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On existence of efficient solutions to vector optimization problems in Banach spaces

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Abstract. In this paper, we present a new characterization of lower semicontinuity of vector-valued mappings and apply it to the solvability of vector optimization problems in Banach spaces. With this aim we introduce a class of vector-valued mappings that is more wider than the class of vector-valued mappings with the "typical" properties of lower semi-continuity including quasi and order lower semi-continuity. We show that in this case the corresponding vector optimization problems have non-empty sets of efficient solutions.

Keywords: lower semicontinuity, vector-valued optimization, efficient solutions, partially ordered spaces

MSC 2000 classification: primary 46B40, 49J45 secondary 90C29, 49N90

Introduction

In this paper, we present a new concept of lower semicontinuity for vector-valued mappings. We consider the case when the mappings take values in a real Banach space Y partially ordered by a closed convex pointed cone Λ. In the vector-value case there are several possible ways to extend the "scalar" notion of lower semicontinuity (see, for example, [1, 2, 3, 6, 9, 11, 12]). Let us mention the lower semicontinuity, quasi lower semicontinuity, and order lower semicontinuity. Usually, in many papers the typical assumption is that the interior of the ordering cone Λ is non-empty. However, in many interesting and important cases, this property does not hold. For instance, in the case when $Y = L^p(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^n$, $p \in [1, +\infty)$, and Λ is the natural cone of non-negative elements of Y, we have $\operatorname{Int}\Lambda = \emptyset$. So, in this paper we make no additional assumptions on the cone Λ and its interior. On the other hand, there are many vector optimization problems with non-empty sets of efficient solutions, for which the corresponding vector-valued mappings satisfy none of the lower semicontinuity concepts mentioned above.

In view of this a new characterization of semicontinuity for such mappings is the main scope of this paper. We introduce, the so-called Λ_τ-lower semicontinuity property for vector-valued mappings in Banach spaces (with respect to the τ-topology of Y) which implies the previous ones. We apply this concept to the study of the vector optimization problems.

Let us describe the contents of the paper. Section 1 provides in details the main notation and ingredients needed in this work. In Section 2, we give the statement of the vector optimization problem in Banach spaces and the definition of its efficient solutions. Section 3 contains a short review of the main definitions of lower semicontinuity of vector-valued mappings, introduced in [3, 6, 7, 13], and some well-known facts concerning these notions. In Section 4, we introduce a new concept of lower semicontinuity for vector-value mappings with respect to different topologies of Banach spaces, and compare this notion with the previous ones. The last section contains our main result concerning the solvability of the vector optimization problem. All main notions and assertions are illustrated by numerous examples.

1 Preliminaries and notation

Throughout this paper, X and Y are two real Banach spaces. We assume that X is reflexive. Let θ_Y be the zero-element of Y. We suppose that these spaces, as topological spaces, are endowed with some topology τ, which usually is associated either with the strong topology ($\tau := s$) or with the weak topology ($\tau := w$). For a subset Y_0 of Y, we denote by $\mathrm{Int}_\tau\, Y_0$, $\mathrm{cl}_\tau\, Y_0$, and $\partial_\tau Y_0$ the interior of Y_0, the closure of Y_0, and its boundary in Y with respect to the τ-topology of Y, respectively. By default τ is always associated with the strong topology of the corresponding space. In this case, we will omit the index if no confusion may occur. Let $\Lambda \subset Y$ be a closed convex cone, which is supposed to be pointed, that is, $\Lambda \cap -\Lambda = \{\theta_Y\}$. No assumption is required on the interior of Λ.

The cone Λ defines a partial order on Y denoted by $\preceq$. For any elements $y, z \in Y$, we will write $y \preceq z$ whenever $z \in y + \Lambda$ and $y \prec z$ for $y, z \in Y$, if $z - y \in \Lambda \setminus \{\theta_Y\}$. We say that a sequence $\{y_k\}_{k=1}^\infty \subset Y$ is non-increasing and we use the notation $y_k \searrow$ whenever, for all $k \in \mathbb{N}$, we have $y_{k+1} \preceq y_k$.

We say that an element $y^* \in Y_0$ is Λ-*minimal* for the set $Y_0 \subset Y$, if there is no $y \in Y_0$ such that $y \prec y^*$, that is,

$$Y_0 \cap (y^* - \Lambda) = \{y^*\}.$$

We denote with $\Lambda\text{-}\mathrm{Min}(Y_0)$ the family of all such elements. We say that an element y^* is the Λ-*ideal minimal point* of the set Y_0, if $y^* \preceq y$ for every $y \in Y_0$. By analogy we can introduce the sets of Λ-maximal and Λ-ideal maximal elements of the set Y_0.

Let us introduce two singular elements $-\infty$ and $+\infty$ in Y. We assume that these elements satisfy the following conditions:

$$1) -\infty \preceq y \preceq +\infty,\ \forall y \in Y; \quad 2) +\infty + (-\infty) = \theta_Y.$$

We use the notation $\overline{Y} = Y \cup \{\pm\infty\}$. Then $+\infty$ is the Λ-greatest element of the set $\overline{Y}$, and the element $-\infty$ is its Λ-smallest element. We denote with $Y^\bullet$ a semi-extended Banach space:

$Y^\bullet = Y \cup \{+\infty\}$. Following [14] we say that for a subset $A \subset Y$ an element $a \in Y$ is called a least upper bound of A when for every $y \in Y$ the following property

$$a \preceq y \ \text{ if and only if } \ b \preceq y \text{ for every } \ b \in A$$

holds true. As usual, we denote by sup A the least upper bound of A. Similarly, the greatest lower bound of A, whenever it exists, inf A, is defined by

$$\text{for every } \ y \in Y, \ y \preceq \inf A \ \text{ if and only if } \ y \preceq z \text{ for every } \ z \in A.$$

Then the next concept is the crucial point in our approach.

Definition 1. We say that a set A is the efficient Λ-infimum of the set $Y_0 \subset Y$ with respect to the τ-topology (or shortly Λ_τ-infimum) if A is the collection of all Λ-minimal elements of the τ-closure Y_0 in the case when this set is non-empty, and A is equal to $\{-\infty\}$ in the opposite case.

