\right)_\delta \to u^0 \text{ strongly in } L^2(\Gamma_0) \text{ as } \delta \to 0 \tag{13}$$

by properties of the classical smoothing. Hence in view of (10), (13), and the fact that the value of ε depends on a given parameter δ, we have

$$\widetilde{u}_{\varepsilon(\delta)} = |C|^{-1}\chi_{\Gamma_{\varepsilon(\delta)}} \left(u^0\right)_\delta \rightharpoonup |C|^{-1}|C|u^0 = u^0 \text{ as } \delta \to 0$$

as a limit of the product of the weak and strong convergent sequences in $L^2(\Gamma_0)$. This implies the required property (4). The proof is complete. □

4 Some auxiliary results

We begin this section with the description of the $(n-1)$-dimensional surface S_ε in the terms of singular measures in $\mathbb{R}^n$ which do not satisfy the regularity property with respect to the corresponding Lebesgue measure.

Let μ_0 be a periodic finite positive Borel measure in $\mathbb{R}^{n-1}$. Let $\square = [0,1)^{n-1}$ be the cell or torus of periodicity for μ_0. We assume that Borel measure μ_0 is the probability measure in $\mathbb{R}^{n-1}$, concentrated and uniformly distributed on the set ∂C, so $\int_\square d\mu_0 = 1$.

Remark 2. By definition we have $\mu_0(\square \setminus \partial C) = 0$. Therefore any functions, taking the same values on the set ∂C, coincide as elements of $L^2(\square,\, d\mu_0)$. Here the Lebesgue space $L^2(\square,\, d\mu_0)$ with respect to the measure μ_0 is defined in a usual way with the corresponding norm

$$\|f\|^2_{L^2(\square,\, d\mu_0)} = \int_\square |f(x)|^2\, d\mu_0$$

(we adopt the standard notation $L^2(\square)$ when μ_0 is the Lebesgue measure).

We set $\square_n = \square \times [0,1) = [0,1)^n$ and consider the following measure $d\mu = d\mu_0 \times dx_n$ in $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1)$, and for any smooth function g we have

$$\int_{\square_n} g\, d\mu = \int_0^1 \int_\square g\, dx_n\, d\mu_0 \;=\; \big[\mathcal{H}^{n-1}(\partial C \times [0,1))\big]^{-1} \int_{\partial C \times [0,1)} g\, d\mathcal{H}^{n-1}.$$

However, as follows from the properties of the Hausdorff measure, we have $\mathcal{H}^{n-1}(\partial C \times [0,1)) = \mathcal{H}^{n-2}(\partial C)$ (see [1]). In what follows, we use the notation $|\partial C|_H = \mathcal{H}^{n-2}(\partial C)$ for $(n-2)$-dimensional Hausdorff measure of the set ∂C. Then the previous relation can be rewritten in the form

$$\int_{\square_n} g\, d\mu = \int_0^1 \int_\square g\, dx_n\, d\mu_0 \;=\; |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g\, d\mathcal{H}^{n-1}. \tag{14}$$

In order to clarify relation (14), we consider the planar thick multi-structure $\Omega_\varepsilon \subset \mathbb{R}^2$. Then $n = 2$ and the set C is some part of the segment $(0,1)$. For instance, let $C = \{x_1 \in (0,1) \;:\; |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed value. In this case $|\partial C|_H = 2$ and the 1-periodic measure μ_0 in $\mathbb{R}^1$ can be defined as

$$\mu_0(B) = \frac{1}{|\partial C|_H} \left(\delta_{M_1} + \delta_{M_2}\right)(B) = \frac{1}{2} \left(\delta_{M_1} + \delta_{M_2}\right)(B)$$

for any Borel set $B \subseteq [0,1)$, where $M_i = \frac{1}{2} + \left(i - \frac{3}{2}\right) h$, $i = 1, 2$. Here δ_{M_i} is the Dirac measures located at the points M_i. Thus, the multiplier $|\partial C|_H^{-1}$ in (14) is equal to $1/2$.

