On existence of efficient solutions to vector optimization problems in Banach spaces

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Abstract. In this paper, we present a new characterization of lower semicontinuity of vector-valued mappings and apply it to the solvability of vector optimization problems in Banach spaces. With this aim we introduce a class of vector-valued mappings that is more wider than the class of vector-valued mappings with the “typical” properties of lower semi-continuity including quasi and order lower semi-continuity. We show that in this case the corresponding vector optimization problems have non-empty sets of efficient solutions.

Keywords: lower semicontinuity, vector-valued optimization, efficient solutions, partially ordered spaces

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Introduction

In this paper, we present a new concept of lower semicontinuity for vector-valued mappings. We consider the case when the mappings take values in a real Banach space $Y$ partially ordered by a closed convex pointed cone $\Lambda$. In the vector-value case there are several possible ways to extend the “scalar” notion of lower semicontinuity (see, for example, [1, 2, 3, 6, 9, 11, 12]). Let us mention the lower semicontinuity, quasi lower semicontinuity, and order lower semicontinuity. Usually, in many papers the typical assumption is that the interior of the ordering cone $\Lambda$ is non-empty. However, in many interesting and important cases, this property does not hold. For instance, in the case when $Y = L^p(\Omega)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $p \in [1, +\infty)$, and $\Lambda$ is the natural cone of non-negative elements of $Y$, we have $\text{Int}\, \Lambda = \emptyset$. So, in this paper we make no additional assumptions on the cone $\Lambda$ and its interior. On the other
hand, there are many vector optimization problems with non-empty sets of efficient solutions, for which the corresponding vector-valued mappings satisfy none of the lower semicontinuity concepts mentioned above.

In view of this a new characterization of semicontinuity for such mappings is the main scope of this paper. We introduce, the so-called \( \Lambda_\tau \)-lower semicontinuity property for vector-valued mappings in Banach spaces (with respect to the \( \tau \)-topology of \( Y \)) which implies the previous ones. We apply this concept to the study of the vector optimization problems.

Let us describe the contents of the paper. Section 1 provides in details the main notation and ingredients needed in this work. In Section 2, we give the statement of the vector optimization problem in Banach spaces and the definition of its efficient solutions. Section 3 contains a short review of the main definitions of lower semicontinuity of vector-valued mappings, introduced in [3, 6, 7, 13], and some well-known facts concerning these notions. In Section 4, we introduce a new concept of lower semicontinuity for vector-value mappings with respect to different topologies of Banach spaces, and compare this notion with the previous ones. The last section contains our main result concerning the solvability of the vector optimization problem. All main notions and assertions are illustrated by numerous examples.

1 Preliminaries and notation

Throughout this paper, \( X \) and \( Y \) are two real Banach spaces. We assume that \( X \) is reflexive. Let \( \theta_Y \) be the zero-element of \( Y \). We suppose that these spaces, as topological spaces, are endowed with some topology \( \tau \), which usually is associated either with the strong topology \( \tau := s \) or with the weak topology \( \tau := w \). For a subset \( Y_0 \) of \( Y \), we denote by \( \text{Int}_\tau Y_0 \), \( \text{cl}_\tau Y_0 \), and \( \partial_\tau Y_0 \) the interior of \( Y_0 \), the closure of \( Y_0 \), and its boundary in \( Y \) with respect to the \( \tau \)-topology of \( Y \), respectively. By default \( \tau \) is always associated with the strong topology of the corresponding space. In this case, we will omit the index if no confusion may occur. Let \( \Lambda \subset Y \) be a closed convex cone, which is supposed to be pointed, that is, \( \Lambda \cap -\Lambda = \{ \theta_Y \} \). No assumption is required on the interior of \( \Lambda \).

The cone \( \Lambda \) defines a partial order on \( Y \) denoted by \( \preceq \). For any elements \( y, z \in Y \), we will write \( y \preceq z \) whenever \( z \in y + \Lambda \) and \( y \prec z \) for \( y, z \in Y \), if \( z - y \in \Lambda \setminus \{ \theta_Y \} \). We say that a sequence \( \{ y_k \}_{k=1}^{\infty} \subset Y \) is non-increasing and we use the notation \( y_k \downarrow \) whenever, for all \( k \in \mathbb{N} \), we have \( y_k + 1 \preceq y_k \).

We say that an element \( y^* \in Y_0 \) is \( \Lambda \)-minimal for the set \( Y_0 \subset Y \), if there is no \( y \in Y_0 \) such that \( y \prec y^* \), that is, \( Y_0 \cap (y^* - \Lambda) = \{ y^* \} \).

We denote with \( \Lambda \)-Min\((Y_0) \) the family of all such elements. We say that an element \( y^* \) is the \( \Lambda \)-ideal minimal point of the set \( Y_0 \), if \( y^* \preceq y \) for every \( y \in Y_0 \). By analogy we can introduce the sets of \( \Lambda \)-maximal and \( \Lambda \)-ideal maximal elements of the set \( Y_0 \).

Let us introduce two singular elements \(-\infty \) and \(+\infty \) in \( Y \). We assume that these elements satisfy the following conditions:

1) \(-\infty \preceq y \preceq +\infty , \forall y \in Y \); 2) \(+\infty + (-\infty) = \theta_Y \).

We use the notation \( \bar{Y} = Y \cup \{ \pm \infty \} \). Then \( +\infty \) is the \( \Lambda \)-greatest element of the set \( \bar{Y} \), and the element \(-\infty \) is its \( \Lambda \)-smallest element. We denote with \( Y^* \) a semi-extended Banach space:
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\( Y^* = Y \cup \{+\infty\} \). Following [14] we say that for a subset \( A \subset Y \) an element \( a \in Y \) is called a least upper bound of \( A \) when for every \( y \in Y \) the following property

\[
\text{for every } y \in Y, a \leq y \text{ if and only if } b \leq y \text{ for every } b \in A
\]

holds true. As usual, we denote by \( \sup \ A \) the least upper bound of \( A \). Similarly, the greatest lower bound of \( A \), whenever it exists, \( \inf \ A \), is defined by

\[
\text{for every } y \in Y, y \leq \inf A \text{ if and only if } y \leq z \text{ for every } z \in A.
\]

Then the next concept is the crucial point in our approach.

**Definition 1.** We say that a set \( A \) is the efficient \( \Lambda \)-infimum of the set \( Y_0 \subset Y \) with respect to the \( \tau \)-topology (or shortly \( \Lambda_{\tau} \)-infimum) if \( A \) is the collection of all \( \Lambda \)-minimal elements of the \( \tau \)-closure \( Y_0 \) in the case when this set is non-empty, and \( A \) is equal to \( \{-\infty\} \) in the opposite case.

