ON APPROXIMATION OF ENTROPY SOLUTIONS FOR ONE SYSTEM OF NONLINEAR HYPERBOLIC CONSERVATION LAWS WITH IMPULSE SOURCE TERMS

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Abstract. We study one class of nonlinear fluid dynamic models with impulse source terms. The model consists of a system of two hyperbolic conservation laws: a nonlinear conservation law for the goods density and an linear evolution equation for the processing rate. We consider the case when influx-rates in the second equation take the form of impulse functions. Using the vanishing viscosity method and the so-called principle of fictitious controls, we show that entropy solutions to the original Cauchy problem can be approximated by optimal solutions of the special optimization problems.

Key words. fluid dynamic model, nonlinear conservation laws, principle of fictitious controls, vanishing viscosity method, impulse controls, entropy solutions

AMS subject classifications. 46B40, 49J45, 90C29, 49N90,76N15

1. Introduction. The main goal of this paper is to approximate entropy solutions to a Cauchy problem for the system of two conservation laws with an impulse source term. Conservation laws, taking the form of hyperbolic partial differential equations, appear in a variety of applications that offer control or identification of parameters, including the control of traffic and water flows, the modeling of supply chains, gas pipelines, blood flows, etc. The analysis of conservation laws is a very active research area. The main difficulty in delaing with them is the fact that the solution of such systems may develop discontinuities (after a finite time), that propagate in time even for smooth initial and boundary conditions (see [1, 6, 7]). Usually such solutions can be formed by the so-called rarefaction or shock waves. Therefore, it makes a sense to consider a more flexible notion of solutions, which are physically meaningful and whose admissibility issue is related to the notions of entropy and energy.

We carry out our analysis for the following initial value problem for the system of nonlinear conservation laws

$$\rho_t + \left(f(\mu,\rho)\right)_x = 0, \quad (t,x) \in \Omega_T = (0,T)\times\mathbb{R}, \tag{1.1}$$

$$\mu_t - \mu_x = u(t,x), \quad (t,x) \in \Omega_T, \tag{1.2}$$

$$\rho(0,x) = \rho_0(x), \quad \mu(0,x) = \mu_0(x), \quad x \in \mathbb{R}. \tag{1.3}$$

Throughout this paper we suppose that the structure of the source term $u(t,x)$ is is prescribed, namely

$$u = u(t,x) = \sum_{i=1}^{N} u_i(t)\delta_{\tau_i}(x), \text{ with } \; -\infty < a < \tau_1 < \cdots < \tau_N < b < +\infty, \tag{1.4}$$

where the functions $\left\{u_i \in L^2(0,T)\right\}_{i=1}^N$ can play the role of control factors, and δ_{τ_i} denote the Dirac measures located at the points τ_i.

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In the recent applications of the model (1.1)–(1.3) to the supply chain problem [3], $\rho = \rho(t, x)$ represents the density of objects or the concentration of a physical quantity processed by the supply region (modelled by a real line $\mathbb{R}$), and $\mu = \mu(t, x)$ is the processing rate. However, to the best knowledge of authors the existence and uniqueness of entropy solutions to the problems of conservation laws with impulse controls, is an open problem even for the simplest situation. Thus our prime interest is to discuss the approximation approach to the construction of entropy solution for the above problem. To this end, we apply the vanishing viscosity method and the so-called principle of fictitious controls. We prove that entropy solutions to Cauchy problem (1.1)–(1.4) can be approximated by optimal solutions of the special optimization problems.

2. Notation and Preliminaries. Let a and b be two fixed constant such that $-\infty < a < b < +\infty$. For a given $T > 0$ we set $\Omega_T = (0, T)\times\mathbb{R}$ and $\Omega = (0, T)\times(a, b)$. Let $L^p_{loc}(\Omega_T)$, with $1 \le p \le \infty$, be the locally convex space of all measurable functions $g : \Omega_T \to \mathbb{R}$ such that $g|_{(0,T)\times K} \in L^p((0, T) \times K)$ for all compact sets $K \subset \mathbb{R}$.

Let $\mathcal{M}(\mathbb{R})$ be the set of all Radon measures on $\mathbb{R}$, that is, $\mu \in \mathcal{M}(\mathbb{R})$ if μ is a countably additive set function defined on the Borel subsets of $\mathbb{R}$ such that μ is finite on every compact subset of $\mathbb{R}$. We say that a sequence of Radon measures $\{\mu_k\}_{k\in\mathbb{N}}$ converges weakly-$*$ to a measure $\mu \in \mathcal{M}(\mathbb{R})$ (in symbols $\mu_k \overset{*}{\rightharpoonup} \mu$) if

$$\lim_{k\to\infty}\int_{\mathbb{R}}\varphi\,d\mu_k = \int_{\mathbb{R}}\varphi\,d\mu \quad \text{for all} \quad \varphi \in C_0(\mathbb{R}).$$

A subset $\mathfrak{M}$ of $\mathcal{M}(\mathbb{R})$ is called to be bounded if for every compact set $K \subset \mathbb{R}$ we have

$$\sup_{\mu\in\mathfrak{M}} |\mu|(K) < +\infty,$$

where $|\mu|$ denotes the total variation of μ. It is well-know the following compactness result for measures.

Proposition 2.1. *Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a bounded sequence of Radon measures on $\mathbb{R}$. Then there exists a subsequence $\{\mu_{k_j}\}_{j\in\mathbb{N}}$ and a Radon measure $\mu \in \mathcal{M}(\mathbb{R})$ such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$.*

According to the Riesz theory, every Radon measure μ on $\mathbb{R}$ can be identified with an element of the dual space $(C_0(\mathbb{R}))'$, that is, μ is a linear form on $C_0(\mathbb{R})$ and for every compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ depending only on K and μ such that

$$|\langle\mu, f\rangle| \le C\|f\|_{C(K)} \quad \forall\, f \in C_0(\mathbb{R}) \;\; \text{with} \;\; \operatorname{supp} f \subseteq K.$$

As an example of a Radon measure on $\mathbb{R}$, we consider the following one. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be two sequences in $\mathbb{R}$ such that $\sum_{k=1}^{\infty}|a_k| \le \mathbf{C} < +\infty$. Let δ_c be the Dirac measure located at the point $c \in \mathbb{R}$, i.e., this measure is defined as follows

$$\langle\delta_c, \varphi\rangle = \int_{\mathbb{R}}\delta_c(x)\varphi(x)\,dx := \varphi(c) \quad \forall\,\varphi \in C_0(\mathbb{R}).$$

Since

$$\left|\sum_{k=1}^{\infty} a_k\varphi(b_k)\right| \le \left(\sum_{k=1}^{\infty}|a_k|\right)\|\varphi\|_{C(\mathbb{R})} \le \mathbf{C}\|\varphi\|_{C(\mathbb{R})}$$

for every continuous functions with compact support $\varphi \in C_0(\mathbb{R})$, it follows that the linear form

$$\mu^* = \sum_{k=1}^{\infty} a_k\delta_{b_k}$$

is continuous on $C_0(\mathbb{R})$. Hence μ^* is an element of the space of Radon measures $\mathcal{M}(\mathbb{R})$.