Let Λ be any Borel set of $\mathbb{R}^n$. We introduce the scaling measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon^n \mu(\varepsilon^{-1}\Lambda)$. This measure has a period ε, and moreover, since $\mu(\varepsilon\square_n) = \varepsilon\, \mu_0(\varepsilon\square)$, it follows that

$$\mu_\varepsilon(\varepsilon\square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon\square} d\mu_0(x'/\varepsilon)\, d(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_\square d\mu_0\, dx_n = \varepsilon^n.$$

As a result, the measure μ_ε weakly converges to Lebesgue measure in $\mathbb{R}^n$ as $\varepsilon \to 0$ (in symbols $d\mu_\varepsilon \rightharpoonup dx$), that is,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi\, d\mu_\varepsilon = \int_{\mathbb{R}^n} \varphi\, dx \quad \text{for all functions} \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

Now, we give the convergence formalism of the sequence of boundary controls $\left\{\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$. First of all we recall that the sequence of Borel measures $\{\mathrm{d}\mu_\varepsilon\}$ weakly converges to Lebesgue measure $\mathrm{d}x$. Let $\left\{\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$ be any bounded sequence, i.e.

$$\limsup_{\varepsilon\to 0} \int_{\Omega^-} \widehat{p}_\varepsilon^2 \, d\mu_\varepsilon < +\infty.$$

Definition 1. *[3]. We say that a bounded sequence $\left\{\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\right\}_{\varepsilon>0}$ is weakly convergent if there exists an element $p^* \in L^2(\Omega^-)$ such that*

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} \varphi \widehat{p}_\varepsilon \, d\mu_\varepsilon = \int_{\Omega^-} \varphi p^* \, dx \quad \text{for any function } \varphi \in C_0^\infty(\mathbb{R}^n).$$

(in symbols we will use the following notation $\widehat{p}_\varepsilon \rightharpoonup p^*$ in $L^2(\Omega^-, d\mu_\varepsilon)$).

Further we make use the observation

$$\begin{aligned}
\varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} &= \varepsilon \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C + \mathbf{k}_j)} \int_{-\beta(x')}^{0} p_\varepsilon \varphi \, d\mathcal{H}^{n-2} \, \mathrm{d}x_n \\
&= \varepsilon |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\Box + \mathbf{k}_j)} \int_{-\beta(x')}^{0} \widehat{p}_\varepsilon \varphi \, \varepsilon^{n-2} \, d\mu_0(x'/\varepsilon) \, dx_n \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\Box + \mathbf{k}_j)} \int_{-\beta(x')}^{0} \widehat{p}_\varepsilon \varphi \; \varepsilon^{n} \, d\mu_0(x'/\varepsilon) \, d(x_n/\varepsilon) \\
&= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\left\{ (x', x_n) \, : \left| \begin{smallmatrix} x' \in \varepsilon(\Box + \mathbf{k}_j) \\ -\beta(x') < x_n < 0 \end{smallmatrix} \right. \right\}} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon \\
&= |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^n). \qquad (15)
\end{aligned}$$

Here a function $\widehat{p}_\varepsilon$ is defined as follows: $\widehat{p}_\varepsilon = p_\varepsilon$ on $S_\varepsilon \equiv \operatorname{supp}(\mu_\varepsilon)$, and $\widehat{p}_\varepsilon = 0$ in $\Omega^- \setminus S_\varepsilon$. Hence, $\widehat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)$. It is clear that any functions, taking the same values on the set S_ε, coincide as elements of $L^2(\Omega^-, \, d\mu_\varepsilon)$. In view of this, any function $\widehat{p}_\varepsilon \in L^2(\Omega^-, \, d\mu_\varepsilon)$ with the property

$$|\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon = \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi \, d\mathcal{H}^{n-1} \qquad \forall \, \varphi \in C_0^\infty(\mathbb{R}^n)$$

will be called a prototype of the function $p_\varepsilon \in L^2(S_\varepsilon)$. We note also that the integral $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$. Indeed, since the set Ω^- is bounded, and $\widehat{p}_\varepsilon \, d\mu_\varepsilon$ is a Radon measure, it follows that $\int_{\Omega^-} \widehat{p}_\varepsilon \varphi \, d\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n; \Gamma_\varepsilon)$.