Hereinafter we denote the efficient Λ_τ-infimum for Y_0 by $\operatorname{Inf}^{\Lambda,\tau} Y_0$. Thus, in view of the definition given above, we have

$$\operatorname{Inf}^{\Lambda,\tau} Y_0 := \begin{cases} \Lambda\text{-}\operatorname{Min}(\operatorname{cl}_\tau Y_0), & \Lambda\text{-}\operatorname{Min}(\operatorname{cl}_\tau Y_0) \neq \emptyset, \\ -\infty, & \Lambda\text{-}\operatorname{Min}(\operatorname{cl}_\tau Y_0) = \emptyset. \end{cases}$$

We conclude this preliminaries by pointing out some basic definitions. Let X_∂ be a subset of the Banach space X, and $f : X_\partial \to Y$ be some mapping. In what follows we always associate the mapping $f : X_\partial \to Y$ with its natural extension $\widehat{f} : X \to Y^\bullet$ to the whole space X, where

$$\widehat{f}(x) = \begin{cases} f(x), & x \in X_\partial, \\ +\infty, & x \notin X_\partial. \end{cases}$$

Given a map $f : X \to Y^\bullet$, its domain is denoted by $\operatorname{Dom} f$ and defined by

$$\operatorname{Dom} f = \{\, x \in X \mid f(x) \prec +\infty \,\}.$$

Further we assume that $\operatorname{Dom} f \neq \emptyset$. A mapping $f : X \to Y^\bullet$ is said to be bounded below if there exists a $z \in Y$ such that $z \le f(x)$ for all $x \in X$.

Definition 2. A subset A of Y is said to be the strong efficient Λ-infimum (resp. the weak efficient Λ-infimum) of a mapping $f : \ X \longrightarrow Y^\bullet$ and is denoted by $\operatorname{Inf}^{\Lambda,s}_{x\in X} f(x)$ (resp. by $\operatorname{Inf}^{\Lambda,w}_{x\in X} f(x)$), if A is the efficient Λ_s-infimum (resp. Λ_w-infimum) of the image $f(X)$ of X in Y, that is,

$$\operatorname{Inf}^{\Lambda,s}_{x\in X} f(x) = \operatorname{Inf}^{\Lambda,\mathrm{s}} \{\, f(x) \mid \forall\, x \in X \,\}$$
$$(\text{resp.} \ \ \operatorname{Inf}^{\Lambda,w}_{x\in X} f(x) = \operatorname{Inf}^{\Lambda,\mathrm{w}} \{\, f(x) \mid \forall\, x \in X \,\}).$$

Remark 1. It is clear now that if $a \in \operatorname{Inf}^{\Lambda,s}_{x\in X} f(x)$ then

$$\operatorname{cl} \{\, f(x) \mid \forall\, x \in X \,\} \cap (a - \Lambda) = \{a\}$$

provided $\Lambda\text{-}\operatorname{Min}\,[\operatorname{cl}\{\, f(x) \mid \forall\, x \in X \,\}] \neq \emptyset$.

Let $\{y_k\}_{k=1}^{\infty}$ be a sequence in Y. Let us denote by $\mathrm{L}^{\tau}\{y_k\}$ the set of all its cluster points with respect to the τ-topology of Y, that is, $y \in \mathrm{L}^{\tau}\{y_k\}$ if there is a subsequence $\{y_{k_i}\}_{i=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $y_{k_i} \stackrel{\tau}{\longrightarrow} y$ in Y as $i \to \infty$. If $\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^{\tau}\{y_k\} = -\infty$, we assume that $\{-\infty\} \in \mathrm{L}^{\tau}\{y_k\}$. If $\mathrm{Sup}^{\Lambda,\tau}\, \mathrm{L}^{\tau}\{y_k\} = +\infty$, we assume that $\{+\infty\} \in \mathrm{L}^{\tau}\{y_k\}$. Let $x_0 \in X$ be a fixed element. In what follows for an arbitrary mapping $f:\ X \longrightarrow Y^{\bullet}$ we make use of the following sets:

$$\begin{aligned}\mathrm{L}_s^{\tau}(f,x_0) &:= \bigcup_{\{x_k\}_{k=1}^{\infty} \in \mathfrak{M}_s(x_0)} \mathrm{L}^{\tau}\{f(x_k)\},\\ \mathrm{L}_w^{\tau}(f,x_0) &:= \bigcup_{\{x_k\}_{k=1}^{\infty} \in \mathfrak{M}_w(x_0)} \mathrm{L}^{\tau}\{f(x_k)\},\end{aligned}$$

where $\mathfrak{M}_s(x_0)$ and $\mathfrak{M}_w(x_0)$ are the sets of all sequences $\{x_k\}_{k=1}^{\infty} \subset X$ such that $x_k \to x_0$ strongly in X and weakly in X, respectively.

Definition 3. We say that a subset $A \subset Y \cup \{\pm\infty\}$ is the Λ-lower sequential limit of the mapping $f : X \longrightarrow Y^{\bullet}$ at the point $x_0 \in X$ with respect to the product of the strong topology of X and the τ-topology of Y, and we use the notation $A = \liminf^{\Lambda,\tau}_{x \to x_0} f(x)$, if

$$\liminf^{\Lambda,\tau}_{x \to x_0} f(x) := \begin{cases} \mathrm{L}^{\tau,s}_{min}(f,x_0,X), & \mathrm{L}^{\tau,s}_{min}(f,x_0,X) \neq \emptyset,\\ \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}_s^{\tau}(f,x_0), & \mathrm{L}^{\tau,s}_{min}(f,x_0,X) = \emptyset,\end{cases} \tag{1}$$

where

$$\mathrm{L}^{\tau,s}_{min}(f,x_0,X) = \mathrm{L}_s^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x).$$

Remark 2. Note that in the scalar case ($f : \mathbb{X} \longrightarrow \overline{\mathbb{R}}$) the sets

$$\mathrm{Inf}^{\Lambda,}_{x \in X} f(x) \quad \text{and} \quad \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}_s^{\tau}(f,x_0)$$

are singletons. So, if $\mathrm{L}_s^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x) \neq \emptyset$, then

$$\mathrm{L}_s^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,}_{x \in X} f(x) \equiv \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}_s^{\tau}(f,x_0),$$

and therefore the choice rules in (1) coincide and give the classical definition of the lower limit.