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}$. Let $f : \mathcal{O} \to \mathbb{R}$ be an element of $L^1(\mathcal{O})$. Define

$$\int_{\mathcal{O}} |Df| = \sup \Big\{ \int_{\mathcal{O}} f\,\varphi'\,dx \;:\; \varphi \in C_0^1(\mathcal{O}),\ |\varphi(x)| \le 1 \text{ for } x \in \mathcal{O} \Big\}.$$

According to the Radon-Nikodym theorem, if $\int_{\mathcal{O}} |Df| < +\infty$ then the distribution Df is a measure and there exist a function $f' \in L^1(\mathcal{O})$ and a measure $D_s f$, singular with respect to the one-dimensional Lebesgue measure $\mathcal{L}\lfloor\mathcal{O}$ restricted to $\mathcal{O}$, such that

$$Df = f'\mathcal{L}\lfloor\mathcal{O} + D_s f.$$

Definition 2.2. *A function $f \in L^1(\mathcal{O})$ is said to have a bounded variation in $\mathcal{O}$ if the derivative Df exists in the sense of distributions and belongs to the class of Radon measures with bounded total variation, i.e. for which $\int_{\mathcal{O}} |Df| < +\infty$. By $BV(\mathcal{O})$ we denote the space of all functions in $L^1(\mathcal{O})$ with bounded variation.*

Under the norm

$$\|f\|_{BV(\mathcal{O})} = \|f\|_{L^1(\mathcal{O})} + \int_{\mathcal{O}} |Df|,$$

$BV(\mathcal{O})$ is a Banach space. It is well-known the following compactness result for BV-functions:

Proposition 2.3. *The uniformly bounded sets in BV-norm are relatively compact in $L^1(\mathcal{O})$, that is, if $\{f_k\}_{k=1}^{\infty} \subset BV(\mathcal{O})$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(\mathcal{O})} < +\infty$, then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ strongly converging in $L^1(\mathcal{O})$ to some $f \in BV(\mathcal{O})$.*

Definition 2.4. *A sequence $\{f_k\}_{k=1}^{\infty} \subset BV(\mathcal{O})$ weakly converges to some $f \in BV(\mathcal{O})$, and we write $f_k \rightharpoonup f$ iff the two following conditions hold: $f_k \to f$ strongly in $L^1(\mathcal{O})$, and $Df_k \rightharpoonup Df$ weakly* in $\mathcal{M}(\mathcal{O})$.*

In the proposition below we give a compactness result related to this convergence, together with the lower semicontinuity property (see [5]):

Proposition 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $BV(\mathcal{O})$ strongly converging to some f in $L^1(\mathcal{O})$ and satisfying $\sup\limits_{k\in\mathbb{N}} \displaystyle\int_{\mathcal{O}} |Df_k| < +\infty$. Then*

(i) $f \in BV(\mathcal{O})$ and $\displaystyle\int_{\mathcal{O}} |Df| \le \liminf_{k\to\infty} \int_{\mathcal{O}} |Df_k|$;

(ii) $f_k \rightharpoonup f$ in $BV(\mathcal{O})$.

3. Statement of Problem and Main Motivation. Let $\{\tau_k\}_{k=1}^{N} \subset \mathbb{R}$ be a given finite family of points such that $a < \tau_1 < \cdots < \tau_N < b$. We focus on the following fluid dynamic model, expressed by the nonlinear inhomogeneous hyperbolic conservation laws

$$\rho_t + \left(f(\mu,\rho)\right)_x = 0, \quad (t,x) \in \Omega_T, \tag{3.1}$$

$$\mu_t - \mu_x = u(t,x), \quad (t,x) \in \Omega_T, \tag{3.2}$$

$$\rho(0,x) = \rho_0(x), \quad \mu(0,x) = \mu_0(x), \quad x \in \mathbb{R}, \tag{3.3}$$

where the source term is subjected to the constraints

$$u(t,x) = u_1(t)\delta_{\tau_1}(x) + u_2(t)\delta_{\tau_2}(x) + \cdots + u_N(t)\delta_{\tau_N}(x), \tag{3.4}$$

$$u_i \in L^2(0,T), \quad \forall\, i = \{1,\dots,N\}. \tag{3.5}$$

Here $u_i \in L^2(0,T)$ are some external distributed sources located at the corresponding points $\tau_i \in (a,b)$, $\rho_0, \mu_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are data functions, and $f = f(\mu,\rho) = f_1(\rho) + f_2(\mu)$ is a flux function.

We note that a particular case of the initial value problem (3.1)–(3.3) is a perturbed model for the supply chain (represented by a real line), where $\rho = \rho(t,x)$ represents the density of objects or the concentration of a physical quantity processed by the supply region (modeled by a real line $\mathbb{R}$), $\mu = \mu(t,x)$ is the processing rate, and $u = u(t,x)$ is a source term associated with an influx-rate.

In order to give a precise description of the set of admissible source terms to the Cauchy problem (3.1)–(3.5), we note that for any function $u(t,x)$ with the structure (3.4), we have

$$\begin{aligned}\int_0^T \|u(t,\cdot)\|^2_{\mathcal{M}(\mathbb{R})}\,dt &= \int_0^T \left(\sup_{\substack{\|\varphi\|_{C(\mathbb{R})}=1\\ \varphi\in C_0(\mathbb{R})}} \langle u(t,\cdot),\varphi(\cdot)\rangle_{\mathcal{M}(\mathbb{R}),C_0(\mathbb{R})}\right)^2 dt\\ &= \int_0^T \left(\sup_{\substack{\|\varphi\|_{C(\mathbb{R})}=1\\ \varphi\in C_0(\mathbb{R})}} \sum_{k=1}^N u_k(t)\int_{\mathbb{R}} \delta_{\tau_k}(x)\varphi(x)\,dx\right)^2 dt \le \int_0^T \left(\sum_{k=1}^N u_k(t)\right)^2 dt\\ &\le N\sum_{k=1}^N \|u_k\|^2_{L^2(0,T)}.\end{aligned}$$

Hence, it is natural to define that class as follows:

$$\mathcal{U}_{ad} = \big\{\, u \in L^2(0,T;\mathcal{M}(\mathbb{R}))\ \big| \ u = u(t,x) \text{ satisfy (3.4)–(3.5) for some } a < \tau_1 < \dots < \tau_N < b \big\}. \tag{3.6}$$

Definition 3.1. *Let $u \in \mathcal{U}_{ad}$ be a fixed source term. We say that a vector value function $Y = \begin{bmatrix}\rho\\ \mu\end{bmatrix} \in \left[L^2(0,T;L^2_{loc}(\mathbb{R}))\right]^2$ is a weak solution to (3.1)–(3.3) if the identities*

$$\int_0^T\int_{\mathbb{R}} \left(Y \odot \frac{\partial\varphi}{\partial t} + F(Y)\odot\frac{\partial\varphi}{\partial x}\right) dx\,dt + \int_0^T \sum_{k=1}^N U_k(t)\odot\varphi(\tau_k)\,dt = 0, \tag{3.7}$$

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t \left|\int_{\mathbb{R}} Y(t,x)\odot\psi(x)\,dx - \int_{\mathbb{R}} Y_0\odot\psi(x)\,dx\right|_{\mathbb{R}^2} dt = 0 \tag{3.8}$$

hold true for every C_0^∞-functions $\varphi : [0,T]\times\mathbb{R}\to\mathbb{R}^2$ and $\psi:\mathbb{R}\to\mathbb{R}^2$ with compact supports in $(0,T)\times\mathbb{R}$ and $\mathbb{R}$, respectively. Here

$$Y_0 = \begin{bmatrix}\rho_0\\ \mu_0\end{bmatrix},\quad U_k(t) = \begin{bmatrix}0\\ u_k(t)\end{bmatrix},\quad F(Y) = \begin{bmatrix} f(\mu,\rho)\\ -\mu\end{bmatrix},$$

and the symbol $\odot$ denotes the tensor product $\begin{bmatrix}a_1\\ a_2\end{bmatrix}\odot\begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}a_1b_1\\ a_2b_2\end{bmatrix}$.

The characteristic feature of the initial value problem (3.1)–(3.3) is that even for arbitrary smooth functions ρ_0, μ_0, and smooth external sources u_k, $k = 1,\dots,N$, a weak solution $Y(t,x) = (\rho(t,x),\mu(t,x))$ to (3.1)–(3.3), is, in general, not unique (see [6, 7]). Hence, in order to select the "physically" relevant solution, some additional conditions must be imposed. Following [4, 6, 7], we can introduce the entropy-admissibility condition, coming from physical considerations.