In a similar manner to (15), we can derive the following relation

$$\varepsilon\|p_\varepsilon\|^2_{L^2(S_\varepsilon)} = \varepsilon\int_{S_\varepsilon} p_\varepsilon^2\, d\mathcal{H}^{n-1}$$
$$= |\partial C|_H \int_{\Omega^-} \widehat{p}_\varepsilon^{\,2}\, d\mu_\varepsilon = |\partial C|_H\, \|\widehat{p}_\varepsilon\|^2_{L^2(\Omega^-,d\mu_\varepsilon)}. \quad (16)$$

5 On smooth S_ε-approximation of $L^2(\Omega^-)$-functions

Let $p^0 \in L^2(\Omega^-)$ be a given function such that $\left\|p^0\right\|_{L^2(\Omega^-)} \le \mathbf{C_p}$ with some constant $\mathbf{C_p} > 0$. Our aim is to construct a sequence of smooth functions $\{p_\varepsilon \in C^\infty(S_\varepsilon)\}_{\varepsilon>0}$ satisfying the following conditions:

$$\|p_\varepsilon\|_{L^2(S_\varepsilon)} \le \sqrt{\frac{|\partial C|_H}{\varepsilon}}\mathbf{C_p}, \quad \forall\, \varepsilon > 0, \quad (17)$$
$$\widetilde{p}_\varepsilon \rightharpoonup p^0 \;\text{ in }\; L^2(\Omega^-, d\mu_\varepsilon) \;\text{ as }\; \varepsilon \to 0. \quad (18)$$

Remark 3. Note that we can not use the restriction of this function to the construction of the sequence $\{p_\varepsilon\}_{\varepsilon>0}$ because a trace of an L^2-function on the $(n-1)$-dimensional surface S_ε is not defined. In view of this we call the sequence $\{p_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon>0}$ with properties (17)–(18) a smooth S_ε-approximation of the function $p^0 \in L^2(\Gamma_0)$.

Let $K_p \in C_0^\infty(\mathbb{R}^n)$ be nonnegative function such that

$$\int_{\mathbb{R}^n} K_p(x)\,\mathrm{d}x = 1, \quad \text{and} \quad K_p(x) = K_p(-x)\ \forall\, x \in \mathbb{R}^n.$$

Let $\widetilde{p}^0$ be the trivial extension by zero of the function $p^0 \in L^2(\Omega^-)$ to $\mathbb{R}^n$. For every fixed $\varepsilon > 0$ we define the following smoothing operator

$$\left(p^0\right)_\varepsilon = \left(\mathcal{K}_p^\varepsilon p^0\right)(x) = \varepsilon^{-n}\int_{\mathbb{R}^n} K_p\left(\frac{x-y}{\varepsilon}\right)\widetilde{p}^0(y)\,\mathrm{d}y.$$

We begin with the following result:

Proposition 3. *Let $p^0 \in L^2(\Omega^-)$ be a given function such that $\left\|p^0\right\|_{L^2(\Omega^-)} \le$ $\mathbf{C_p}$. Then there exists $\varepsilon_0 > 0$ such that the functions*

$$p_\varepsilon = \left(p^0\right)_\varepsilon\Big|_{S_\varepsilon} \quad (19)$$

satisfy the estimate (17) *for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Since $\left(p^0\right)_\varepsilon \in C(\overline{\Omega^-})$ it is clear that $p_\varepsilon^{sub} = \left(p^0\right)_\varepsilon\big|_{S_\varepsilon} \in L^2(S_\varepsilon, \mathcal{H}^{n-1})$. By analogy with (8) it is easy to show the validity of the following inequality

$$\left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(\Omega^-)} \le \int_{\Omega^-}\left(p^0\right)^2\,\mathrm{d}x = \|p^0\|^2_{L^2(\Omega^-)} \le \mathbf{C_p}^2. \quad (20)$$

Thus, taking into account (20), we have to prove that the estimate (17) is valid at least for ε small enough. In order to verify this fact we will show that the value

$$\sigma = \left| \varepsilon |\partial C|_H^{-1} \left\| \left(p^0\right)_\varepsilon \right\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})} - \left\| \left(p^0\right)_\varepsilon \right\|^2_{L^2(\Omega^-)} \right|$$

can be done as small as is wished for ε small enough.