By analogy, we can introduce the notion of the Λ-*lower sequential limit* of $f : X \longrightarrow Y^{\bullet}$ at $x_0 \in X$ with respect to the product of the weak topology of X and the τ-topology of Y. In this case we have

$$\liminf^{\Lambda,\tau}_{x \rightharpoonup x_0} f(x) := \begin{cases} \mathrm{L}_w^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x), & \mathrm{L}_w^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x) \neq \emptyset,\\ \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}_w^{\tau}(f,x_0), & \mathrm{L}_w^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x) = \emptyset.\end{cases}$$

In particular, if τ is associated with the strong topology of Y, then following our previous conventions, we will use the notation

$$\liminf^{\Lambda,s}_{x \rightharpoonup x_0} f(x) := \begin{cases} \mathrm{L}_w^{s}(f,x_0) \cap \mathrm{Inf}^{\Lambda,s}_{x \in X} f(x), & \mathrm{L}_w^{s}(f,x_0) \cap \mathrm{Inf}^{\Lambda,s}_{x \in X} f(x) \neq \emptyset,\\ \mathrm{Inf}^{\Lambda,s}\, \mathrm{L}_w^{s}(f,x_0), & \mathrm{L}_w^{s}(f,x_0) \cap \mathrm{Inf}^{\Lambda,s}_{x \in X} f(x) = \emptyset.\end{cases}$$

To illustrate the crucial role of the conditions

$$\mathrm{L}_s^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x) \neq \emptyset \quad \text{and} \quad \mathrm{L}_s^{\tau}(f,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x \in X} f(x) = \emptyset$$

of Definition 3, we give the following example.

Example 1. Let $X = Y = \mathbb{R}^2$, $X_\partial = X_\partial^1 \cup X_\partial^2$,

$$X_\partial^1 = \left\{ x \in \mathbb{R}^2 \mid (x_1 - 6)^2 + (x_2 - 6)^2 \le 25,\ x_1 + x_2 \le 7 \right\}, \tag{2}$$

$$X_\partial^2 = \left\{ x \in \mathbb{R}^2 \mid x_1 + x_2 > 7,\ x_1 + x_2 \le 8,\ x_1 \ge 1,\ x_2 \ge 1 \right\}, \tag{3}$$

and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements. Then the strong and weak topologies in X and Y coincide. We define a vector-value mapping $f : X_\partial \to Y$ as follows:

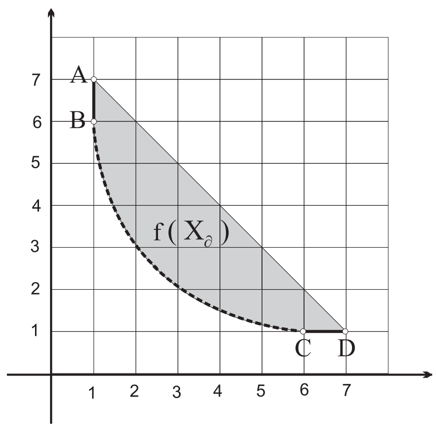

Figure 1. The image of X_∂ in Example 1

$$f(x) = \begin{cases} x, & x \notin X_0, \\ \left[\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}\right], & x \in X_0' \cup \{A, C\}, \\ \left[\begin{smallmatrix} 2 \\ 6 \end{smallmatrix}\right], & x \in X_0'' \cup \{B, D\}, \end{cases} \tag{4}$$

where $A = \left[\begin{smallmatrix} 1 \\ 7 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 6 \\ 1 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 7 \\ 1 \end{smallmatrix}\right]$, $X_0 = X_0' \cup X_0'' \cup \{A, B, C, D\}$,

$$X_0' = \left\{ x \in X_\partial \mid (x_1 - 6)^2 + (x_2 - 6)^2 = 25,\ 1 < x_1 \le x_2 \right\},$$

$$X_0'' = \left\{ x \in X_\partial \mid (x_1 - 6)^2 + (x_2 - 6)^2 = 25,\ x_2 < x_1 < 6 \right\}.$$

Let us find the Λ-lower sequential limit of $f : X_\partial \longrightarrow Y$ at two points: firstly at $x_0 = A$, and after at $x_0 = C$. To begin with, we note that

$$\mathrm{Inf}^{\Lambda,s}_{x \in X} \widehat{f}(x) = X_0' \cup X_0'' \cup \{B, C\}$$

(see Fig.1). Then, in the case when $x_0 = A$, we have

$$\mathrm{L}_s^s(\widehat{f}, x_0) = \left\{ A, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}.$$

Hence, since $\mathrm{L}_s^s(\widehat{f}, x_0) \cap \mathrm{Inf}^{\Lambda,s}_{x \in X} \widehat{f}(x) = \emptyset$, by Definition 3, we conclude that

$$\liminf^{\Lambda,s}_{x \to x_0} \widehat{f}(x) = \mathrm{Inf}^{\Lambda,\mathrm{s}}\, \mathrm{L}_s^s(\widehat{f}, x_0) = \left\{ A, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}.$$

At the same time, if we take $x_0 = C$, then $\mathrm{L}^s_s(\widehat{f}, x_0) = \left\{C, \left[\begin{smallmatrix}6\\2\end{smallmatrix}\right]\right\}$. Hence,

$$\liminf^{\Lambda,s}_{x\to x_0} \widehat{f}(x) = \mathrm{L}^s_s(\widehat{f}(x_0)) \cap \mathrm{Inf}^{\Lambda,s}_{x\in X} \widehat{f}(x) = \{C\}\,.$$

2 The statement of the vector optimization problem

Let X_∂ be a non-empty bounded weakly closed subset of the reflexive Banach space X. Let $F : X_\partial \to Y$ be a given mapping. The vector optimization problem we consider can be stated as follows:

$$\left.\begin{array}{c}\text{Minimize } \ F(x) \ \text{ with respect to the cone } \ \Lambda\\ \text{subject to } \ x \in X_\partial.\end{array}\right\} \tag{5}$$

In view of this we will associate the vector optimization problem (5) with the following triplet

$$\langle X_\partial, F, \Lambda\rangle\,, \tag{6}$$

where the set X_∂ is called the set of admissible solutions to the problem (5).

Definition 4. We say that $x_0 \in X_\partial$ is a Λ_s-efficient solution of the problem (5) if x_0 realizes the strong efficient Λ-infimum of the mapping $F : X_\partial \to Y$, that is,

$$F(x_0) \in \mathrm{Inf}^{\Lambda,s}_{x\in X_\partial} F(x).$$

Definition 5. An element $x_0 \in X_\partial$ is said to be a Λ_w- efficient solution to the problem (5) if x_0 realizes the weak efficient Λ-infimum of the mapping $F : X_\partial \to Y$, that is,

$$F(x_0) \in \mathrm{Inf}^{\Lambda,w}_{x\in X_\partial} F(x).$$

We denote by $\mathrm{Sol}_w(X_\partial;\, F;\, \Lambda)$ and $\mathrm{Sol}(X_\partial;\, F;\, \Lambda)$, respectively, the sets of all weak efficient solutions and all strong efficient solutions to the above vectorial problem. So, by definition, we have

$$\mathrm{Sol}(X_\partial;\, F;\, \Lambda) = \left\{ x_0 \in X_\partial \mid F(x_0) \in \mathrm{Inf}^{\Lambda,s}_{x\in X_\partial} F(x) \right\},$$
$$\mathrm{Sol}_w(X_\partial;\, F;\, \Lambda) = \left\{ x_0 \in X_\partial \mid F(x_0) \in \mathrm{Inf}^{\Lambda,w}_{x\in X_\partial} F(x) \right\}.$$

Remark 3. It is clear that Definitions 4 and 5 are identical in the case when the set $F(X_\partial) = \{\, F(x) \mid \forall\, x \in X_\partial \,\}$ is convex. To specify these definitions more exactly, we say that the vector optimization problem $\langle X_\partial, F, \Lambda\rangle$ has a $\Lambda_{(\tau,\mu)}$-efficient solution $x_0 \in X_\partial$ if x_0 is a Λ_τ-efficient solution and $x_k \to x_0$ in the μ-topology of X whenever $F(x_k) \to F(x_0)$ with respect to the τ topology of Y, that is, every τ-minimizing sequence is μ-convergent.