Hereinafter we denote the efficient \( \Lambda_{\tau} \)-infimum for \( Y_0 \) by \( \text{Inf}^{\Lambda_{\tau}} Y_0 \). Thus, in view of the definition given above, we have

\[
\text{Inf}^{\Lambda_{\tau}} Y_0 := \begin{cases}
\Lambda\text{-Min(cl}_{\tau} Y_0), & \Lambda\text{-Min(cl}_{\tau} Y_0) \neq \emptyset, \\
-\infty, & \Lambda\text{-Min(cl}_{\tau} Y_0) = \emptyset.
\end{cases}
\]

We conclude this preliminaries by pointing out some basic definitions. Let \( X_\partial \) be a subset of the Banach space \( X \), and \( f : X_\partial \to Y \) be some mapping. In what follows we always associate the mapping \( f : X_\partial \to Y \) with its natural extension \( \hat{f} : X \to Y^* \) to the whole space \( X \), where

\[
\hat{f}(x) = \begin{cases}
f(x), & x \in X_\partial, \\
+\infty, & x \notin X_\partial.
\end{cases}
\]

Given a map \( f : X \to Y^* \), its domain is denoted by \( \text{Dom} \ f \) and defined by

\[
\text{Dom} f = \{ x \in X \mid f(x) \prec +\infty \}.
\]

Further we assume that \( \text{Dom} f \neq \emptyset \). A mapping \( f : X \to Y^* \) is said to be bounded below if there exists a \( z \in Y \) such that \( z \leq f(x) \) for all \( x \in X \).

**Definition 2.** A subset \( A \) of \( Y \) is said to be the strong efficient \( \Lambda \)-infimum (resp. the weak efficient \( \Lambda \)-infimum) of a mapping \( f : X \to Y^* \) and is denoted by \( \text{Inf}^{\Lambda_{s}} Y_0 \) (resp. \( \text{Inf}^{\Lambda_{w}} Y_0 \)), if \( A \) is the efficient \( \Lambda_{s} \)-infimum (resp. \( \Lambda_{w} \)-infimum) of the image \( f(X) \) of \( X \) in \( Y \), that is,

\[
\text{Inf}^{\Lambda_{s}} Y_0 := \{ x \in X \mid f(x) \prec +\infty \}.
\]

**Remark 1.** It is clear now that if \( a \in \text{Inf}^{\Lambda_{s}} Y_0 \) then

\[
\text{cl} \{ f(x) \mid \forall x \in X \} \cap (a - \Lambda) = \{a\}
\]

provided \( \Lambda\text{-Min [cl} \{ f(x) \mid \forall x \in X \}] \neq \emptyset \).
Let \( \{ y_k \}_{k=1}^{\infty} \) be a sequence in \( Y \). Let us denote by \( L^r \{ y_k \} \) the set of all its cluster points with respect to the \( \tau \)-topology of \( Y \), that is, \( y \in L^r \{ y_k \} \) if there is a subsequence \( \{ y_{k_i} \}_{i=1}^{\infty} \subset \{ y_k \}_{k=1}^{\infty} \) such that \( y_{k_i} \xrightarrow{i} y \) in \( Y \) as \( i \to \infty \). If \( \text{Inf}^{\Lambda, \tau} L^r \{ y_k \} = -\infty \), we assume that \( \{ -\infty \} \in L^r \{ y_k \} \). If \( \text{Sup}^{\Lambda, \tau} L^r \{ y_k \} = +\infty \), we assume that \( \{ +\infty \} \in L^r \{ y_k \} \). Let \( x_0 \in X \) be a fixed element. In what follows for an arbitrary mapping \( f : X \to Y^* \) we make use of the following sets:

\[
L^*_s(f, x_0) := \bigcup_{(x_k)_{k=1}^{\infty} \in \mathcal{M}_s(x_0)} L^r \{ f(x_k) \},
\]

\[
L^*_w(f, x_0) := \bigcup_{(x_k)_{k=1}^{\infty} \in \mathcal{M}_w(x_0)} L^r \{ f(x_k) \},
\]

where \( \mathcal{M}_s(x_0) \) and \( \mathcal{M}_w(x_0) \) are the sets of all sequences \( \{ x_k \}_{k=1}^{\infty} \subset X \) such that \( x_k \to x_0 \) strongly in \( X \) and weakly in \( X \), respectively.

**Definition 3.** We say that a subset \( A \subset Y \cup \{ \pm \infty \} \) is the \( \Lambda \)-lower sequential limit of the mapping \( f : X \to Y^* \) at the point \( x_0 \in X \) with respect to the product of the strong topology of \( X \) and the \( \tau \)-topology of \( Y \), and we use the notation \( A = \text{inf}^{\Lambda, \tau}_{x \to x_0} f(x) \), if

\[
\lim \inf_{x \to x_0}^{\Lambda, \tau} f(x) := \begin{cases} L^*_{\min} (f, x_0, X), & L^*_{\min} (f, x_0, X) \neq \emptyset, \\ L^*_{\min} (f, x_0), & L^*_{\min} (f, x_0, X) = \emptyset, \end{cases}
\]

(1)

where

\[
L^*_{\min} (f, x_0, X) = L^*_s(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x).
\]

**Remark 2.** Note that in the scalar case \( (f : X \to \mathbb{R}) \) the sets

\[
\text{Inf}^{\Lambda, \tau}_{x \in X} f(x) \quad \text{and} \quad \text{Inf}^{\Lambda, \tau}_{x \in X} L^*_w(f, x_0)
\]

are singletons. So, if \( L^*_s(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) \neq \emptyset \), then

\[
L^*_s(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) = \text{Inf}^{\Lambda, \tau}_{x \in X} L^*_w(f, x_0),
\]

and therefore the choice rules in (1) coincide and give the classical definition of the lower limit.