To begin with, we note that

$$\left\| \left(p^0\right)_\varepsilon \right\|^2_{L^2(\Omega^-)} = \sum_{j=1}^{N^{n-1}} \int_{\left\{ x=(x',x_n)\ : \left| \begin{smallmatrix} x'\in\varepsilon(\Box+\mathbf{k}_j) \\ -\beta_0<x_n<0 \end{smallmatrix} \right. \right\}} \left(p^0\right)^2_\varepsilon \, \mathrm{d}x. \tag{21}$$

Here $\Box = [0,1)^{n-1}$. Further we make use the following notation:

$$L^{\mathbf{k}}_\varepsilon = \{(x',x_n)\ :\ x' \in \varepsilon\Box + \varepsilon\mathbf{k},\ -\beta_0 < x_n \le 0\},$$
$$G^{\mathbf{k}}_\varepsilon = \{(x',x_n)\ :\ x' \in \varepsilon C + \varepsilon\mathbf{k},\ -\beta_0 < x_n \le 0\},$$
$$\Delta_i S^{\mathbf{k}}_\varepsilon = \{(x',x_n)\ :\ x' \in \varepsilon\partial C + \varepsilon\mathbf{k},\ -\beta_0 + (i-1)\Delta < x_n \le -\beta_0 + i\Delta\},.$$

where $\Delta = \beta_0/J$.

Since $\left(p^0\right)_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ it means that a wild oscillation of this function is excluded on the cross sections of thin cylinders $L^{\mathbf{k}}_\varepsilon$ for ε small enough. Hence for a given $\eta > 0$ there exist an $\varepsilon_\eta > 0$, an integer $J \in \mathbb{N}$, and a collection of points $\left\{x^{\mathbf{k}}_{\varepsilon,i} \in \Delta_i S^{\mathbf{k}}_\varepsilon\right\}$ such that

$$\left| \int_{L^{\mathbf{k}}_\varepsilon} \left(p^0\right)^2_\varepsilon \mathrm{d}x - \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \mathcal{L}^n(\varepsilon\Box)\Delta \right| < \frac{\eta}{2N^{n-1}} \quad \forall\, \varepsilon \in (0,\varepsilon_\eta).$$

Taking into account the chain of transformations

$$\begin{aligned}
&\sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \mathcal{L}^n(\varepsilon\Box)\Delta \\
&\quad = \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \varepsilon^n \Delta = \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \varepsilon^n \frac{\mathcal{H}^{n-1}(\Delta_i S^{\mathbf{k}}_\varepsilon)}{H^{n-2}(\varepsilon\partial C)} \\
&\quad = \frac{\varepsilon}{H^{n-2}(\partial C)} \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \mathcal{H}^{n-1}(\Delta_i S^{\mathbf{k}}_\varepsilon) \\
&\quad = \varepsilon |\partial C|_H^{-1} \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \mathcal{H}^{n-1}(\Delta_i S^{\mathbf{k}}_\varepsilon)
\end{aligned}$$

and the fact that

$$\left| \sum_{i=1}^{J} \left(p^0\right)^2_\varepsilon (x^{\mathbf{k}}_{\varepsilon,i}) \mathcal{H}^{n-1}(\Delta_i S^{\mathbf{k}}_\varepsilon) - \int_{S^{\mathbf{k}}_\varepsilon} \left(p^0\right)^2_\varepsilon \, \mathrm{d}\mathcal{H}^{n-1} \right| < \frac{\eta |\partial C|_H}{2\varepsilon N^{n-1}}$$

for J large enough, we deduce: for a given $\eta > 0$ there exists an $\varepsilon_\eta > 0$ such that

$$\begin{aligned}\Big|\int_{L_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}x - \varepsilon|\partial C|_H^{-1} \int_{S_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1}\Big| \\ \le \Big|\int_{L_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}x - \varepsilon|\partial C|_H^{-1} \sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}})\mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}})\Big| \\ + \varepsilon|\partial C|_H^{-1} \Big|\sum_{i=1}^{J} \left(p^0\right)_\varepsilon^2 (x_{\varepsilon,i}^{\mathbf{k}})\mathcal{H}^{n-1}(\Delta_i S_\varepsilon^{\mathbf{k}}) \\ - \int_{S_\varepsilon^{\mathbf{k}}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1}\Big| < \frac{\eta}{N^{n-1}} \end{aligned} \quad (22)$$

for all $\varepsilon \in (0, \varepsilon_\eta)$.