Definition 3.2. *A C^1-function $\eta : \mathbb{R}^2 \to \mathbb{R}$ is an entropy for the system* (3.1)–(3.2)*, if it is convex and there exists a C^1-function $q : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$D\eta(v) \cdot DF(v) = Dq(v) \quad \forall\, v \in \mathbb{R}^2. \tag{3.9}$$

The function $q : \mathbb{R}^2 \to \mathbb{R}$ is said an entropy flux for η. The pair (η, q) is said a entropy-entropy flux pair for the system (3.1)–(3.2)*.*

Remark 3.3. *Note that the C^1-functions η, q in Definition 3.2 form a special family of convex entropy pairs. However, any convex function η defined on an open set is locally Lipschitz, and therefore $D\eta$ is defined almost everywhere. This allow us to call a C^0-function η an entropy, if there exists a sequence of C^1-entropies $\left\{\eta^\nu : \mathbb{R}^2 \to \mathbb{R}\right\}_{\nu=1}^{\infty}$ converging to η locally uniformly as $\nu \to \infty$. Moreover a C^0-function q is a corresponding entropy flux, if there exists a sequence $\{q^\nu\}_{\nu=1}^{\infty}$ C^1-entropy fluxes of η^ν, converging to q locally uniformly.*

As a result, an entropy solution of (3.1)–(3.3) for a given $u \in \mathcal{U}_{ad}$ can be defined as follows.

Definition 3.4. *Let $u \in \mathcal{U}_{ad}$ be a given source term with prescribed location $a < \tau_1 < \cdots < \tau_N < b$. A weak solution $Y = \begin{bmatrix} \rho \\ \mu \end{bmatrix} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ to the Cauchy problem* (3.1)–(3.3) *is said entropy admissible if for any constants $k, l \in \mathbb{R}$ the entropy inequalities*

$$\int_0^T \int_{\mathbb{R}} \left(\nu_l(\rho)\psi_t + g_l(\rho)\psi_x\right)\, dxdt - \int_0^T \int_{\mathbb{R}} \operatorname{sign}(\rho - l)\left(f_2(\mu)\right)_x \psi\, dxdt \geq 0, \tag{3.10}$$

$$\int_0^T \int_{\mathbb{R}} \left(\nu_k(\mu)\varphi_t - q_k(\mu)\varphi_x\right)\, dxdt + \sum_{i=1}^{T} \int_0^T \operatorname{sign}\left(\mu(t,\tau_i) - k\right) u_i(t)\varphi(t,\tau_i)\, dt \geq 0 \tag{3.11}$$

hold true for all positive functions $\varphi, \psi \in C_0^\infty(\Omega_T)$ provided

$$\nu_k(\mu) := |\mu - k|,\ q_k(\mu) := (\mu - k)\operatorname{sign}(\mu - k),\ g_l(\rho) := (f_1(\rho) - f_1(l))\operatorname{sign}(\rho - l).$$

Remark 3.5. *Note that the existence and differential properties of entropy solutions to the Cauchy problem* (3.1)–(3.3) *with impulse influx-rate* (3.4) *in the sense of Definition 3.4 are unknown in general. To the best knowledge of authors, the problems* (3.1)–(3.4) *with measure data in the right hand side are not covered by the the classical theory of nonlinear hyperbolic conservation laws. Moreover, we can not assert that entropy admissible solutions $(\rho(u), \mu(u))$ to the above problem are the elements of the class*

$$\left[C([0,T]; L^1(a,b)) \cap L^\infty(\Omega) \cap L^\infty(0,T; BV(a,b))\right]^2$$

which is a natural functional space for the scalar hyperbolic conservation laws (see [2, 6, 10]). Usually these properties essentially depend not only on the flux function $f(\mu, \rho)$, but also on the properties of the admissible source terms $u(t,x)$, which typically, in contrast to our case, are supposed to be bounded in $L^\infty(\Omega_T)$ and closed in $L^1(\Omega_T)$ [10].

Taking this motivation into account, it is reasonably to introduce the following concept:

Definition 3.6. *Let $u \in \mathcal{U}_{ad}$ be a given source term. We say that a vector value function $Y = \begin{bmatrix} \rho \\ \mu \end{bmatrix} \in \left[L^2(0,T; L^2_{loc}(\mathbb{R}))\right]^2$ is the approximately entropy solution to the Cauchy problem* (3.1)–(3.3) *in a domain $(0,T) \times \mathcal{O}$, if $Y = \begin{bmatrix} \rho \\ \mu \end{bmatrix} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ is a weak solution in the sense of Definition 3.1 and there exists a sequence $\left\{Y^\varepsilon = \begin{bmatrix} \rho^\varepsilon \\ \mu^\varepsilon \end{bmatrix}\right\}_{\varepsilon > 0} \subset \left[L^2(0,T; L^2(\mathcal{O}))\right]^2$ such that*

(B1) $\rho^\varepsilon \rightharpoonup \rho$ *and* $\mu^\varepsilon \rightharpoonup \mu$ *in* $L^2(0,T;L^2(\mathcal{O}))$ *as* $\varepsilon \to 0$*;*

(B2) for any constants $k,l \in \mathbb{R}$ *and for all positive concave functions* $\varphi \in C_0^\infty((0,T)\times\mathcal{O})$ *the entropy inequalities*

$$\int_0^T \int_{\mathcal{O}} \left(\nu_l(\rho^\varepsilon)\varphi_t + g_l(\rho^\varepsilon)\varphi_x\right)\,dxdt \\ + \int_0^T \left\langle \operatorname{sign}\left(\rho^\varepsilon(t,\cdot)-l\right)\left(f_2(\mu^\varepsilon(t,\cdot))_x\right), \varphi(t,\cdot)\right\rangle_{\mathcal{M}(\mathcal{O}),C_0(\mathcal{O})}\,dt \ge 0, \quad (3.12)$$

$$\int_0^T \int_{\mathcal{O}} \left(\nu_k(\mu^\varepsilon)\varphi_t - q_k(\mu^\varepsilon)\varphi_x\right)\,dxdt \\ + \sum_{i=1}^N \int_0^T \operatorname{sign}\left(\mu^\varepsilon(t,\tau_i)-k\right)u_i(t)\varphi(t,\tau_i)\,dt \ge 0 \quad (3.13)$$

hold true for every $\varepsilon > 0$ *with* $\nu_l(\rho) := |\rho - l|$, $g_l(\rho) := (f_1(\rho) - f_1(l))\operatorname{sign}(\rho - l)$, $\nu_k(\mu) := |\mu - k|$ *and* $q_k(\mu) := (\mu - k)\operatorname{sign}(\mu - k)$.

4. A Perturbation Framework. As was mentioned above the existence and uniqueness of entropy solutions for nonlinear hyperbolic conservation laws (3.1)–(3.3) with source terms (3.4), where $u_i \in L^2(0,T)$ for all $i = 1,\dots,N$, and with initial distributions $\rho_0, \mu_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, is not covered by the the classical theory. In view of this, we apply in this section the scheme of vanishing viscosity method and the principle of fictitious controls.

To begin with, we impose the following assumptions on the flux function

$$f(\mu,\rho) = f_1(\rho) + f_2(\mu)$$

(A1) the function $f_1 : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, i.e.,

$$|f_1(\rho_1) - f_1(\rho_2)| \le L_\rho|\rho_1 - \rho_2| \quad \forall\, \rho_1,\rho_2 \in [-M_\rho, M_\rho], \quad (4.1)$$

$f_1(0) = 0$, $f_2 \in C^1_{loc}(\mathbb{R})$, and $f_2' : \mathbb{R} \mapsto \mathbb{R}$ is a piecewise linear mapping.