By analogy, we can introduce the notion of the \( \Lambda \)-lower sequential limit of \( f : X \to Y^* \) at \( x_0 \in X \) with respect to the product of the weak topology of \( X \) and the \( \tau \)-topology of \( Y \). In this case we have

\[
\lim \inf_{x \to x_0}^{\Lambda, \tau} f(x) := \begin{cases} L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x), & L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) \neq \emptyset, \\ \text{Inf}^{\Lambda, \tau}_{x \in X} L^*_w(f, x_0), & L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) = \emptyset. \end{cases}
\]

In particular, if \( \tau \) is associated with the strong topology of \( Y \), then following our previous conventions, we will use the notation

\[
\lim \inf_{x \to x_0}^{\Lambda, s} f(x) := \begin{cases} L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, s}_{x \in X} f(x), & L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, s}_{x \in X} f(x) \neq \emptyset, \\ \text{Inf}^{\Lambda, s}_{x \in X} L^*_w(f, x_0), & L^*_w(f, x_0) \cap \text{Inf}^{\Lambda, s}_{x \in X} f(x) = \emptyset. \end{cases}
\]

To illustrate the crucial role of the conditions

\[
L^*_s(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) \neq \emptyset \quad \text{and} \quad L^*_s(f, x_0) \cap \text{Inf}^{\Lambda, \tau}_{x \in X} f(x) = \emptyset
\]

of Definition 3, we give the following example.
Example 1. Let $X = Y = \mathbb{R}^2$, $X_0 = X_0^1 \cup X_0^2$, 
\begin{align}
X_0^1 &= \{ x \in \mathbb{R}^2 \mid (x_1 - 6)^2 + (x_2 - 6)^2 \leq 25, \ x_1 + x_2 \leq 7 \}, \\
X_0^2 &= \{ x \in \mathbb{R}^2 \mid x_1 + x_2 > 7, \ x_1 + x_2 \leq 8, \ x_1 \geq 1, \ x_2 \geq 1 \},
\end{align}
and let $\Lambda = \mathbb{R}^2$ be the cone of positive elements. Then the strong and weak topologies in $X$ and $Y$ coincide. We define a vector-value mapping $f : X_0 \to Y$ as follows:

$$f(x) = \begin{cases} x, & x \notin X_0, \\ [2], & x \in X_0^1 \cup \{A, C\}, \\ [2], & x \in X_0^2 \cup \{B, D\}, \end{cases}$$

where $A = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$, $X_0 = X_0^1 \cup X_0^2 \cup \{A, B, C, D\}$,

$$X_0^1 = \{ x \in X_0 \mid (x_1 - 6)^2 + (x_2 - 6)^2 = 25, \ 1 < x_1 \leq x_2 \}$$
$$X_0^2 = \{ x \in X_0 \mid (x_1 - 6)^2 + (x_2 - 6)^2 = 25, \ x_2 < x_1 < 6 \}.$$

Let us find the $\Lambda$-lower sequential limit of $f : X_0 \to Y$ at two points: firstly at $x_0 = A$, and after at $x_0 = C$. To begin with, we note that

$$\text{Inf}^{\Lambda,s}_{x \in X} \tilde{f}(x) = X_0^1 \cup X_0^2 \cup \{B, C\}$$
(see Fig.1). Then, in the case when $x_0 = A$, we have

$$L^s_*(\tilde{f}, x_0) = \left\{ A, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}.$$ 

Hence, since $L^s_*(\tilde{f}, x_0) \cap \text{Inf}^{\Lambda,s}_{x \in X} \tilde{f}(x) = \emptyset$, by Definition 3, we conclude that

$$\liminf_{x \to x_0} \Lambda^{s} L^s_*(\tilde{f}, x_0) = \text{Inf}^{\Lambda,s}_{x \in X} \tilde{f}(x) = \left\{ A, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}.$$
At the same time, if we take \( x_0 = C \), then \( L_s^* \left( \hat{f}, x_0 \right) = \{ C, \left[ \frac{6}{12} \right] \} \). Hence,

\[
\lim \inf_{x \to x_0} L_s^* \left( \hat{f}(x) \right) = \{ C, \left[ \frac{6}{12} \right] \}
\]

Hence,

\[
\lim \inf_{x \to x_0} \hat{f}(x) = L_s^* \left( \hat{f}(x_0) \right) \cap \text{Inf}_{x \in X} \hat{f}(x) = \{ C \}.
\]

2 The statement of the vector optimization problem

Let \( X_\partial \) be a non-empty bounded weakly closed subset of the reflexive Banach space \( X \). Let \( F : X_\partial \to Y \) be a given mapping. The vector optimization problem we consider can be stated as follows:

\[
\text{Minimize } F(x) \text{ with respect to the cone } \Lambda \text{ subject to } x \in X_\partial.
\]

In view of this we will associate the vector optimization problem (5) with the following triplet

\[
\langle X_\partial, F, \Lambda \rangle,
\]

where the set \( X_\partial \) is called the set of admissible solutions to the problem (5).

Definition 4. We say that \( x_0 \in X_\partial \) is a \( \Lambda \)-efficient solution of the problem (5) if \( x_0 \) realizes the strong efficient \( \Lambda \)-infimum of the mapping \( F : X_\partial \to Y \), that is,

\[
F(x_0) \in \text{Inf}_{x \in X_\partial}^\Lambda F(x).
\]

Definition 5. An element \( x_0 \in X_\partial \) is said to be a \( \Lambda_w \)-efficient solution to the problem (5) if \( x_0 \) realizes the weak efficient \( \Lambda \)-infimum of the mapping \( F : X_\partial \to Y \), that is,

\[
F(x_0) \in \text{Inf}_{x \in X_\partial}^{\Lambda_w} F(x).
\]

We denote by \( \text{Sol}(X_\partial; F; \Lambda) \) and \( \text{Sol}(X_\partial; F; \Lambda) \), respectively, the sets of all weak efficient solutions and all strong efficient solutions to the above vectorial problem. So, by definition, we have

\[
\text{Sol}(X_\partial; F; \Lambda) = \left\{ x_0 \in X_\partial \mid F(x_0) \in \text{Inf}_{x \in X_\partial}^\Lambda F(x) \right\},
\]

\[
\text{Sol}(X_\partial; F; \Lambda) = \left\{ x_0 \in X_\partial \mid F(x_0) \in \text{Inf}_{x \in X_\partial}^{\Lambda_w} F(x) \right\}.
\]

Remark 3. It is clear that Definitions 4 and 5 are identical in the case when the set \( F(X_\partial) = \{ F(x) \mid \forall x \in X_\partial \} \) is convex. To specify these definitions more exactly, we say that the vector optimization problem \( \langle X_\partial, F, \Lambda \rangle \) has a \( \Lambda_{(\tau, \mu)} \)-efficient solution \( x_0 \in X_\partial \) if \( x_0 \) is a \( \Lambda \)-efficient solution and \( x_k \to x_0 \) in the \( \mu \)-topology of \( X \) whenever \( F(x_k) \to F(x_0) \) with respect to the \( \tau \)-topology of \( Y \), that is, every \( \tau \)-minimizing sequence is \( \mu \)-convergent.