Remark 4. It should be emphasized the difference between the notion of Λ_τ- efficient solutions to the vector optimization problem (5) and the "classical" definition of the weak efficient solutions. Let us recall that an element $x^* \in X_\partial$ is said to be a weakly efficient solution to the problem (5) if $\mathrm{Int}\,\Lambda \neq \emptyset$ and $F(x^*)$ is a minimal element of the set

$$F(X_\partial) := \{\, y \in Y \mid y = F(x)\ \forall\, x \in X_\partial \,\}$$

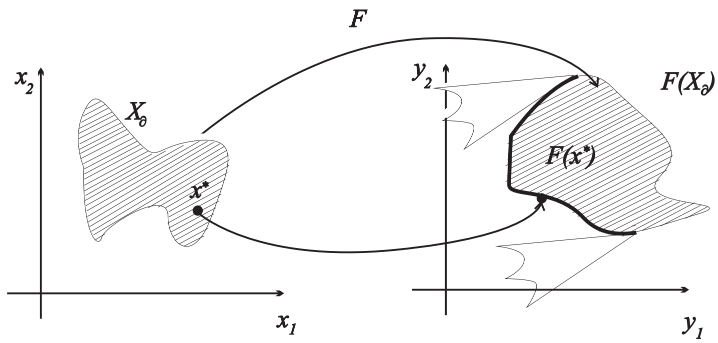

Figure 2. Efficient solutions of the vector optimization problem

with respect to the cone $\{\theta_Y\} \cup \mathrm{Int}\,\Lambda$, i.e., if there is no $y \in F(X_\partial)$ such that $F(x^*) \neq y$ and $F(x^*) - y \in \mathrm{Int}\,\Lambda$ (see [5]).

It is easy to show that each Λ_τ- efficient solution is a weak efficient solution to this problem, but the converse is not true in general. Indeed, let x_0 be any element of $\mathrm{Sol}(X_\partial;\, F;\, \Lambda)$. We assume that the cone Λ has a non-empty interior. Then $F(x_0) \in \Lambda\text{-}\,\mathrm{Min}\,(\mathrm{cl}_s\, F(X_\partial))$. Hence, $F(x_0) - y \notin \Lambda$ for all $y \in \mathrm{cl}_s\, F(X_\partial)$. So, $F(x_0) - F(x) \notin \Lambda$ for all $x \in X_\partial$. It immediately leads us to the conclusion:

$$F(x_0) - F(x) \notin \mathrm{Int}\,\Lambda, \quad \forall\, x \in X_\partial.$$

Thus x_0 is a weak efficient solution to the problem (5), and we obtain the required: $\mathrm{Sol}(X_\partial; F; \Lambda)$ belongs to the set of weak efficient solutions.

The main question is to obtain an existence theorem of the Λ_τ-efficient solutions for a vector optimization problem $\langle X_\partial, F, \Lambda\rangle$, that is, to find sufficient conditions which guarantee the relation $\mathrm{Sol}_\tau(X_\partial;\, F;\, \Lambda) \neq \emptyset$. The main interest here is in the proof of the relation $\mathrm{Sol}_\tau(X_\partial;\, F;\, \Lambda) \neq \emptyset$ without using scalarization process of vector optimization problem (5). We begin with the following obvious result (see, for instance, [15]):

Theorem 1. *If* $\Lambda\text{-}\,\mathrm{Min}\,\{\, F(x) \mid \forall\, x \in X_\partial \,\}$ *is compact with respect to the strong topology of* Y *then* $\mathrm{Sol}(X_\partial;\, F;\, \Lambda) \neq \emptyset$.

However, the strong compactness property of subsets in Banach spaces is a very restrictive assumption. So, we recall some additional notions and results from the non-smooth analysis of vector-valued mappings.

3 Lower semicontinuity for vector-valued mappings

It is well known that the concept of lower semicontinuity property, that was introduced for scalar functions by R. Baire, is a fundamental notion of mathematical analysis. Thanks to efforts of D. Hilbert and L. Tonelli, the main field of its application is the Calculus of Variations and the scalar minimization theory. A very natural and challenging question is, therefore, to determine a concept of lower semicontinuity for vector-valued mappings. However, in the vector-valued case there are several possible extensions of the "scalar" notion of lower semicontinuity (see, for example, [1, 2, 3, 6, 11, 12]). We recall now a few main definitions of lower semicontinuity of a vector-valued mapping with respect to the strong topologies of X and Y, introduced in [3, 6, 7, 13].

Definition 6. [13] A mapping $F : X \to Y^\bullet$ is said to be lower semicontinuous (lsc) at $x_0 \in X$, if for any neighborhood V of $F(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $F(U) \subset V + \Lambda \cup \{+\infty\}$.

Definition 7. [6] A mapping $F : X \to Y^\bullet$ is said to be sequentially lower semicontinuous (s-lsc) at $x_0 \in X$, if for any $b \in Y$ satisfying $b \preceq F(x_0)$ and for any sequence $\{x_k\}_{k=1}^\infty$ of X which converges to x_0, there exists a sequence $\{b_k\}_{k=1}^\infty$ (in Y) converging to b and satisfying $b_k \preceq F(x_k)$, for any $k \in \mathbb{N}$.

Remark 5. For $x_0 \in X$, the Definition 7 can be expressed as follows. For each sequence $\{x_k\}_{k=1}^\infty$ converging to x_0, there exists a sequence $\{b_k\}_{k=1}^\infty$ converging to $F(x_0)$ such that $b_k \preceq F(x_k)$ for all $k \in \mathbb{N}$.

Note also that, Definitions 6 and 7 coincide whenever X and Y are metrizable spaces (it has been proved in [6]).