Let ε be a small positive parameter which we associate with a viscosity coefficient. Then instead of the fluid dynamic system (3.1)–(3.3), we focus on the following singular perturbed system of nonlinear PDEs

$$\rho^\varepsilon_t(t,x) - \varepsilon\rho^\varepsilon_{xx}(t,x) + \left(f_1(\rho^\varepsilon(t,x))\right)_x = v(t,x), \quad (t,x) \in \Omega_T, \quad (4.2)$$

$$\mu^\varepsilon_t(t,x) - \varepsilon\mu^\varepsilon_{xx}(t,x) - \mu^\varepsilon_x(t,x) = u(t,x) \quad (t,x) \in \Omega_T, \quad (4.3)$$

$$\rho^\varepsilon(0,x) = \rho_0(x), \quad \mu^\varepsilon(0,x) = \mu_0(x), \quad x \in \mathbb{R} \quad (4.4)$$

subjected to the constraints

$$u(t,x) = u_1(t)\delta_{\tau_1}(x) + u_2(t)\delta_{\tau_2}(x) + \cdots + u_N(t)\delta_{\tau_N}(x), \quad (4.5)$$

$$u_i \in L^2(0,T) \quad \forall\, i = \{1,\dots,N\}, \quad (4.6)$$

$$v \in L^2(0,T;\mathcal{M}(\mathbb{R})), \quad (4.7)$$

where $v = v(t,x)$ is a fictitious control. By $\mathcal{V}_{ad}$ we denote the set of all fictitious controls satisfying conditions (4.7).

Since $\rho_0, \mu_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it is natural to assume that there is a compact interval $I \subset \mathbb{R}$ such that $\rho_0 = 0$ and $\mu_0 = 0$ almost everywhere in $\mathbb{R} \setminus I$. Then having taken a sufficiently big open bounded interval $\mathcal{O} \subset \mathbb{R}$ including the interval I, we can suppose that the rate processing μ^ε and the density ρ^ε vanish at the ends of $\mathcal{O}$. As a result, we can supplement the model (4.2)–(4.7) by the following boundary conditions

$$\rho^\varepsilon(t,x) = 0 \quad \text{and} \quad \mu^\varepsilon(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\mathcal{O}. \tag{4.8}$$

Since by the initial assumptions the influx-rate $u = \sum_{k=1}^N u_k(t)\delta_k$ and fictitious control v belong to the space of measure data $L^2(0,T;\mathcal{M}(\mathcal{O}))$, we make precise the notion of solution for the problem (4.2)–(4.8). To this end, we give the following theorem which plays an important role in the study of partial differential equations (see [8])

Theorem 4.1. *Let us define the Banach spaces*

$$\mathcal{W} = \left\{ y \;:\; y \in L^2(0,T;H_0^1(\mathcal{O})), \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\mathcal{O})) \right\},$$
$$\mathcal{W}_1 = \left\{ y \;:\; y \in L^2(0,T;L^2(\mathcal{O})), \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\mathcal{O})) \right\},$$

equipped with the norm of the graph. Then, the following properties holds true:

(1) the embeddings $\mathcal{W} \hookrightarrow L^2(0,T;L^2(\mathcal{O}))$, $\mathcal{W}_1 \hookrightarrow L^2(0,T;H^{-1}(\mathcal{O}))$ are compact;

(2) one has the embedding

$$\mathcal{W} \hookrightarrow C([0,T];L^2(\mathcal{O})), \quad \mathcal{W}_1 \hookrightarrow C([0,T];H^{-1}(\mathcal{O})), \tag{4.9}$$

where, for $\mathbf{X} = L^2(\mathcal{O})$ or $\mathbf{X} = H^{-1}(\mathcal{O})$, one denotes by $C([0,T];\mathbf{X})$ the space of measurable functions on $[0,T] \times \mathcal{O}$ such that $y(t,\cdot) \in \mathbf{X}$ for any $t \in [0,T]$ and such that the map $t \in [0,T] \mapsto y(t,\cdot) \in \mathbf{X}$ is continuous;

(3) for any $u, v \in \mathcal{W}$ one has

$$\frac{d}{dt}\int_{\mathcal{O}} u(t,x)v(t,x)\,dx = \langle u'(t,\cdot), v(t,\cdot)\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})} + \langle v'(t,\cdot), u(t,\cdot)\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})};$$

(4) let $y \in L^2\left(0,T;H_0^1(\mathcal{O})\right) \cap C\left([0,T];L^2(\mathcal{O})\right)$. Then the following density result holds: for any $\delta > 0$ there exists $\Phi \in C^\infty([0,T];C_0^\infty(\mathcal{O}))$, such that

$$\|y - \Phi\|_{C([0,T];L^2(\mathcal{O}))} \le \delta, \quad \|\nabla y - \nabla\Phi\|_{L^2((0,T)\times\mathcal{O})} \le \delta.$$

Further we note that by the Fridrichs inequality, we have

$$\left|\int_{\mathbb{O}} y\varphi'\,dx\right| \le c\int_{\mathbb{O}} y^2\,dx \int_{\mathbb{O}} |\varphi'|^2\,dx \le c_1\int_{\mathbb{O}} |y'|^2\,dx \int_{\mathbb{O}} |\varphi'|^2\,dx \quad \forall\, y,\varphi \in H_0^1(\mathcal{O}).$$

Hence the bilinear form $\int_{\mathbb{O}} y\varphi'\,dx$ is bounded on $H_0^1(\mathcal{O})$. Moreover, this form is skew-symmetric by the identity

$$\int_{\mathbb{O}} y\varphi'\,dx = \int_{\mathbb{O}} (y\varphi)'\,dx - \int_{\mathbb{O}} y'\varphi\,dx = -\int_{\mathbb{O}} y'\varphi\,dx \quad \forall\, y,\varphi \in C_0^\infty(\mathcal{O}),$$

which remains valid for all $y,\varphi \in H_0^1(\mathcal{O})$ by continuity. Hence, we come to the following classical result (see [8, 9].

THEOREM 4.2. *Assume that $\mu_0 \in BV(\mathcal{O}) \cap L^\infty(\mathcal{O})$ and Hypothesis (A1) is valid. Then for every $\varepsilon > 0$ the initial-boundary value problem (4.2)–(4.8) admits a unique solution $(\rho^\varepsilon, \mu^\varepsilon) \in \mathcal{W} \times \mathcal{W}$ satisfying the integral identities*

$$\int_0^T \int_{\mathcal{O}} \left[\frac{\partial \rho^\varepsilon}{\partial t} \psi + \varepsilon \frac{\partial \rho^\varepsilon}{\partial x} \frac{\partial \psi}{\partial x} - f_1(\rho^\varepsilon) \frac{\partial \psi}{\partial x} \right] dx\, dt$$
$$= \int_0^T \langle v(t,\cdot), \psi(t,\cdot) \rangle_{H^{-1}(\mathcal{O}), H_0^1(\mathcal{O})} \, dt \quad \forall\, \psi \in L^2(0,T; H_0^1(\mathcal{O})), \tag{4.10}$$

$$\int_0^T \int_{\mathcal{O}} \left[\frac{\partial \mu^\varepsilon}{\partial t} \varphi + \varepsilon \frac{\partial \mu^\varepsilon}{\partial x} \frac{\partial \varphi}{\partial x} + \mu^\varepsilon \frac{\partial \varphi}{\partial x} \right] dx\, dt$$
$$= \int_0^T \langle u(t,\cdot), \varphi(t,\cdot) \rangle_{H^{-1}(\mathcal{O}), H_0^1(\mathcal{O})} \, dt \quad \forall\, \varphi \in L^2(0,T; H_0^1(\mathcal{O})), \tag{4.11}$$

with a priori estimates

$$\int_0^T \int_{\mathcal{O}} \left[|\rho^\varepsilon(t,x)|^2 + \varepsilon\, |\rho^\varepsilon_x(t,x)|^2 + |\rho^\varepsilon_t(t,x)|^2 \right] dxdt$$
$$\le C \int_0^T \int_{\mathcal{O}} \left[|\widehat{g}_t(t,x)|^2 + |\widehat{g}_x(t,x)|^2 \right] dxdt + C \|v\|^2_{L^2(0,T;H^{-1}(\mathcal{O}))}, \tag{4.12}$$

$$\int_0^T \int_{\mathcal{O}} \left[|\mu^\varepsilon(t,x)|^2 + \varepsilon\, |\mu^\varepsilon_x(t,x)|^2 + |\mu^\varepsilon_t(t,x)|^2 \right] dxdt$$
$$\le C \int_0^T \int_{\mathcal{O}} \left[|g_t(t,x)|^2 + |g_x(t,x)|^2 \right] dxdt + C \|u\|^2_{L^2(0,T;H^{-1}(\mathcal{O}))}, \tag{4.13}$$