Remark 4. It should be emphasized the difference between the notion of \( \Lambda \)-efficient solutions to the vector optimization problem (5) and the “classical” definition of the weak efficient solutions. Let us recall that an element \( x^* \in X_\partial \) is said to be a weakly efficient solution to the problem (5) if Int \( \Lambda \neq \emptyset \) and \( F(x^*) \) is a minimal element of the set

\[
F(X_\partial) := \{ y \in Y \mid y = F(x) \forall x \in X_\partial \}.
\]
Figure 2. Efficient solutions of the vector optimization problem

with respect to the cone \( \{ \theta_y \} \cup \text{Int } \Lambda \), i.e., if there is no \( y \in F(X_\partial) \) such that \( F(x^*) \neq y \) and \( F(x^*) - y \in \text{Int } \Lambda \) (see [5]).

It is easy to show that each \( \Lambda_\tau \)-efficient solution is a weak efficient solution to this problem, but the converse is not true in general. Indeed, let \( x_0 \) be any element of \( \text{Sol}(X_\partial; F; \Lambda) \). We assume that the cone \( \Lambda \) has a non-empty interior. Then \( F(x_0) \in \Lambda\text{-Min} (\text{cl } F(X_\partial)) \). Hence, \( F(x_0) - y \notin \Lambda \) for all \( y \in \text{cl } F(X_\partial) \). So, \( F(x_0) - F(x) \notin \Lambda \) for all \( x \in X_\partial \). It immediately leads us to the conclusion:

\[
F(x_0) - F(x) \notin \text{Int } \Lambda, \quad \forall x \in X_\partial.
\]

Thus \( x_0 \) is a weak efficient solution to the problem (5), and we obtain the required: \( \text{Sol}(X_\partial; F; \Lambda) \neq \emptyset \).

The main question is to obtain an existence theorem of the \( \Lambda_\tau \)-efficient solutions for a vector optimization problem \( (X_\partial, F, \Lambda) \), that is, to find sufficient conditions which guarantee the relation \( \text{Sol}_\tau(X_\partial; F; \Lambda) \neq \emptyset \). The main interest here is in the proof of the relation \( \text{Sol}_\tau(X_\partial; F; \Lambda) \neq \emptyset \) without using scalarization process of vector optimization problem (5).

We begin with the following obvious result (see, for instance, [15]):

**Theorem 1.** If \( \Lambda\text{-Min} \{ F(x) \mid \forall x \in X_\partial \} \) is compact with respect to the strong topology of \( Y \) then \( \text{Sol}(X_\partial; F; \Lambda) \neq \emptyset \).

However, the strong compactness property of subsets in Banach spaces is a very restrictive assumption. So, we recall some additional notions and results from the non-smooth analysis of vector-valued mappings.

**3 Lower semicontinuity for vector-valued mappings**

It is well known that the concept of lower semicontinuity property, that was introduced for scalar functions by R. Baire, is a fundamental notion of mathematical analysis. Thanks to efforts of D. Hilbert and L. Tonelli, the main field of its application is the Calculus of Variations and the scalar minimization theory. A very natural and challenging question is,
such that $(s-lsc)$ at $x$ has been proved in [6]).

In the vector-valued case there are several possible extensions of the “scalar” notion of lower semicontinuity (see, for example, [1, 2, 3, 6, 11, 12]). We recall now a few main definitions of lower semicontinuity of a vector-valued mapping with respect to the strong topologies of $X$ and $Y$, introduced in [3, 6, 7, 13].

Definition 6. [13] A mapping $F : X \to Y^*$ is said to be lower semicontinuous (lsc) at $x_0 \in X$, if for any neighborhood $V$ of $F(x_0)$ in $Y$, there is a neighborhood $U$ of $x_0$ in $X$ such that $F(U) \subset V + \Lambda \cup \{+\infty\}$.

Definition 7. [6] A mapping $F : X \to Y^*$ is said to be sequentially lower semicontinuous (s-lsc) at $x_0 \in X$, if for any $b \in Y$ satisfying $b \not\leq F(x_0)$ and for any sequence $\{x_k\}_{k=1}^{\infty}$ of $X$ which converges to $x_0$, there exists a sequence $\{b_k\}_{k=1}^{\infty}$ in $Y$ converging to $b$ and satisfying $b_k \not\leq F(x_k)$, for any $k \in \mathbb{N}$.

Remark 5. For $x_0 \in X$, the Definition 7 can be expressed as follows. For each sequence $\{x_k\}_{k=1}^{\infty}$ converging to $x_0$, there exists a sequence $\{b_k\}_{k=1}^{\infty}$ converging to $F(x_0)$ such that $b_k \not\leq F(x_k)$ for all $k \in \mathbb{N}$.

Note also that, Definitions 6 and 7 coincide whenever $X$ and $Y$ are metrizable spaces (it has been proved in [6]).

Definition 8. [3] A mapping $F : X \to Y^*$ is said to be quasi lower semicontinuous (q-lsc) at $x_0 \in X$, if for each $b \in Y$ such that $b \not\leq F(x_0)$, there exists a neighborhood $U$ of $x_0$ in $X$ such that $b \not\leq F(x)$ for each $x \in U$.

Definition 9. [7] A mapping $F : X \to Y^*$ is said to be order lower semicontinuous (o-lsc) at $x_0 \in X$, if for each sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converging to $x_0$ for which there exists a sequence $\{\epsilon_k\}_{k=1}^{\infty} \subset Y$ converging to $\theta_Y$ such that the sequence $\{F(x_k) + \epsilon_k\}_{k=1}^{\infty}$ is non-increasing, there exists a sequence $\{g_k\}_{k=1}^{\infty} \subset Y$ converging to $\theta_Y$ such that

$$F(x_0) \leq F(x_k) + g_k$$

for all $k \in \mathbb{N}$,

in symbols, $x_k \to x_0$ and $F(x_k) + O(1) \searrow \Rightarrow F(x_0) \leq F(x_k) + o(1)$.

A mapping $F$ is lsc (resp., q-lsc, o-lsc) if $F$ is lsc (resp., q-lsc, o-lsc) at each point of $X$.

Let us give some well-known facts concerning these notions.

(1) Whenever $X$ is metrizable and $Y = \mathbb{R}$, the s-lc continuity coincides with the classical lower semicontinuity property. In this case, a function $F : X \to \mathbb{R}$ is s-lc at every point $x_0$ if, and only if its epigraph

$$\text{epi } F := \{ (x, y) \in X \times Y \mid y \in F(x) + \Lambda \}$$

is closed in $X \times \mathbb{R}$.

(2) A mapping $F$ is lsc at $x_0$ if and only if $\liminf_{x \to x_0, a \in \Lambda} \|F(x_0) + a - F(x)\|_Y = 0$.