Definition 8. [3] A mapping $F : X \to Y^\bullet$ is said to be quasi lower semicontinuous (q-lsc) at $x_0 \in X$, if for each $b \in Y$ such that $b \not\preceq F(x_0)$, there exists a neighborhood U of x_0 in X such that $b \not\preceq F(x)$ for each x in U.

Definition 9. [7] A mapping $F : X \to Y^\bullet$ is said to be order lower semicontinuous (o-lsc) at $x_0 \in X$, if for each sequence $\{x_k\}_{k=1}^\infty \subset X$ converging to x_0 for which there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset Y$ converging to θ_Y such that the sequence $\{F(x_k) + \varepsilon_k\}_{k=1}^\infty$ is non-increasing, there exists a sequence $\{g_k\}_{k=1}^\infty \subset Y$ converging to θ_Y such that

$$F(x_0) \leq F(x_k) + g_k \quad \text{for all} \quad k \in \mathbb{N},$$

$$\text{in symbols, } x_k \to x_0 \text{ and } \; F(x_k) + o(1) \searrow\!\!\Longrightarrow F(x_0) \leq F(x_k) + o(1).$$

A mapping F is lsc (resp., q-lsc, o-lsc) if F is lsc (resp., q-lsc, o-lsc) at each point of X. Let us give some well-known facts concerning these notions.

(1) Whenever X is metrizable and $Y = \mathbb{R}$, the s-ls continuity coincides with the classical lower semicontinuity property. In this case, a function $F : X \to \mathbb{R}$ is s-lsc at every point x_0 if, and only if its epigraph

$$\operatorname{epi} F := \{\, (x, y) \in X \times Y \mid y \in F(x) + \Lambda \,\}$$

is closed in $X \times \mathbb{R}$.

(2) A mapping F is lsc at x_0 if and only if $\lim\limits_{x \to x_0} \inf\limits_{a \in \Lambda} \|F(x_0) + a - F(x)\|_Y = 0$.

(3) A mapping F is q-lsc at x_0 if and only if for each $b \in Y$, the set

$$\{F \preceq b\} := \{\, x \in X \mid F(x) \preceq b \,\}$$

is closed in X.

(4) A lsc mapping at x_0 is both q-lsc and o-lsc at this point.

(5) A q-lsc mapping is o-lsc at this point if either $F : X \to Y^\bullet$ is bounded below and the dimension of Y is finite, or the pair (Y, Λ) has the monotone bounds property (BMP), i.e., any sequence $\{y_k\}_{k=1}^\infty \subset Y$ converging to θ_Y has a subsequence $\{y_{k_i}\}_{i=1}^\infty$ for which there exists a non-increasing sequence $\{\overline{y}_i\}_{i=1}^\infty \subset Y$ converging to θ_Y such that $y_{k_i} \leq \overline{y}_i$ for all $i \in \mathbb{N}$ (see [7]).

(6) Every lsc mapping has a closed epigraph (see [4]), but the converse is not true as the following counterexample in [12] shows: the mapping $F : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\begin{cases} F(x) = (-1, 1/|x|), & \text{for } x \neq 0, \\ F(x) = (0, 0), & \text{otherwise,} \end{cases}$$

is not lsc at 0 while its epigraph (with respect to the cone $\Lambda = \mathbb{R}^2_+$) is closed. At the same time, as immediately follows from Definitions 8 and 9, this mapping is both q-lsc and o-lsc at 0.

(7) The notion of lsc, q-lsc, and o-lsc coincide for the case when $Y = \mathbb{R}$, but not in general. Indeed, as shown in the previous example, the mapping $F : \mathbb{R} \to \mathbb{R}^2$ is both q-lsc and o-lsc but not lsc at 0. On the other hand, without BMP, the implication q-lsc $\Longrightarrow$ o-lsc is false as well. Let us take $Y = \mathbb{R}^2$ and $\Lambda = \{(x, 0) \in \mathbb{R}^2 \ : \ x \geq 0\}$. Then the mapping $F : \mathbb{R} \to \mathbb{R}^2$ defined by

$$F(x) = \begin{cases} (0, 0) & \text{if } x = 0, \\ (|x|, |x| + 1) & \text{otherwise} \end{cases}$$

is q-lsc but not o-lsc at $x = 0$.

(8) If epi F is closed then F is quasi-lsc. The converse is true if the interior of Λ is non-empty.

We end up this section by the examples of a vector-valued mapping for which both the quasi-ls continuity property and the order-ls continuity property do not hold at some point x_0. Moreover, as we will see later, this point is a Λ_s-efficient solution to the corresponding vector optimization problem. This is the main reason to introduce a new notion of lower semicontinuity weaker than the others three.

Example 2. [10] Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements. To state a vector optimization problem $\langle X_\partial, F, \Lambda \rangle$, we define the set of admissible solutions X_∂ and the mapping $F : X_\partial \to Y$ as follows:

$$X_\partial = \left\{ x \in \mathbb{R}^1 \mid -3 \leq x \leq -1 \right\}, \tag{7}$$

$$F(x) = \begin{bmatrix} -x \\ 2 \end{bmatrix}, \text{ for all } x \neq -1, \quad F(-1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{8}$$

Let $x_0 = -1$. Then

$$F(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \liminf^{\Lambda, s}_{x \to x_0} \widehat{F}(x) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(see Fig. 3). Let us take $b = \left[\begin{smallmatrix} 1{,}5 \\ 3 \end{smallmatrix}\right]$. Obviously $b \not\geq F(x_0)$ and there is no neighborhood of the point x_0 such that $b \not\geq F(x)$ for all x from this neighborhood. Hence, this mapping is not q-lsc at the point x_0. On the other hand, every sequence $\{x_k\}_{k=1}^\infty$, which is converging to 0 and such that $2 > x_k \neq 0$ for all $k \in \mathbb{N}$, is o-admissible, that is, there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset Y$ converging to $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ such that the sequence $\{F(x_k) + \varepsilon_k\}_{k=1}^\infty$ is non-increasing (see Definition 9).

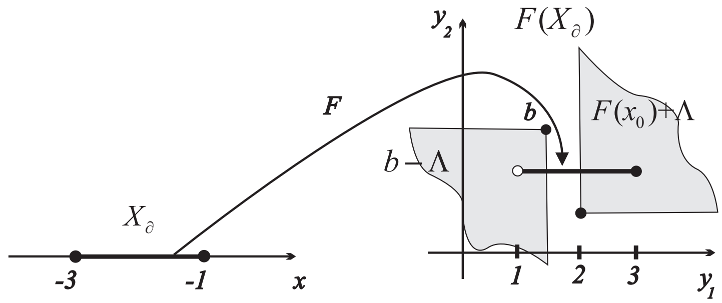

Figure 3. The example of neither q-lsc nor o-lsc mapping

Moreover, $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ is its limit. However, in this case there is no sequence $\{g_k\}_{k=1}^{\infty} \subset Y$ converging to $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ such that

$$\begin{bmatrix}2\\1\end{bmatrix} = F(x_0) \le F(x_k) + g_k \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, the order lower semicontinuity property for the mapping F is failed at the point x_0.