As a result, combining this estimate with the representation (21), we come to the conclusion:

$$\begin{aligned}\Big|\left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(\Omega^-)} - \varepsilon|\partial C|_H^{-1} \left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})}\Big| \\ = \Big|\sum_{j=1}^{N^{n-1}} \Big(\int_{L_\varepsilon^{\mathbf{k}_j}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}x - \varepsilon|\partial C|_H^{-1} \int_{S_\varepsilon^{\mathbf{k}_j}} \left(p^0\right)_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{n-1}\Big)\Big| < \eta. \end{aligned} \quad (23)$$

Since relation (20) turns into the equality only in the case when p^0 is a constant, it follows that choosing an appropriate value $\eta > 0$ we can find an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta)$ the inequality

$$\varepsilon|\partial C|_H^{-1} \left\|\left(p^0\right)_\varepsilon\right\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})} \le \left\|p^0\right\|^2_{L^2(\Omega^-)} \le \mathbf{C_p}^2$$

holds true. Hence the estimate (17) is valid for $\varepsilon > 0$ small enough. □

It remains to substantiate the convergence property (18).

Proposition 4. *Assume the functions $p_\varepsilon \in C^\infty(S_\varepsilon)$ are defined by (19). Then the property (18) holds true.*

Proof. Let $\{\zeta_\varepsilon\}$ be any sequence of smooth functions such that $\zeta_\varepsilon \to p$ strongly in $L^2(\Omega^-)$ as $\varepsilon \to 0$. Let us show that

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \,\varphi \,\mathrm{d}\mu_\varepsilon = \int_{\Omega^-} p\,\varphi \,\mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n). \quad (24)$$

Indeed, let us partition Ω^- into cubes with edges ε and denote these cubes by the symbols $\varepsilon\square_n^j$. Then

$$\int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon = \sum_j \int_{\varepsilon\square_n^j} \zeta_\varepsilon(x)\, \varphi(x)\, \mathrm{d}\mu_\varepsilon + \sum \int_{\Omega^- \cap\, \varepsilon\square_n^j} \zeta_\varepsilon(x)\, \varphi(x)\, \mathrm{d}\mu_\varepsilon = \sum_j \zeta_\varepsilon(x_j)\, \varphi(x_j) \int_{\varepsilon\square_n^j} \mathrm{d}\mu_\varepsilon + \sum \int_{\Omega^- \cap\, \varepsilon\square_n^j} \zeta_\varepsilon(x)\, \varphi(x)\, \mathrm{d}\mu_\varepsilon,$$

where x_j is a Lebesgue point of p in the cube $\varepsilon\square_n^j$ and the second sum is calculated over the set of boundary cubes. By definition of the measure μ_ε we have

$$\int_{\varepsilon\square_n^j} \mathrm{d}\mu_\varepsilon = \varepsilon^n \int_{\square_n} \mathrm{d}\mu = \varepsilon^n.$$

Moreover,

$$\left| \sum \int_{\Omega^- \cap\, \varepsilon\square_n^j} \zeta_\varepsilon(x)\, \varphi(x)\, \mathrm{d}\mu_\varepsilon \right| \le \sup_{x \in \Omega^-} |\zeta_\varepsilon(x) \varphi(x)| \varepsilon^n \cdot D(\varepsilon),$$

where $D(\varepsilon)$ is a quantity of boundary cubes, and $\varepsilon^n D(\varepsilon) \to 0$ by Jordan's measurability of the set $\partial\Omega^-$. Thus, summarizing the above cited facts, we have