(3) A mapping $F$ is q-lsc at $x_0$ if and only if for each $b \in Y$, the set

$$\{ F \leq b \} := \{ x \in X \mid F(x) \leq b \}$$

is closed in $X$.
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(4) A lsc mapping at $x_0$ is both q-lsc and o-lsc at this point.

(5) A q-lsc mapping is o-lsc at this point if either $F : X \to Y^*$ is bounded below and the dimension of $Y$ is finite, or the pair $(Y, \Lambda)$ has the monotone bounds property (BMP), i.e., any sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ converging to $\theta_Y$ has a subsequence $\{y_{k_i}\}_{i=1}^{\infty}$ for which there exists a non-increasing sequence $\{y_i\}_{i=1}^{\infty} \subset Y$ converging to $\theta_Y$ such that $y_{k_i} \leq y_i$, for all $i \in \mathbb{N}$ (see [7]).

(6) Every lsc mapping has a closed epigraph (see [4]), but the converse is not true as the following counterexample in [12] shows: the mapping $F : \mathbb{R} \to \mathbb{R}^2$ defined by

$$
F(x) = \begin{cases} 
(0, 0), & \text{if } x = 0, \\
(0, 1/|x|), & \text{otherwise},
\end{cases}
$$

is not lsc at 0 while its epigraph (with respect to the cone $\Lambda = \mathbb{R}_+^2$) is closed. At the same time, as immediately follows from Definitions 8 and 9, this mapping is both q-lsc and o-lsc at 0.

(7) The notion of lsc, q-lsc, and o-lsc coincide for the case when $Y = \mathbb{R}$, but not in general. Indeed, as shown in the previous example, the mapping $F : \mathbb{R} \to \mathbb{R}^2$ is both q-lsc and o-lsc but not lsc at 0. On the other hand, without BMP, the implication q-lsc $\implies$ o-lsc is false as well. Let us take $Y = \mathbb{R}^2$ and $\Lambda = \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}$. Then the mapping $F : \mathbb{R} \to \mathbb{R}^2$ defined by

$$
F(x) = \begin{cases} 
(0, 0), & \text{if } x = 0, \\
(|x|, |x| + 1), & \text{otherwise},
\end{cases}
$$

is q-lsc but not o-lsc at $x = 0$.

(8) If epi $F$ is closed then $F$ is quasi-lsc. The converse is true if the interior of $\Lambda$ is non-empty.

We end up this section by the examples of a vector-valued mapping for which both the quasi-ls continuity property and the order-ls continuity property do not hold at some point $x_0$. Moreover, as we will see later, this point is a $\Lambda$-efficient solution to the corresponding vector optimization problem. This is the main reason to introduce a new notion of lower semicontinuity weaker than the others three.

**Example 2.** [10] Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}_+^2$ be the cone of positive elements. To state a vector optimization problem $(X, \theta, F, \Lambda)$, we define the set of admissible solutions $X_\theta$ and the mapping $F : X_\theta \to Y$ as follows:

$$
X_\theta = \{ x \in \mathbb{R}^1 | -3 \leq x \leq -1 \},
$$

(7)

$$
F(x) = \begin{bmatrix} -x \\ 2 \end{bmatrix}, \text{ for all } x \neq -1, \quad F(-1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
$$

(8)

Let $x_0 = -1$. Then

$$
F(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \liminf_{r \to 0}^{\text{loc}} F(x) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}
$$

(see Fig. 3). Let us take $b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Obviously $b \not\in F(x_0)$ and there is no neighborhood of the point $x_0$ such that $b \not\in F(x)$ for all $x$ from this neighborhood. Hence, this mapping is not q-lsc at the point $x_0$. On the other hand, every sequence $\{x_k\}_{k=1}^{\infty}$, which is converging to 0 and such that $2 > x_k \neq 0$ for all $k \in \mathbb{N}$, is $\theta$-admissible, that is, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset Y$ converging to $\left[0 \right]$ such that the sequence $\{F(x_k) + x_k\}_{k=1}^{\infty}$ is non-increasing (see Definition 9).
Moreover, \([1, 2]\) is its limit. However, in this case there is no sequence \(\{g_k\}_{k=1}^\infty \subseteq Y\) converging to \([0, 0]\) such that
\[
\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{F}(x_0) \leq \mathbf{F}(x_k) + g_k \quad \text{for all} \quad k \in \mathbb{N}.
\]
Hence, the order lower semicontinuity property for the mapping \(\mathbf{F}\) is failed at the point \(x_0\).

The next example indicates the case when a vector-value mapping \(\mathbf{F} : X_\partial \rightarrow Y\) is not quasi lower semicontinuous at any point of \(X_\partial\), whereas it possesses a \(\Lambda\)-lower semicontinuity property on the whole of the domain \(X_\partial\).

**Example 3.** Let \(X_\partial\) be a bounded closed subset of a reflexive Banach space \(X\), let \(Y = \mathbb{R}^2\), and let \(\Lambda = \mathbb{R}_+^2\) be the cone of positive elements in \(\mathbb{R}^2\). Let us consider the mapping \(\mathbf{F} : X_\partial \rightarrow \mathbb{R}^2\) defined as follows

\[
\mathbf{F}(x) = \begin{bmatrix} \|x\| \\ -\|x\| \end{bmatrix}, \quad \forall x \in X_\partial.
\]

Then \(\mathbf{F}(X_\partial)\) is a segment

\[
D = \left\{ y \in \mathbb{R}^2 \mid y = \alpha \begin{bmatrix} m \\ -m \end{bmatrix} + (1 - \alpha) \begin{bmatrix} M \\ -M \end{bmatrix}, \alpha \in [0, 1] \right\},
\]

where \(m = \min_{x \in X_\partial} \|x\|\) and \(m = \max_{x \in X_\partial} \|x\|\). Since \(\text{Inf}_{x \in X_\partial}^{\Lambda, \tau} \mathbf{F}(x) = D\), it follows that each element of \(X_\partial\) is a \(\Lambda\)-efficient solution to the corresponding problem \((X_\partial, F, \Lambda)\). However, because of the fact that

\[
\liminf_{k \to \infty} \|x_k\| \geq \|x\|, \quad \forall x_k \rightharpoonup x \quad \text{in} \quad X,
\]

the lower and quasi lower semicontinuity properties for \(\mathbf{F} : X_\partial \rightarrow \mathbb{R}^2\) are broken at all points \(x \in X_\partial\).