$$\int_0^T \|\rho^\varepsilon_t(t,\cdot)\|^2_{H^{-1}(\mathcal{O})} \, dt \le C \|\widehat{g}\|^2_{W^{-1,2}((0,T)\times\mathcal{O})} + C \|v\|^2_{L^2(0,T;H^{-1}(\mathcal{O}))}, \tag{4.14}$$
$$\int_0^T \|\mu^\varepsilon_t(t,\cdot)\|^2_{H^{-1}(\mathcal{O})} \, dt \le C \|g\|^2_{W^{-1,2}((0,T)\times\mathcal{O})} + C \|u\|^2_{L^2(0,T;H^{-1}(\mathcal{O}))}, \tag{4.15}$$

where $C > 0$ is a constant independent of ε, and $g, \widehat{g} \in W^{1,2}((0,T)\times\mathcal{O})$ are such that $g|_{\partial\mathcal{O}} = 0$, $\widehat{g}|_{\partial\mathcal{O}} = 0$, $\widehat{g}(0,\cdot) = \rho_0$ and $g(0,\cdot) = \mu_0$ in $\mathcal{O}$ (the so-called compatibility condition).

Note that in this case

$$\rho^\varepsilon \in C([0,T]; L^2(\mathcal{O})) \quad \text{and} \quad \mu^\varepsilon \in C([0,T]; L^2(\mathcal{O}))$$

by the embedding (4.9), and the terms in the right hand sides of (4.10)–(4.11) is well defined because $H_0^1(\mathcal{O}) \subset C_0(\overline{\mathcal{O}})$ by the classical Sobolev Embedding Theorem. Moreover, in the one-dimensional case every Radon measure $\nu \in \mathcal{M}(\mathcal{O})$ can be identified with an element of $H^{-1}(\mathcal{O})$, i.e., $\mathcal{M}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$. As a result, the integral identity (4.11) with a source term

$$u(t,x) = \sum_{k=1}^N u_k(t) \delta_k(x)$$

can be rewritten as follows

$$\int_0^T \int_{\mathcal{O}} \left[\frac{\partial \mu^\varepsilon}{\partial t} \varphi + \varepsilon \frac{\partial \mu^\varepsilon}{\partial x} \frac{\partial \varphi}{\partial x} + \mu^\varepsilon \frac{\partial \varphi}{\partial x} \right] dx\, dt \\ = \sum_{k=1}^{N} \int_0^T u_k(t) \varphi(t, \tau_k)\, dt \quad \forall\, \varphi \in L^2(0,T; H_0^1(\mathcal{O})). \quad (4.16)$$

We are now in a position to state the following entropy property of the weak solutions to the initial-boundary value problem (4.2)–(4.8).

LEMMA 4.3. *Let* $u^* = \sum_{k=1}^N u_k^*(t)\delta_{\tau_k} \in \mathcal{U}_{ad}$ *a given source term with prescribed location* $a < \tau_1 < \cdots < \tau_N < b$. *Let* $\{(\rho^\varepsilon, \mu^\varepsilon)\}_{\varepsilon>0}$ *be a sequence of the corresponding weak solutions to the initial boundary value problem* (4.2)–(4.8)(4.2)–(4.8) *when the small parameter* $\varepsilon > 0$ *varies in a strictly decreasing sequence of positive numbers converging to* 0. *Let* $\left\{v^\varepsilon \in L^2(0,T; \mathcal{M}(\mathbb{R}))\right\}_{\varepsilon>0}$ *be a bounded sequence of fictitious controls. Assume that supposition (A1) holds true. Then for every* $\varepsilon > 0$, $k, l \in \mathbb{R}$, *and for all positive concave functions* $\varphi \in C_0^\infty((0,T) \times \mathcal{O})$, *each of the pairs* $((\rho^\varepsilon, \mu^\varepsilon))$ *satisfies the following integral inequalities*

$$\int_0^T \int_{\mathcal{O}} \left(\nu_l(\rho^\varepsilon)\varphi_t + g_l(\rho^\varepsilon)\varphi_x\right) dxdt \\ + \int_0^T \langle \operatorname{sign}(\rho^\varepsilon(t,\cdot) - l) v^\varepsilon(t,\cdot), \varphi(t,\cdot)\rangle_{\mathcal{M}(\mathcal{O}), C_0(\mathcal{O})}\, dt \geq 0, \quad (4.17)$$

$$\int_0^T \int_{\mathcal{O}} \left(\nu_k(\mu^\varepsilon)\varphi_t - q_k(\mu^\varepsilon)\varphi_x\right) dxdt \\ + \sum_{i=1}^{N} \int_0^T \operatorname{sign}(\mu^\varepsilon(t,\tau_i) - k) u_i^*(t) \varphi(t,\tau_i)\, dt \geq 0 \quad (4.18)$$

with $\nu_l(\rho) := |\rho - l|$, $g_l(\rho) := (f_1(\rho) - f_1(l))\operatorname{sign}(\rho - l)$, $\nu_k(\mu) := |\mu - k|$ *and* $q_k(\mu) := (\mu - k)\operatorname{sign}(\mu - k)$.

Proof. Let $E = E(\rho) \in C^2(\mathbb{R})$ be any convex function. We multiply equation (4.2) by $E'(\rho)$. Then the equalities

$$E'(\rho)\rho_t = \frac{\partial E(\rho(t,x))}{\partial t}, \quad f_1'(\rho)E'(\rho)\rho_x = \frac{\partial}{\partial x}\left(\int_k^{\rho(t,x)} f_1'(\xi)E'(\xi)\, d\xi\right), \\ E'(\rho)\rho_{xx} = (E(\rho))_{xx} - E''(\rho)\rho_x^2,$$

imply to the following relation

$$(E(\rho^\varepsilon))_t + \left(\int_k^{\rho^\varepsilon} f_1'(\xi)E'(\xi)\, d\xi\right)_x \\ = \varepsilon\left(E(\rho^\varepsilon)\right)_{xx} - \varepsilon E''(\rho^\varepsilon)\left(\rho^\varepsilon\right)_x^2 + E'(\rho^\varepsilon)v^\varepsilon \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}). \quad (4.19)$$

By the initial assumptions, for every $\varepsilon > 0$ the functions $\rho^\varepsilon \in \mathcal{W}$ can be zero-extended to the domain $\Omega_T = (0,T) \times \mathbb{R}$. Now let us multiply equality (4.19) by a test function

$\varphi \in C_0^\infty(\Omega_T)$ and integrate it over Ω_T. Using the integration by parts and the fact that $\varepsilon > 0$ and $E''(\rho^\varepsilon) \geq 0$ a.e. in Ω_T, we transfer all derivatives to the test function φ:

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}} \left[E(\rho^\varepsilon)\,\varphi_t + \int_k^{\rho^\varepsilon} f_1'(\xi)E'(\xi)\,d\xi\,\varphi_x \right] dxdt = \varepsilon \int_0^T \int_{\mathbb{R}} \left(E(\rho^\varepsilon)\right)\,\varphi_{xx}\,dxdt \\
- \varepsilon \int_0^T \int_{\mathbb{R}} E''(\rho^\varepsilon)\,(\rho^\varepsilon)_x^2\,\varphi\,dxdt + \int_0^T \int_{\mathbb{R}} E'(\rho^\varepsilon)v^\varepsilon\,\varphi\,dxdt \\
\leq \varepsilon \int_0^T \int_{\mathbb{R}} E(\rho^\varepsilon)\,\varphi_{xx}\,dxdt + \int_0^T \int_{\mathbb{R}} E'(\rho^\varepsilon)v^\varepsilon\,\varphi\,dxdt. \quad (4.20)
\end{aligned}$$