The next example indicates the case when a vector-value mapping $F : X_\partial \to Y$ is not quasi lower semicontinuous at any point of X_∂, whereas it possesses a Λ_τ-lower semicontinuity property on the whole of the domain X_∂.

Example 3. Let X_∂ be a bounded closed subset of a reflexive Banach space X, let $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements in $\mathbb{R}^2$. Let us consider the mapping $F : X_\partial \to \mathbb{R}^2$ defined as follows

$$F(x) = \begin{bmatrix} \|x\| \\ -\|x\| \end{bmatrix}, \quad \forall\, x \in X_\partial.$$

Then $F(X_\partial)$ is a segment

$$\mathfrak{D} = \left\{ y \in \mathbb{R}^2 \mid y = \alpha \begin{bmatrix} m \\ -m \end{bmatrix} + (1-\alpha) \begin{bmatrix} M \\ -M \end{bmatrix},\ \alpha \in [0,1] \right\},$$

where $m = \min\limits_{x \in X_\partial} \|x\|$ and $m = \max\limits_{x \in X_\partial} \|x\|$. Since $\mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} F(x) = \mathfrak{D}$, it follows that each element of X_∂ is a Λ_τ-efficient solution to the corresponding problem $\langle X_\partial, F, \Lambda\rangle$. However, because of the fact that

$$\liminf_{k \to \infty} \|x_k\| \ge \|x\|, \quad \forall\, x_k \rightharpoonup x \ \text{ in } \ \mathbb{X}, \tag{9}$$

the lower and quasi lower semicontinuity properties for $F : X_\partial \to \mathbb{R}^2$ are broken at all points $x \in X_\partial$.

At the same time for every $x_0 \in X_\partial$ we have

$$\mathrm{L}^\tau_w(\widehat{F}, x_0) := \bigcup_{x_k \rightharpoonup x_0} \mathrm{L}^\tau\{\widehat{F}(x_k)\} \subset \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} F(x),$$

and whence $F(x_0) \in \mathrm{Inf}^{\Lambda,\tau}_{x \in X_\partial} F(x)$ due to the property (9). Thus, the objective function $F : X_\partial \to Y$ is sequentially Λ_τ-lower semicontinuous at each point of X_∂.

4 Λ_τ-lower semicontinuity property

In this section, we introduce a new concept of lower semicontinuity for vector-value mapping with respect to the strong and weak topologies of the spaces X and Y. We compare this notion with the previous ones and give some examples. Let $\widehat{F} : X \to Y^\bullet$ denote the natural extension of $F : X_\partial \longrightarrow Y$ to the whole space X.

Definition 10. We say that a mapping $F : X_\partial \longrightarrow Y$ is Λ_τ-lower semicontinuous (Λ_s-lsc) at the point $x_0 \in X_\partial$ (with respect to the strong topology of X) if

$$F(x_0) \in \liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x).$$

Definition 11. A mapping $F : X_\partial \longrightarrow Y$ is said to be weakly Λ_τ-lower semicontinuous (Λ_τ-wlsc) at the point $x_0 \in X_\partial$ if

$$F(x_0) \in \liminf^{\Lambda,\tau}_{x\rightharpoonup x_0} \widehat{F}(x).$$

Compare Definitions 10 and 11 with Definition 3.

A mapping F is Λ_τ-lsc (resp., Λ_τ-wlsc) if F is Λ_τ-lsc (resp., Λ_τ-wlsc) at each point of X_∂. As immediately follows from Definitions 10 and 11, the following result is obvious.

Proposition 1. *The weakly Λ_τ-lower semicontinuity of a mapping $F : X_\partial \longrightarrow Y$ implies its Λ_τ-lower semicontinuity.*

To characterize the properties of Λ_τ-lower semicontinuity more precisely, we begin with the following assertion.

Lemma 1. *If a mapping $F : X_\partial \longrightarrow Y$ is q-lower semicontinuous at $x_0 \in X_\partial$ with respect to the τ-topology of Y and the strong topology of X, then F is Λ_τ-lower semicontinuous at this point.*

Proof. Let $F : X_\partial \to Y$ be a q-lower semicontinuous mapping at the point $x_0 \in X_\partial$, and let $\widehat{F} : X \to Y^\bullet$ be its natural extension. Let $\{x_k\}_{k=1}^\infty$ be a sequence strongly converging to x_0, i.e., $\{x_k\}_{k=1}^\infty \in \mathfrak{M}_s(x_0)$. Let us assume that there exist a subsequence $\{F(x_{k_i})\}_{i=1}^\infty$ and an index $i^* \in \mathbb{N}$ such that $F(x_{k_i}) \not\succeq F(x_0)$ for all $i \geq i^*$. Then, in view of the definition of the quasi-lower semicontinuity, we just conclude that $\{+\infty\} \in \mathrm{L}_s^\tau(\widehat{F}, x_0)$. So, to characterize the set $\liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x)$, we suppose that the corresponding image sequence $\{F(x_k)\}_{k=1}^\infty$ is bounded above with respect to the cone Λ. In this case there can be found an index k^* such that

$$F(x_k) \succeq F(x_0), \quad \forall k \geq k^*.$$

Hence, for any $y^* \in \mathrm{L}_s^\tau(\widehat{F}, x_0)$, we have $F(x_0) \preceq y^*$. It means that

$$\{F(x_0)\} \in \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}_s^\tau(\widehat{F}, x_0).$$

Thus, due to Definition 3, we deduce: $F(x_0) \in \liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x)$. This concludes the proof.

As a consequence of this result and the properties of quasi-lower semicontinuity, we have: if F is lsc then F is Λ_s-lsc. However, in general, for vector-value mappings, Λ_s-ls continuity does not imply q-lsc. Indeed, let us consider the mapping $F : X_\partial \to Y$ defined in example 2.

As it was shown before, this mapping is neither q-lsc nor o-lsc mapping at the point $x_0 = -1$. However, taking into account the fact that

$$F(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \liminf^{\Lambda,s}_{x\to x_0} \widehat{F}(x) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\},$$

we just obtain the fulfillment of the inclusion

$$F(x_0) \in \liminf^{\Lambda,s}_{x\to x_0} \widehat{F}(x).$$

Hence, F is a Λ_s-lower semicontinuous mapping at $x_0 = -1$.