At the same time for every \(x_0 \in X_\partial\) we have

\[
L_w^*(\hat{F}, x_0) := \bigcup_{x_k \rightharpoonup x_0} L^*(\hat{F}(x_k)) \subseteq \text{Inf}_{x \in X_\partial}^{\Lambda, \tau} \mathbf{F}(x),
\]

and whence \(\mathbf{F}(x_0) \in \text{Inf}_{x \in X_\partial}^{\Lambda, \tau} \mathbf{F}(x)\) due to the property (9). Thus, the objective function \(\mathbf{F} : X_\partial \rightarrow Y\) is sequentially \(\Lambda\)-lower semicontinuous at each point of \(X_\partial\).
4 Λτ-lower semicontinuity property

In this section, we introduce a new concept of lower semicontinuity for vector-value mapping with respect to the strong and weak topologies of the spaces X and Y. We compare this notion with the previous ones and give some examples. Let \( \hat{F} : X \to Y^\ast \) denote the natural extension of \( F : X_0 \to Y \) to the whole space \( X \).

**Definition 10.** We say that a mapping \( F : X_0 \to Y \) is \( \Lambda_\tau \)-lower semicontinuous (\( \Lambda_\tau \)-lsc) at the point \( x_0 \in X_0 \) (with respect to the strong topology of \( X \)) if
\[
F(x_0) \in \liminf_{x \to x_0} ^{\Lambda_\tau} \hat{F}(x).
\]

**Definition 11.** A mapping \( F : X_0 \to Y \) is said to be weakly \( \Lambda_\tau \)-lower semicontinuous (\( \Lambda_\tau \)-wlsc) at the point \( x_0 \in X_0 \) if
\[
F(x_0) \in \liminf_{x \to x_0} ^{\Lambda_\tau} \hat{F}(x).
\]

Compare Definitions 10 and 11 with Definition 3.

A mapping \( F \) is \( \Lambda_\tau \)-lsc (resp., \( \Lambda_\tau \)-wlsc) if \( F \) is \( \Lambda_\tau \)-lsc (resp., \( \Lambda_\tau \)-wlsc) at each point of \( X_0 \). As immediately follows from Definitions 10 and 11, the following result is obvious.

**Proposition 1.** The weakly \( \Lambda_\tau \)-lower semicontinuity of a mapping \( F : X_0 \to Y \) implies its \( \Lambda_\tau \)-lower semicontinuity.

To characterize the properties of \( \Lambda_\tau \)-lower semicontinuity more precisely, we begin with the following assertion.

**Lemma 1.** If a mapping \( F : X_0 \to Y \) is \( q \)-lower semicontinuous at \( x_0 \in X_0 \) with respect to the \( \tau \)-topology of \( Y \) and the strong topology of \( X \), then \( F \) is \( \Lambda_\tau \)-lower semicontinuous at this point.

Proof. Let \( F : X_0 \to Y \) be a \( q \)-lower semicontinuous mapping at the point \( x_0 \in X_0 \), and let \( \hat{F} : X \to Y^\ast \) be its natural extension. Let \( \{ x_k \} _{k=1}^\infty \) be a sequence strongly converging to \( x_0 \), i.e., \( \{ x_k \} _{k=1}^\infty \in \mathbb{M}_s(x_0) \). Let us assume that there exist a subsequence \( \{ F(x_{k_i}) \} _{i=1}^\infty \) and an index \( i^* \in \mathbb{N} \) such that \( F(x_{k_i}) \not\in F(x_0) \) for all \( i \geq i^* \). Then, in view of the definition of the quasi-lower semicontinuity, we just conclude that \( \{ +\infty \} \in L_\tau^\ast (\hat{F}, x_0) \). To characterize the set \( \liminf_{x \to x_0} ^{\Lambda_\tau} \hat{F}(x) \), we suppose that the corresponding image sequence \( \{ F(x_k) \} _{k=1}^\infty \) is bounded above with respect to the cone \( \Lambda \). In this case there can be found an index \( k^* \) such that
\[
F(x_k) \geq F(x_0), \quad \forall k \geq k^*.
\]

Hence, for any \( y^* \in L_\tau^\ast (\hat{F}, x_0) \), we have \( F(x_0) \not\leq y^* \). It means that
\[
\{ F(x_0) \} \in \text{Inf}^{\Lambda_\tau} L_\tau^\ast (\hat{F}, x_0).
\]

Thus, due to Definition 3, we deduce: \( F(x_0) \in \liminf_{x \to x_0} ^{\Lambda_\tau} \hat{F}(x) \). This concludes the proof.

As a consequence of this result and the properties of quasi-lower semicontinuity, we have: if \( F \) is lsc then \( F \) is \( \Lambda_\tau \)-lsc. However, in general, for vector-value mappings, \( \Lambda_\tau \)-ls continuity does not imply \( q \)-lsc. Indeed, let us consider the mapping \( F : X_0 \to Y \) defined in example 2.
As it was shown before, this mapping is neither q-lsc nor o-lsc mapping at the point $x_0 = -1$. However, taking into account the fact that

$$ F(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} $$

we just obtain the fulfillment of the inclusion

$$ F(x_0) \in \liminf_{\tau \to x_0} \hat{F}(x). $$

Hence, $F$ is a $\Lambda_\tau$-lower semicontinuous mapping at $x_0 = -1$.

**Lemma 2.** If a mapping $F : X_\partial \to Y$ is order-lower semicontinuous at $x_0 \in X_\partial$ with respect to the $\tau$-topology of $Y$ and the strong topology of $X$, then $F$ is $\Lambda_\tau$-lower semicontinuous at this point.

Proof. Let $\{x_k\}_{k=1}^{\infty} \subset X_\partial$ be an $\alpha$-admissible sequence strongly converging to $x_0$ in $X$ for which the set $\{F(x_k)\}_{k=1}^{\infty}$ is relatively $\tau$-compact in $Y^\tau$. Then for $\{x_k\}_{k=1}^{\infty}$ there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty} \subset Y$ $\tau$-converging to $\theta_Y$ such that the sequence $\{F(x_k) + \varepsilon_k\}_{k=1}^{\infty}$ is non-increasing. Hence, the corresponding image sequence $\{F(x_k)\}_{k=1}^{\infty}$ is bounded above, that is, there are elements $y \in Y$ and $k^* \in \mathbb{N}$ such that $F(x_k) \leq y$ for all $k \geq k^*$. Since the mapping is $\alpha$-lsc at $x_0$, it follows that there exists a sequence $\{g_k\}_{k=1}^{\infty} \subset Y$ $\tau$-converging to $\theta_Y$ such that $F(x_0) \leq F(x_k) + g_k$ for all $k \in \mathbb{N}$. Hence the sequence $\{F(x_k)\}_{k=1}^{\infty}$ is bounded below. So, we may suppose that there exists an element $y^* \in Y$ such that $F(x_k) \to y^*$ with respect to the $\tau$-topology of $Y$. This fact can be written as $F(x_k) + g_k(1) = y^*$ when $k \to \infty$. Since $F$ is $\alpha$-lsc at $x_0$, that implies $F(x_0) \leq F(x_k) + g_k(1) = y^* + g_k(1)$, and since $\Lambda$ is closed, it follows that passing to the limit in the last inequality as $k \to \infty$, we obtain $F(x_0) \leq y^*$, where $y^* \in \Lambda^\tau(\hat{F}, x_0) \cap Y$. Since $\{x_k\}_{k=1}^{\infty} \subset X_\partial$ is an arbitrary $\alpha$-admissible sequence strongly converging to $x_0$, it follows that