Since $\rho^\varepsilon \in L^2(0,T;H_0^1(\mathcal{O}))$ for all $\varepsilon > 0$ and $H_0^1(\mathcal{O}) \hookrightarrow C(\overline{\mathcal{O}})$ by the classical Sobolev Embedding Theorem, it follows that the following term is well defined

$$\int_0^T \int_{\mathbb{R}} E'(\rho^\varepsilon)v^\varepsilon\,\varphi\,dxdt = \int_0^T \langle E'(\rho^\varepsilon)v^\varepsilon, \varphi\rangle_{\mathcal{M}(\mathbb{R}),C_0(\mathbb{R})}\,dt.$$

Further we use the well-known trick. Let $\{E_m\}_{m\in\mathbb{N}}$ be a sequence of C^2-functions approximating the function $\xi \mapsto |\xi - k|$ uniformly on $\mathbb{R}$. Substitute $E = E_m(\rho)$ in the inequality (4.20) and pass to the limit as $m \to \infty$. Note that we can choose E_m in such way that E_m' is bounded and $E_m'(\xi) \to \mathrm{sign}(\xi - k)$ for all $\xi \in \mathbb{R}$, $\xi \neq k$. Since $\varphi_x x \leq 0$ in in Ω_T and

$$\begin{aligned}
\int_k^{\rho^\varepsilon} f_1'(\xi)E_m'(\xi)\,d\xi \to \int_k^{\rho^\varepsilon} f_1'(\xi)\mathrm{sign}(\xi - k)\,d\xi \\
= \mathrm{sign}(\rho^\varepsilon - k)\int_k^{\rho^\varepsilon} f_1'(\xi)\,d\xi = \mathrm{sign}(\rho^\varepsilon - k)(f_1(\rho^\varepsilon) - f_1(k)),
\end{aligned}$$

it immediately leads us to entropy inequality (4.17) from (4.20). The verification of inequality (4.18) can be done by the similar arguments. □

In what follows, for every $\varepsilon > 0$ and a given influx-rate $u^\varepsilon \in \mathcal{U}_{ad}$, we associate with the singular perturbed initial-boundary value problem (4.2)–(4.8) the following penalized optimization problem

$$I_\varepsilon(v^\varepsilon, \rho^\varepsilon) = \|v^\varepsilon\|^2_{L^2(0,T;\mathcal{M}(\mathcal{O}))} + \varepsilon^{-1}\left\|(f_2(\mu^\varepsilon))_x - v^\varepsilon\right\|_{L^2(0,T;H^{-1}(\mathcal{O}))} \longrightarrow \inf \quad (4.21)$$

$$\text{subject to the constraints (4.2)–(4.8).} \quad (4.22)$$

Definition 4.4. *We say that a pair $(v^\varepsilon, \rho^\varepsilon)$ is admissible to optimization problem* (4.21)–(4.22) *if $v^\varepsilon \in L^2(0,T;\mathcal{M}(\mathbb{R}))$ and $\rho^\varepsilon = \rho^\varepsilon(v^\varepsilon) \in \mathcal{W}$ is the corresponding weak solution to the initial boundary value problem* (4.2), $(4.4)_1$, $(4.8)_1$.

Let Ξ_ε be the set of all admissible solutions to the perturbed problem (4.21)–(4.22). As follows from Theorem 4.1, for every $\varepsilon > 0$, Ξ_ε is a nonempty subset of the space

$$\mathcal{Y} = L^2(0,T;\mathcal{M}(\mathcal{O})) \times L^2((0,T) \times \mathcal{O}). \quad (4.23)$$

Remark 4.5. *We note that the cost functional* (4.22) *is well defined on Ξ_ε for every $\varepsilon > 0$. Indeed, let $(v^\varepsilon, \rho^\varepsilon)$ be any representative of Ξ_ε. By supposition (A1), we have: $f_2' : \mathbb{R} \to \mathbb{R}$ is a piecewise linear mapping and $\mu^\varepsilon \in \mathcal{W}$. Hence $(f_2'(\mu^\varepsilon))\,\mu_x^\varepsilon$ is in $L^2((0,T) \times \mathcal{O})$, and $v^\varepsilon \in L^2(0,T;\mathcal{M}(\mathcal{O}))$ by the definition of the class $\mathcal{V}_{ad}$. Since $L^2((0,T) \times \mathcal{O}) \subset L^2(0,T;\mathcal{M}(\mathcal{O}))$, we come to the required conclusion.*

We define the τ-topology on $\mathcal{Y}$ as follows: τ is the product of the weak-$*$ topology of $L^2(0,T;\mathcal{M}(\mathcal{O}))$ and the topology of norm in $L^2((0,T)\times\mathcal{O})$. Then we have the following property concerning the topological properties of the set Ξ_ε of admissible solutions to the perturbed optimization problem (4.21)–(4.22).

LEMMA 4.6. *Assume that supposition (A1) holds true. Then set Ξ_ε is nonempty and sequentially τ-closed for every $\varepsilon > 0$.*

Proof. For a fixed $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ be an arbitrary pair of source terms. Then Theorem 4.2 implies the existence of a unique pair $(\rho^\varepsilon, \mu^\varepsilon)$ such that $\rho^\varepsilon = \rho^\varepsilon(v^\varepsilon)$ and $\mu^\varepsilon = \mu^\varepsilon(u^\varepsilon)$ are the corresponding weak solutions to the initial boundary value problem (4.2)–(4.4), (4.8). Since

$$\mu^\varepsilon, \rho^\varepsilon \in \mathcal{W} := \left\{ y \;:\; y \in L^2(0,T;H_0^1(\mathcal{O})), \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\mathcal{O})) \right\}$$

and $\mathcal{W} \hookrightarrow L^2((0,T)\times\mathcal{O})$, we conclude $(v^\varepsilon, \rho^\varepsilon) \in \Xi_\varepsilon$ and hence $\Xi_\varepsilon \neq \emptyset$.

To establish the τ-closedness of Ξ_ε, we fix an arbitrary τ-converging sequence of admissible solutions to the perturbed problem (4.2)–(4.8), (4.21) $\{(v_k^\varepsilon, \rho_k^\varepsilon) \in \Xi_\varepsilon\}_{k=1}^\infty$ and show that $(v_*^\varepsilon, \rho_*^\varepsilon) \in \Xi_\varepsilon$, where $(v_*^\varepsilon, \rho_*^\varepsilon)$ is its τ-limit.

We have: $v_k^\varepsilon \overset{*}{\rightharpoonup} v_*^\varepsilon$ in $L^2(0,T;\mathcal{M}(\mathcal{O}))$ and $\rho_k^\varepsilon \to \rho_*^\varepsilon$ in $L^2((0,T)\times\mathcal{O})$. Hence $v_*^\varepsilon \in \mathcal{V}_{ad}$ and it remains to show that ρ_*^ε is the corresponding weak solution of the initial-boundary value problem (4.2),$(4.4)_1$, $(4.8)_1$. Indeed, in view of the a priori estimate (4.12), it is easy to see that $L^2((0,T)\times\mathcal{O}$-limit function ρ_*^ε belongs to the space $\mathcal{W}$ and satisfies conditions:

$$f_1(\rho_k^\varepsilon) \to f_1(\rho_*^\varepsilon) \quad \text{in} \quad L^2(0,T;L^2(\mathcal{O})) \quad \text{as} \quad k\to\infty, \tag{4.24}$$

$$(\rho_k^\varepsilon)_x \rightharpoonup (\rho_*^\varepsilon)_x \quad \text{in} \quad L^2(0,T;L^2(\mathcal{O})) \quad \text{as} \quad k\to\infty, \tag{4.25}$$

$$(\rho_k^\varepsilon)_t \rightharpoonup (\rho_*^\varepsilon)_t \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})) \quad \text{as} \quad k\to\infty, \tag{4.26}$$

$$\rho_k^\varepsilon, \rho_*^\varepsilon \in C([0,T];L^2(\mathcal{O})), \quad \rho_k^\varepsilon(0,x) = \rho_0(x) \quad \text{in} \quad \mathcal{O}\ \forall k\in\mathbb{N}. \tag{4.27}$$

This enable us to pass to the limit in the integral identity (4.10) as $k\to\infty$ with $\rho^\varepsilon = \rho_k^\varepsilon$ and $v = v_k^\varepsilon$, and eo ipso to show that the limit function ρ_*^ε is a weak solution to parabolic problem (4.2),$(4.4)_1$, $(4.8)_1$.