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}\mu_\varepsilon = \lim_{\varepsilon\to 0} \left[\sum_j \zeta_\varepsilon(x_j)\, \varphi(x_j) \varepsilon^n - \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] + \lim_{\varepsilon\to 0} \int_{\Omega^-} \zeta_\varepsilon \, \varphi \, \mathrm{d}x = \lim_{\varepsilon\to 0} \sum_j \left[\zeta_\varepsilon(x_j)\, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] \varepsilon^n$$
$$+ \lim_{\varepsilon\to 0} \sum \int_{\Omega^- \cap\, \varepsilon\square_n^j} \zeta_\varepsilon(x)\, \varphi(x)\, \mathrm{d}x + \int_{\Omega^-} p \, \varphi \, \mathrm{d}x$$
$$= \lim_{\varepsilon\to 0} \sum_j \left[\zeta_\varepsilon(x_j)\, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right] \varepsilon^n + \int_{\Omega^-} p \, \varphi \, \mathrm{d}x.$$

Let us suppose that

$$\lim_{\varepsilon\to 0} \sum_j \left| \zeta_\varepsilon(x_j)\, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right| \varepsilon^n > 0.$$

Then there exist a constant $C^* > 0$ and a value $\varepsilon^* > 0$ such that

$$\left| \zeta_\varepsilon(x_j)\, \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon\square_n^j} \zeta_\varepsilon \, \varphi \, \mathrm{d}x \right| \ge C^* \ \forall\, \varepsilon \in (0, \varepsilon^*)$$

(possibly except, for a finite number of indices j for every fixed ε). Hence extremely wild oscillations are present in the sequence $\{\zeta_\varepsilon \, \varphi\}$. However (see

[1],[2]), if we have very rapid fluctuations in the functions $\{\zeta_\varepsilon\,\varphi\}$, then the convergence $\zeta_\varepsilon\,\varphi \to p\,\varphi$ almost everywhere in Ω^- is excluded. This fact immediately reflects the failure of the strong convergence $\zeta_\varepsilon\,\varphi \to p\,\varphi$ in $L^2(\Omega^-)$ as $\varepsilon \to 0$ (see Valadier's Theorem [2]). So, our supposition was wrong and we get

$$\lim_{\varepsilon\to 0}\sum_j \left|\zeta_\varepsilon(x_j)\,\varphi(x_j) - \varepsilon^{-n}\int_{\varepsilon\square_n^j}\zeta_\varepsilon\,\varphi\,\mathrm{d}x\right|\varepsilon^n = 0.$$

As a result, we come to the relation (24).

To conclude the proof it remains to use the main property of classical smoothing

$$\left(p^0\right)_\varepsilon \to p^0 \ \text{ strongly in } \ L^2(\Omega^-) \ \text{ as } \ \varepsilon \to 0$$

and apply the relation (24). □

As the direct consequence of Proposition 3, which we feel to be interesting per se, we present one observation concerning L^2-functions defined on thick junctions.

Corollary 1. *Let Ω_ε be a thick multi-structure in $\mathbb{R}^n$, which consists of some domain Ω^+ and a large number of thin cylinders $G_\varepsilon^{\mathbf{k}}$ with a small cross section of the size εC and ε-periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 1 for 3-d example). Let $\Omega^- = \Sigma \times (a,b)$ be a domain which is filled up by the thin cylinders in the limit passage as $\varepsilon \to 0$. Let $f \in L^2_{loc}(\mathbb{R}^n)$ a given function. Then*

$$\|f\|^2_{L^2(\Omega^-)} \geq \frac{\varepsilon}{\mathcal{H}^{n-2}(\partial C)}\,\|(f)_\varepsilon\|^2_{L^2(S_\varepsilon;\mathcal{H}^{n-1})}$$

for ε small enough, where $(f)_\varepsilon$ denotes the direct smoothing of the function $\chi_{\Omega^-}f$, i.e.

$$(f)_\varepsilon = \varepsilon^{-n}\int_{\mathbb{R}^n} K\left(\frac{x-y}{\varepsilon}\right)\chi_{\Omega^-}(y)f(y)\,\mathrm{d}y.$$

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