$$ F(x_0) \in \Lambda^\tau(\hat{F}, x_0) \cap Y \quad \text{and} \quad F(x_0) \leq y^*, \quad \forall y^* \in \Lambda^\tau(\hat{F}, x_0) \cap Y. \tag{10} $$

As a result, we conclude that the set $\text{Inf}^\tau \Lambda^\tau(\hat{F}, x_0) \cap Y$ consists of the unique element $F(x_0)$. Indeed, if we suppose the converse, then (10) just leads us to a contradiction with the definition of $\tau$-efficient $\Lambda$-infimum. Therefore, taking into account Definition 3, we get

$$ \liminf_{\tau \to x_0} \hat{F}(x) = \{ F(x_0) \}. $$

Thus, the $\Lambda_\tau$-lower semicontinuity property of the mapping $F$ at $x_0$ is proved.

It is well-known that for real-valued mappings $F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup +\infty$ the notions of lsc, q-lsc, and $\alpha$-lsc are equivalent (see [7]). However, as immediately follows from Definition 10, for real-value mappings the condition

$$ F(x_0) \in \liminf_{\tau \to x_0} \hat{F}(x) $$

is identical to the following one $F(x_0) \leq \liminf \hat{F}(x)$. Hence, in this case, lsc and $\Lambda_\tau$-lsc are identical properties. As a result, we come to the following conclusion:

**Lemma 3.** For real-value mapping $F : X \to \overline{\mathbb{R}}$ the four notions of lower semicontinuity given above are equivalent.

To conclude this section we give the following observation concerning the property of two $\Lambda_\tau$-lsc mappings. It is well known that the sum of two q-lsc (resp. $\alpha$-lsc) mappings is not a q-lsc (resp. $\alpha$-lsc) mapping in general. Due to the following example, we can give a similar conclusion for the $\Lambda_\tau$-lsc mappings.
Example 4. Let \( X = \mathbb{R}, Y = \mathbb{R}^2 \), and let \( \Lambda = \mathbb{R}^2_+ \) be the cone of positive elements. Let us consider the mappings \( F : \mathbb{R} \rightarrow \mathbb{R}^2 \) and \( G : \mathbb{R} \rightarrow \mathbb{R}^2 \) defined by

\[
F(x) = \begin{cases} \begin{bmatrix} 0 \\ -2+|x|^{-1} \end{bmatrix} & \text{if } x = 0, \\ \begin{bmatrix} -2+|x|^{-1} \\ -2|x|^{-1} \end{bmatrix} & \text{if } x \neq 0, \end{cases} \quad G(x) = \begin{cases} \begin{bmatrix} 0 \\ -|x|^{-1} \end{bmatrix} & \text{if } x = 0, \\ \begin{bmatrix} -|x|^{-1} \\ 2|x|^{-1} \end{bmatrix} & \text{if } x \neq 0. \end{cases}
\]

It is easy to see that each of these mappings is \( q \)-lsc at \( x_0 = 0 \) since for all \( b \in \mathbb{R}^2 \) such that \( b \leq [0] \) it is impossible to find any sequence \( \{x_k\}_{k=1}^\infty \) converging to 0 and satisfying condition \( F(x_k) \leq b \) (resp. \( G(x_k) \leq b \)) for all \( k \in \mathbb{N} \). So, due to Lemma 1, these mappings are \( \Lambda \)-lsc at 0. However, for the mapping \( F + G \) we have

\[
L^*_s(F(0) + G(0)) = \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \quad \text{and} \quad \text{Inf}_{x \in X}^{\Lambda,s} [F(x) + G(x)] = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}.
\]

Hence,

\[
\text{Inf}_{x \in X_0}^{\Lambda,s} [F(x) + G(x)] = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} \neq F(0) + G(0),
\]

and we obtain the required conclusion: the sum of two \( \Lambda \)-lsc mappings is not a \( \Lambda \)-lsc mapping in general.

Remark 6. We conclude this section with the following observation. As follows from the definition of the \( \Lambda \)-lower semicontinuity for vector-value mappings \( F : X_0 \rightarrow Y \), this property essentially depends on the domain \( X_0 \subset X \). In fact, the assertion: “if \( F : X \rightarrow Y \) is a \( \Lambda \)-lower semicontinuous mapping then its restriction on any bounded subset \( X_0 \subset X \) preserves this property at every point of \( X_0 \)” can be wrong in general. However such situation is both natural and typical in the vectorial case. Indeed, for the different sets of admissible solutions \( X_0^1, X_0^2 \subseteq X_0^3 \cap X_0^4 \neq \emptyset \) and any point \( x_0 \) such that \( x_0 \in X_0^1 \cap X_0^2 \), the sets \( \text{Inf}_{x \in X_0}^{\Lambda,s} L^*_s(F, x_0) \) and \( \text{Inf}_{x \in X_0}^{\Lambda,s} F(x) \) are not singletons in general. So, the sets

\[
\text{Inf}_{x \in X_0}^{\Lambda,s} L^*_s(F, x_0) \cap \text{Inf}_{x \in X_0}^{\Lambda,s} F(x) \quad \text{and} \quad \text{Inf}_{x \in X_0}^{\Lambda,s} L^*_s(F, x_0) \cap \text{Inf}_{x \in X_0}^{\Lambda,s} F(x)
\]

can be drastically different as well. Thus, in view of Definitions 3 and 10, the mappings \( F : X_0^1 \rightarrow Y \) and \( F : X_0^2 \rightarrow Y \) can be distinguished by a \( \Lambda \)-lower semicontinuity property at the point \( x_0 \in X_0^1 \cap X_0^2 \).