Thus, the pair $(v_*^\varepsilon, \rho_*^\varepsilon)$ is an admissible solutions to the perturbed optimization problem (4.2)–(4.8), (4.21). The proof is complete. $\square$

In conclusion of this section, we prove that penalized problem (4.2)–(4.8), (4.21) has a nonempty set of optimal solutions.

THEOREM 4.7. *Assume that supposition (A1) holds true. Then for every $\varepsilon > 0$ and $u^\varepsilon \in \mathcal{U}_{ad}$ there exists at least one pair $(v_0^\varepsilon, \rho_0^\varepsilon) \in \Xi_\varepsilon$ such that*

$$I_\varepsilon(v_0^\varepsilon, \rho_0^\varepsilon) = \inf_{(v^\varepsilon,\rho^\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(v^\varepsilon, \rho^\varepsilon),$$

i.e. the problem (4.2)–(4.8), (4.21) *is solvable.*

Proof. Since $\Xi_\varepsilon \neq \emptyset$ and the cost functional I_ε is bounded below on Ξ_ε, it follows that there exists a sequence $\{(v_k^\varepsilon, \rho_k^\varepsilon)\}_{k\in\mathbb{N}} \subset \Xi_\varepsilon$ such that

$$I_\varepsilon(v_k^\varepsilon, \rho_k^\varepsilon) \xrightarrow[k\to\infty]{} I_\varepsilon^{\min} \equiv \inf_{(v^\varepsilon,\rho^\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(v^\varepsilon, \rho^\varepsilon) \geq 0, \tag{4.28}$$

i.e. $\{(v_k^\varepsilon, \rho_k^\varepsilon)\}_{k\in\mathbb{N}} \subset \Xi_\varepsilon$ is a minimizing sequence for the problem (4.2)–(4.8), (4.21).

To begin with, we show that for any $\lambda > 0$ the set

$$\Xi_\varepsilon^\lambda = \{(v^\varepsilon, \rho^\varepsilon) \in \Xi_\varepsilon \;:\; I_\varepsilon(v^\varepsilon, \rho^\varepsilon) \leq \lambda\}$$

is bounded in $L^2(0,T;\mathcal{M}(\mathcal{O}))\times\mathcal{W}$. Indeed, as follows from inequality (4.28), the sequence of fictitious controls $\{v_k^\varepsilon\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;\mathcal{M}(\mathcal{O}))$. Hence, we may assume that there exists an element $v_0^\varepsilon\in\mathcal{V}_{ad}$ such that $v_k^\varepsilon \overset{*}{\rightharpoonup} v_0^\varepsilon$ in $L^2(0,T;\mathcal{M}(\mathcal{O}))$ as $k\to\infty$, i.e.

$$\lim_{k\to\infty}\int_0^T \langle v_k^\varepsilon(t,\cdot),\varphi\rangle_{M(\mathcal{O}),C_0(\mathcal{O})}\,\psi(t)\,dt \\ = \int_0^T \langle v_0^\varepsilon(t,\cdot),\varphi\rangle_{M(\mathcal{O}),C_0(\mathcal{O})}\,\psi(t)\,dt \quad \forall\,\varphi\in C_0(\mathcal{O}),\ \forall\,\psi\in C_0^\infty(0,T).$$

Then having used a priori estimate (4.12), we see that $\{\rho_k^\varepsilon=\rho^\varepsilon(v_k^\varepsilon)\}_{k\in\mathbb{N}}$ form a uniformly bounded sequence in $\mathcal{W}$. Hence, we may again assume that, up to a subsequence, there exists an element $\rho_0^\varepsilon\in\mathcal{W}$ such that $\rho_k^\varepsilon\rightharpoonup\rho_0^\varepsilon$ weakly in $\mathcal{W}$ and strongly in $L^2((0,T)\times\mathcal{O})$. As a result, $(v_0^\varepsilon,\rho_0^\varepsilon)\in\Xi_\varepsilon$ by Lemma 4.6.

Let us show that the τ-limit pair $(v_0^\varepsilon,\rho_0^\varepsilon)$ is an optimal solution to penalized problem (4.2)–(4.8), (4.21). Indeed, taking into account supposition (A1) and Theorem 4.2, we have

$$\Big[(f_2(\mu^\varepsilon))_x - v_k^\varepsilon\Big] \overset{*}{\rightharpoonup} (f_2(\mu^\varepsilon))_x - v_0^\varepsilon \quad \text{in}\ \ L^2(0,T;\mathcal{M}(\mathcal{O})).$$

Using the property of lower semi-continuity for I_ε with respect to the τ-topology, we get

$$0\le I_\varepsilon(v_0^\varepsilon,\rho_0^\varepsilon)\le \lim_{k\to\infty} I(v^\varepsilon,\rho^\varepsilon)=I_\varepsilon^{\min}.$$

Thus the pair $(u_0^\varepsilon,\rho_0^\varepsilon)$ is optimal for problem (4.2)–(4.8), (4.21). □

5. Approximation Properties of the Perturbed Optimization Problem. The aim of this section is to study the asymptotic behavior of the optimal solutions to the penalized optimization problem (4.2)–(4.8), (4.21) as the small parameter ε tends to zero. To begin with, we note that for every $\varepsilon>0$ the set of admissible solutions Ξ_ε is embedded in the topological space $(\mathcal{Y}_1,\sigma)$, where

$$\mathcal{Y}_1=L^2(0,T;\mathcal{M}(\mathcal{O}))\times L^2(0,T;H^{-1}(\mathcal{O})), \tag{5.1}$$

and σ is the product of the weak-$*$ topology of $L^2(0,T;\mathcal{M}(\mathcal{O}))$ and the strong topology of $L^2(0,T;H^{-1}(\mathcal{O}))$. So, we can take σ as the main topology for the asymptotic analysis.

LEMMA 5.1. *Let $u^*=\sum_{k=1}^N u_k^*(t)\delta_{\tau_k}\in\mathcal{U}_{ad}$ a given source term with prescribed location $a<\tau_1<\dots<\tau_N<b$. Let $\{\mu^\varepsilon\}_{\varepsilon>0}$ be a sequence of the corresponding weak solutions to the initial boundary value problem (4.3), $(4.4)_2$, $(4.8)_2$ when the small parameter $\varepsilon>0$ varies in a strictly decreasing sequence of positive numbers converging to 0. Let $\{(v_0^\varepsilon,\rho_0^\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of optimal solutions to the penalized problem (4.2)–(4.8), (4.21). Assume that the fictitious controls $\{v_0^\varepsilon\}_{\varepsilon>0}$ are bounded in $L^2(0,T;\mathcal{M}(\mathcal{O}))$ and supposition (A1) holds true. Then there can be extracted subsequences of $\{\mu^\varepsilon\}_{\varepsilon>0}$ and of $\{(v_0^\varepsilon,\rho_0^\varepsilon)\}_{\varepsilon>0}$, still denoted by the suffix ε, such that*

(a) $v_0^\varepsilon\overset{*}{\rightharpoonup} v^*$ in $L^2(0,T;\mathcal{M}(\mathcal{O}))$;

(b) $\rho_0^\varepsilon\to\rho^*$ and $\mu^\varepsilon\to\mu^*$ weakly in $L^2((0,T)\times\mathcal{O})$ and strongly in $L^2(0,T;H^{-1}(\mathcal{O}))$;