Lemma 2. *If a mapping $F : X_\partial \longrightarrow Y$ is order-lower semicontinuous at $x_0 \in X_\partial$ with respect to the τ-topology of Y and the strong topology of X, then F is Λ_s-lower semicontinuous at this point.*

Proof. Let $\{x_k\}_{k=1}^\infty \subset X_\partial$ be an o-admissible sequence strongly converging to x_0 in X for which the set $\{F(x_k)\}_{k=1}^\infty$ is relatively τ-compact in $Y^\bullet$. Then for $\{x_k\}_{k=1}^\infty$ there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset Y$ τ-converging to θ_Y such that the sequence $\{F(x_k) + \varepsilon_k\}_{k=1}^\infty$ is non-increasing. Hence, the corresponding image sequence $\{F(x_k)\}_{k=1}^\infty$ is bounded above, that is, there are elements $z \in Y$ and $k^* \in \mathbb{N}$ such that $F(x_k) \preceq z$ for all $k \geq k^*$. Since the mapping is o-lsc at x_0, it follows that there exists a sequence $\{g_k\}_{k=1}^\infty \subset Y$ τ-converging to θ_Y such that $F(x_0) \leq F(x_k) + g_k$ for all $k \in \mathbb{N}$. Hence the sequence $\{F(x_k)\}_{k=1}^\infty$ is bounded below. So, we may suppose that there exists an element $y^* \in Y$ such that $F(x_k) \to y^*$ with respect to the τ-topology of Y. This fact can be written as $F(x_k) + o_\tau(1) = y^*$ when $k \to \infty$. Since F is o-lsc at x_0, that implies $F(x_0) \preceq F(x_k) + o_\tau(1) = y^* + o_\tau(1)$, and since Λ is closed, it follows that passing to the limit in the last inequality as $k \to \infty$, we obtain $F(x_0) \preceq y^*$, where $y^* \in \mathrm{L}^\tau_s(\widehat{F}, x_0) \cap Y$. Since $\{x_k\}_{k=1}^\infty \subset X_\partial$ is an arbitrary o-admissible sequence strongly converging to x_0, it follows that

$$F(x_0) \in \mathrm{L}^\tau_s(\widehat{F}, x_0) \cap Y \quad \text{and} \quad F(x_0) \preceq y^*, \quad \forall\, y^* \in \mathrm{L}^\tau_s(\widehat{F}, x_0) \cap Y. \tag{10}$$

As a result, we conclude that the set $\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau_s(\widehat{F}, x_0) \cap Y$ consists of the unique element $F(x_0)$. Indeed, if we suppose the converse, then (10) just leads us to a contradiction with the definition of τ-efficient Λ-infimum. Therefore, taking into account Definition 3, we get

$$\liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x) = \{\, F(x_0)\, \}.$$

Thus, the Λ_τ-lower semicontinuity property of the mapping F at x_0 is proved.

It is well-known that for real-valued mappings $F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup +\infty$ the notions of lsc, q-lsc, and o-lsc are equivalent (see [7]). However, as immediately follows from Definition 10, for real-value mappings the condition

$$F(x_0) \in \liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x)$$

is identical to the following one $F(x_0) \leq \liminf\limits_{x\to x_0} \widehat{F}(x)$. Hence, in this case, lsc and Λ_τ-lsc are identical properties. As a result, we come to the following conclusion:

Lemma 3. *For real-value mapping $F : X \to \overline{\mathbb{R}}$ the four notions of lower semicontinuity given above are equivalent.*

To conclude this section we give the following observation concerning the property of two Λ_τ-lsc mappings. It is well known that the sum of two q-lsc (resp. o-lsc) mappings is not a q-lsc (resp. o-lsc) mapping in general. Due to the following example, we can give a similar conclusion for the Λ_τ-lsc mappings.

Example 4. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements. Let us consider the mappings $F : \mathbb{R} \to \mathbb{R}^2$ and $G : \mathbb{R} \to \mathbb{R}^2$ defined by

$$F(x) = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } x = 0, \\ \begin{bmatrix} -2+|x|^{-1} \\ -2|x|^{-1} \end{bmatrix} & \text{if } x \neq 0, \end{cases} \qquad G(x) = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } x = 0, \\ \begin{bmatrix} -|x|^{-1} \\ 2|x|^{-1} \end{bmatrix} & \text{if } x \neq 0. \end{cases}$$

It is easy to see that each of these mappings is q-lsc at $x_0 = 0$ since for all $b \in \mathbb{R}^2$ such that $b \preceq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ it is impossible to find any sequence $\{x_k\}_{k=1}^{\infty}$ converging to 0 and satisfying condition $F(x_k) \preceq b$ (resp. $G(x_k) \preceq b$) for all $k \in \mathbb{N}$. So, due to Lemma 1, these mappings are Λ_s-lsc at 0. However, for the mapping $F + G$ we have

$$\mathrm{L}^s_s(F(0) + G(0)) = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \quad \text{and} \quad \mathrm{Inf}^{\Lambda,s}_{x\in X} \left[F(x) + G(x)\right] = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}.$$

Hence,

$$\mathrm{Inf}^{\Lambda,s}_{x\in X_\partial} \left[F(x) + G(x)\right] = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} \not\ni F(0) + G(0),$$

and we obtain the required conclusion: the sum of two Λ_s-lsc mappings is not a Λ_s-lsc mapping in general.

Remark 6. We conclude this section with the following observation. As follows from the definition of the Λ_τ-lower semicontinuity for vector-value mappings $F : X_\partial \to Y$, this property essentially depends on the domain $X_\partial \subset X$. In fact, the assertion: " if $F : X \to Y$ is a Λ_τ-lower semicontinuous mapping then its restriction on any bounded subset $X_\partial \subset X$ preserves this property at every point of X_∂" can be wrong in general. However such situation is both natural and typical in the vectorial case. Indeed, for the different sets of admissible solutions $X^1_\partial, X^2_\partial$ ($X^1_\partial \cap X^2_\partial \neq \emptyset$) and any point x_0 such that $x_0 \in X^1_\partial \cap X^2_\partial$, the sets $\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau_s(F, x_0)$ and $\mathrm{Inf}^{\Lambda,\tau}_{x\in X^i_\partial} F(x)$ are not singletons in general. So, the sets

$$\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau_s(F, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X^1_\partial} F(x) \quad \text{and} \quad \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau_s(F, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X^2_\partial} F(x)$$

can be drastically different as well. Thus, in view of Definitions 3 and 10, the mappings $F : X^1_\partial \to Y$ and $F : X^2_\partial \to Y$ can be distinguished by a Λ_τ-lower semicontinuity property at the point $x_0 \in X^1_\partial \cap X^2_\partial$.