5 Existence theorem of the \( \Lambda \)-efficient solutions for vector optimization problems

We begin with the following supposition: assume that the ordering cone \( \Lambda \) possesses the so-called \( D \)-property, that is, every decreasing sequence in \( Y \) is \( \tau \)-convergent if and only if this sequence is \( \Lambda \)-lower bounded. For instance, the ordering cone of positive elements in \( L^p(\Omega) \) \((1 < p < +\infty)\) which is defined as

\[
\Lambda_{L^p(\Omega)} = \{ f \in L^p(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega \}
\]

satisfies this property with respect to both the weak and the strong topologies of \( L^p(\Omega) \) (see [8]).
Theorem 2. Let $X$ and $Y$ be Banach spaces, and let $\Lambda \subseteq Y$ be a closed convex ordering pointed cone, which is supposed to be with $D$-property. Assume that $X_0$ is a compact subset of $X$ (with respect to the strong topology), and $F : X_0 \rightarrow Y$ is a $\Lambda$-lower semicontinuous mapping. Then the vector optimization problem $\langle X_0, F, \Lambda \rangle$ has a non-empty set of $\Lambda$-efficient solutions $\text{Sol}_v(X_0; F; \Lambda)$.

Proof. To begin with, we prove that $\{-\infty\} \not\in \text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x)$. To do so, it is sufficient to show that if $\{x_k\}_{k=1}^{\infty} \subseteq X_0$ is a sequence such that its image $\{F(x_k)\}_{k=1}^{\infty} \subseteq Y$ is a decreasing sequence in $Y$, then there is an element $z \in Y$ such that $z \preceq F(x_k)$ for all $k \in \mathbb{N}$. Let us assume the converse. Then there are sequences $\{\hat{z}_k\}_{k=1}^{\infty} \subseteq X_0$ and $\{\hat{z}_k\}_{k=1}^{\infty} \subseteq Y$ such that $\hat{z}_{k+1} \preceq \hat{z}_k \ \forall k \in \mathbb{N}$, and

$$\text{Inf}_{x \in X_0}^{\Lambda, \tau} \{\hat{z}_k\}_{k=1}^{\infty} = \{-\infty\}, \quad F(\hat{z}_k) \preceq \hat{z}_k \ \forall k \in \mathbb{N}. \quad (11)$$

By the initial assumptions, the family $\{\hat{z}_k\}_{k=1}^{\infty} \subseteq X_0$ is compact, so we may suppose that $\hat{z}_k \rightarrow x^*$ in $X$, where $x^*$ is some element of $X_0$. Then, by monotonicity of $\{\hat{z}_k\}_{k=1}^{\infty} \subseteq Y$ and $D$-property of $\Lambda$, we can pass to the limit in $F(\hat{z}_k) \preceq \hat{z}_k$ as $k \rightarrow \infty$. As a result, we have

$$\xi \leq -\infty, \quad \forall \xi \in L^* \{F(\hat{z}_k)\}, \quad (12)$$

where $L^* \{F(\hat{z}_k)\}$ is the set of all cluster points of $\{F(\hat{z}_k)\}_{k=1}^{\infty}$ with respect to the $\tau$-topology of $Y$. On the other hand, in view of Definition 3 and the $\Lambda$-lower semicontinuity of $F$, we have

$$F(x^*) \in \lim \inf_{x \rightarrow x^*, \tau} F(x), \quad \text{and hence} \quad F(x^*) \not\in \xi, \quad \forall \xi \in L^* \{F(\hat{z}_k)\}. \quad (13)$$

Combining this result with (12), we obtain $F(x^*) \not\in -\infty$. However this contradicts (11). Hence $\text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x) \not\in \{-\infty\}$.

Let $\xi$ be any element of $\text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x)$. Then, by definition of the $\Lambda$-efficient infimum, there exists a sequence $\{y_k\}_{k=1}^{\infty} \subseteq Y$ such that $y_k \nearrow \xi$ in $Y$. We define a sequence $\{x_k\}_{k=1}^{\infty} \subseteq X_0$ as follows $F(x_k) = y_k$ for all $k \in \mathbb{N}$. Since the set $X_0$ is compact, we may suppose that there exists $x_0 \in X_0$ such that $x_k \rightarrow x_0$ in $X$. Hence $\xi \in L^*_\tau(F, x_0)$, and we get

$$L^*_\tau(F, x_0) \cap \text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x) \neq \emptyset.$$  

Then, due to the $\Lambda$-lower semicontinuity of the mapping $F$ on $X_0$ and Definition 3, we obtain

$$F(x_0) \in \lim \inf_{x \rightarrow x_0, \tau} F(x) = L^*_\tau(F, x_0) \cap \text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x).$$

Hence, $F(x_0) \in L^*_\tau(F, x_0)$, which implies we may assume

$$F(x_0) = \xi, \quad \text{and} \quad \xi \in \text{Inf}_{x \in X_0}^{\Lambda, \tau} F(x).$$

Thus, $x_0 \in \text{Sol}_v(X_0; F; \Lambda)$ and this concludes the proof.

However, the compactness property of the set of admissible solutions $X_0$ is a very restrictive assumption. In view of this we use the Banach-Alaoglu Theorem in reflexive Banach spaces, which leads us to the following generalization of the previous theorem.

Theorem 3. Let $X_0$ be a bounded weakly closed subset of a reflexive Banach space $X$, let $Y$ be a Banach space partially ordered by a closed convex pointed cone $\Lambda$, and let $F : X_0 \rightarrow Y$ be a weakly $\Lambda$-lower semicontinuous mapping. Then the vector optimization problem $\langle X_0, F, \Lambda \rangle$ has a non-empty set of the $\Lambda$-efficient solutions $\text{Sol}_v(X_0; F; \Lambda)$. 

Proof. We will only deal with that part of the previous proof concerning the compactness property of the sequences $\{\hat{x}_k\}_{k=1}^\infty$ and $\{x_k\}_{k=1}^\infty$. Indeed, taking into account the initial suppositions and the Banach-Alaoglu Theorem, the subset $X_0$ is sequentially compact with respect to the weak topology of $X$. Hence we may suppose that, passing to subsequences if necessary, each of the above sequences is weakly convergent to some elements of $X_0$. To conclude the proof, we can use motivations similar to the proof of the previous theorem changing the components $L^s_{\tau}(F,x_0)$ and $\liminf_{x\rightarrow x_0} \Lambda^{s,\tau}_{\hat{F}}(x)$ onto $L^w_{\tau}(F,x_0)$ and $\liminf_{x\rightharpoonup x_0} \Lambda^{s,\tau}_{\hat{F}}(x)$, respectively.

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