5 Existence theorem of the Λ_τ-efficient solutions for vector optimization problems

We begin with the following supposition: assume that the ordering cone Λ possesses the so-called D-property, that is, every decreasing sequence in Y is τ-convergent if and only if this sequence is Λ-lower bounded. For instance, the ordering cone of positive elements in $L^p(\Omega)$ ($1 < p < +\infty$) which is defined as

$$\Lambda_{L^p(\Omega)} = \{\, f \in L^p(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega \,\}$$

satisfies this property with respect to both the weak and the strong topologies of $L^p(\Omega)$ (see [8]).

Theorem 2. *Let X and Y be Banach spaces, and let $\Lambda \subset Y$ be a closed convex ordering pointed cone, which is supposed to be with D-property. Assume that X_∂ is a compact subset of X (with respect to the strong topology), and $F : X_\partial \to Y$ is a Λ_τ-lower semicontinuous mapping. Then the vector optimization problem $\langle X_\partial, F, \Lambda\rangle$ has a non-empty set of Λ_τ-efficient solutions* $\mathrm{Sol}_\tau(X_\partial;\, F;\, \Lambda)$.

Proof. To begin with, we prove that $\{-\infty\} \notin \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x)$. To do so, it is sufficient to show that if $\{x_k\}_{k=1}^\infty \subset X_\partial$ is a sequence such that its image $\{F(x_k)\}_{k=1}^\infty \subset Y$ is a decreasing sequence in Y, then there is an element $z \in Y$ such that $z \preceq F(x_k)$ for all $k \in \mathbb{N}$. Let us assume the converse. Then there are sequences $\{\widehat{x}_k\}_{k=1}^\infty \subset X_\partial$ and $\{\widehat{z}_k\}_{k=1}^\infty \subset Y$ such that $\widehat{z}_{k+1} \preceq \widehat{z}_k\ \forall\, k \in \mathbb{N}$, and

$$\mathrm{Inf}^{\Lambda,\tau}\{\widehat{z}_k\}_{k=1}^\infty = \{-\infty\}, \quad F(\widehat{x}_k) \preceq \widehat{z}_k \quad \forall\, k \in \mathbb{N}. \tag{11}$$

By the initial assumptions, the family $\{\widehat{x}_k\}_{k=1}^\infty \subset X_\partial$ is compact, so we may suppose that $\widehat{x}_k \to x^*$ in X, where x^* is some element of X_∂. Then, by monotonicity of $\{z_k\}_{k=1}^\infty \subset Y$ and D-property of Λ, we can pass to the limit in $F(\widehat{x}_k) \preceq \widehat{z}_k$ as $k \to \infty$. As a result, we have

$$\xi \preceq -\infty, \quad \forall\, \xi \in L^\tau\{F(\widehat{x}_k)\}, \tag{12}$$

where $L^\tau\{F(\widehat{x}_k)\}$ is the set of all cluster points of $\{F(\widehat{x}_k)\}_{k=1}^\infty$ with respect to the τ-topology of Y. On the other hand, in view of Definition 3 and the Λ_τ-lower semicontinuity of F, we have

$$F(x^*) \in \liminf^{\Lambda,\tau}_{x\to x^*} \widehat{F}(x), \text{ and hence } F(x^*) \not\succ \xi, \quad \forall\, \xi \in L^\tau\{F(\widehat{x}_k)\}.$$

Combining this result with (12), we obtain $F(x^*) \not\succ -\infty$. However this contradicts (11). Hence $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x) \not\ni \{-\infty\}$.

Let ξ be any element of $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x)$. Then, by definition of the Λ_τ-efficient infimum, there exists a sequence $\{y_k\}_{k=1}^\infty \subset Y$ such that $y_k \xrightarrow{\tau} \xi$ in Y. We define a sequence $\{x_k\}_{k=1}^\infty \subset X_\partial$ as follows $F(x_k) = y_k$ for all $k \in \mathbb{N}$. Since the set X_∂ is compact, we may suppose that there exists $x_0 \in X_\partial$ such that $x_k \to x_0$ in X. Hence $\xi \in \mathrm{L}^\tau_s(F, x_0)$, and we get

$$\mathrm{L}^\tau_s(F, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x) \neq \emptyset.$$

Then, due to the Λ_τ-lower semicontinuity of the mapping F on X_∂ and Definition 3, we obtain

$$F(x_0) \in \liminf^{\Lambda,\tau}_{x\to x_0} \widehat{F}(x) = \mathrm{L}^\tau_s(F, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x).$$

Hence, $F(x_0) \in \mathrm{L}^\tau_s(F, x_0)$, which implies we may assume

$$F(x_0) = \xi, \quad \text{and} \quad \xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} F(x).$$

Thus, $x_0 \in \mathrm{Sol}_\tau(X_\partial;\, F;\, \Lambda)$ and this concludes the proof.

However, the compactness property of the set of admissible solutions X_∂ is a very restrictive assumption. In view of this we use the Banach-Alaoglu Theorem in reflexive Banach spaces, which leads us to the following generalization of the previous theorem.

Theorem 3. *Let X_∂ be a bounded weakly closed subset of a reflexive Banach space X, let Y be a Banach space partially ordered by a closed convex pointed cone Λ, and let $F : X_\partial \to Y$ be a weakly Λ_τ-lower semicontinuous mapping. Then the vector optimization problem $\langle X_\partial, F, \Lambda\rangle$ has a non-empty set of the Λ_τ-efficient solutions* $\mathrm{Sol}_\tau(X_\partial;\, F;\, \Lambda)$.

Proof. We will only deal with that part of the previous proof concerning the compactness property of the sequences $\{\widehat{x}_k\}_{k=1}^{\infty}$ and $\{x_k\}_{k=1}^{\infty}$. Indeed, taking into account the initial suppositions and the Banach-Alaoglu Theorem, the subset X_∂ is sequentially compact with respect to the weak topology of X. Hence we may suppose that, passing to subsequences if necessary, each of the above sequences is weakly convergent to some elements of X_∂. To conclude the proof, we can use motivations similar to the proof of the previous theorem changing the components $\mathrm{L}_s^\tau(F, x_0)$ and $\liminf_{x \to x_0}^{\Lambda,\tau} \widehat{F}(x)$ onto $\mathrm{L}_w^\tau(F, x_0)$ and $\liminf_{x \rightharpoonup x_0}^{\Lambda,\tau} \widehat{F}(x)$, respectively.

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