(c) (ρ^*,μ^*) is a weak solution in $\left[L^2((0,T)\times\mathcal{O})\right]^2$ of the Cauchy problem

$$\rho_t^*+(f_1(\rho^*))_x=v^*,\quad \rho^*(0,\cdot)=\rho_0, \tag{5.2}$$

$$\mu_t^*-\mu_x^*=u^*,\quad \mu^*(0,\cdot)=\mu_0. \tag{5.3}$$

Proof. As follows from a priori estimates (4.14)–(4.15) the sequences $\{\rho_0^\varepsilon\}_{\varepsilon>0}$, and $\{\mu^\varepsilon\}_{\varepsilon>0}$ are bounded in

$$\mathcal{W}_1 = \left\{ y \;:\; y \in L^2(0,T;L^2(\mathcal{O})), \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\mathcal{O})) \right\}.$$

Hence the compactness properties (a)-(b) of the sequences $\{v_0^\varepsilon\}_{\varepsilon>0}$, $\{\rho_0^\varepsilon\}_{\varepsilon>0}$, and $\{\mu^\varepsilon\}_{\varepsilon>0}$ is a direct consequence of the initial suppositions, Banach-Alaoglu Theorem, and the compactness embedding $\mathcal{W}_1 \hookrightarrow L^2(0,T;H^{-1}(\mathcal{O}))$. Moreover, as follows from estimates (4.12)–(4.13), the sequence $\{(\rho_0^\varepsilon, \mu^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\mathcal{O}))$. So, we can suppose that

$$\rho_0^\varepsilon \to \rho^* \quad \text{and} \quad \mu^\varepsilon \to \mu^* \quad \text{as} \quad \varepsilon \to 0 \tag{5.4}$$

strongly in $L^2(0,T;H^{-1}(\mathcal{O}))$ and weakly in $L^2(0,T;L^2(\mathcal{O}))$. In view of estimates (4.12)–(4.13), there are element $\eta, \widehat{\eta} \in L^2((0,T)\times\mathcal{O})$ such that, up to subsequences, we have

$$\sqrt{\varepsilon}\,(\rho_0^\varepsilon)_x \rightharpoonup \eta \quad \text{and} \quad \sqrt{\varepsilon}\,(\mu^\varepsilon)_x \rightharpoonup \widehat{\eta} \quad \text{in} \quad L^2((0,T)\times\mathcal{O}) \quad \text{as} \quad \varepsilon \to 0. \tag{5.5}$$

In order to verify the item (c), we note that the integral identity (4.10) leads us to the following relation

$$\begin{aligned}\int_0^T \int_{\mathcal{O}} \left[-\rho_0^\varepsilon \frac{\partial \varphi}{\partial t} + \sqrt{\varepsilon}\left(\sqrt{\varepsilon}\frac{\partial \rho_0^\varepsilon}{\partial x}\right)\frac{\partial \varphi}{\partial x} - f_1(\rho_0^\varepsilon)\frac{\partial \varphi}{\partial x}\right] dx\,dt \\ = \int_0^T \langle v_0^\varepsilon(t,\cdot), \varphi(t,\cdot)\rangle_{\mathcal{M}(\mathcal{O}),C_0(\mathcal{O})}\,dt \end{aligned} \tag{5.6}$$

which holds true for every $\varepsilon > 0$ and any test function $\varphi \in C_0^\infty((0,T)\times\mathcal{O})$. Since $v_0^\varepsilon \overset{*}{\rightharpoonup} v^*$ in $L^2(0,T;\mathcal{M}(\mathcal{O}))$ as $\varepsilon \to 0$, we can pass to the limit is (5.6) using the property $(5.4)_1$–$(5.5)_1$. As a result, we come to the relation

$$\int_0^T \int_{\mathcal{O}} \left[-\rho^* \frac{\partial \varphi}{\partial t} - f_1(\rho^*)\frac{\partial \varphi}{\partial x}\right] dx\,dt = \int_0^T \langle v^*(t,\cdot), \varphi(t,\cdot)\rangle_{\mathcal{M}(\mathcal{O}),C_0(\mathcal{O})}\,dt,$$

which gives us the weak formulation of hyperbolic conservation law $(5.3)_1$. As for the initial condition $(5.3)_1$, we note that by continuity property (4.9) the following identity

$$\lim_{t\to 0+} \frac{1}{t}\int_0^t \left| \int_{\mathcal{O}} (\rho_0^\varepsilon(s,\cdot) - \rho_0)\psi\,dx \right| ds = 0 \quad \forall\,\psi \in C_0^\infty(\mathcal{O}) \tag{5.7}$$

is valid for every $\varepsilon > 0$. So, we can pass to the limit in (5.7) as $\varepsilon \to 0$ using the weak convergence of $\rho_0^\varepsilon \to \rho^*$ in $L^2((0,T)\times\mathcal{O})$. As a result, the initial condition for the limit function ρ^* is satisfied in the following sense

$$\lim_{t\to 0+} \frac{1}{t}\int_0^t \left| \int_{\mathcal{O}} (\rho^*(s,\cdot) - \rho_0)\psi\,dx \right| ds = 0 \quad \forall\,\psi \in C_0^\infty(\mathcal{O}).$$

Thus, $\rho^* \in L^2((0,T)\times\mathcal{O})$ is a weak solution to the Cauchy problem (5.2). By analogy it can be proved the similar properties for the limit function μ^*. This concludes the proof. ∎

The next result is a crucial in this paper. We show that approximately entropy weak solutions to the system of nonlinear conservation laws with impulse controls can be constructed by optimal solutions to the penalized problem (4.2)–(4.8), (4.21).

Theorem 5.2. *Let $u = \sum_{k=1}^N u_k(t)\delta_{\tau_k} \in \mathcal{U}_{ad}$ be a given source term with prescribed location $a < \tau_1 < \cdots < \tau_N < b$. Assume that there exists a sequence of pairs $\{(\widehat{v}^\varepsilon, \widehat{\rho}^\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$*

satisfying the relation

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\widehat{v}^\varepsilon, \widehat{\rho}^\varepsilon) < +\infty. \tag{5.8}$$

Let $\{(v_0^\varepsilon, \rho_0^\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *be a sequence of optimal solutions to the penalized problem* (4.2)–(4.8), (4.21). *Then, under supposition (A1), for every* σ*-cluster point* $(v^*, \rho^*) \in \mathcal{Y}_1$ *of the sequence* $\{(v_0^\varepsilon, \rho_0^\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *we have: the triplet* (u^*, ρ^*, μ^*) *is an approximately entropy solution to the Cauchy problem* (3.1)–(3.3) *in a domain* $(0,T)\times\mathcal{O}$ *and the equality* $v^* = (f_2(\mu^*))_x$ *is valid almost everywhere in* $(0,T)\times\mathcal{O}$*. Here the distribution* μ^* *is defined by* (5.3).

Proof. As Lemma 5.1 indicates, the sequence $\{(\mu^\varepsilon) \in \mathcal{W}_1\}_{\varepsilon>0}$ is relatively compact with respect to the strong convergence in $L^2(0,T;H^{-1}(\mathcal{O}))$ and the weak convergence in $L^2(0,T;L^2(\mathcal{O}))$. So, passing to a subsequence, when the occasion requires, we get

$$\mu^\varepsilon \to \mu^* \ \text{ in } \ L^2(0,T;H^{-1}(\mathcal{O})), \quad \mu^\varepsilon \rightharpoonup \mu^* \ \text{ in } \ L^2(0,T;L^2(\mathcal{O})), \tag{5.9}$$

where $\mu^* \in L^2((0,T)\times\mathcal{O})$ is a weak solution to the Cauchy problem (5.3). For our further analysis we have to show that

$$(f_2(\mu^\varepsilon))_x \rightharpoonup (f_2(\mu^*))_x \quad \text{ in } \ L^2(0,T;H^{-1}(\mathcal{O})). \tag{5.10}$$

Indeed, let $\varphi \in C_0^\infty((0,T)\times\mathcal{O})$ be a fixed test function. Then the following estimate takes a place

$$\begin{aligned}\left|\int_0^T\int_{\mathcal{O}} \mu_x^\varepsilon \varphi\, dxdt\right| &= \left|\int_0^T\int_{\mathcal{O}} (\mu_t^\varepsilon\varphi + \mu_x^\varepsilon\varphi_x - u\varphi)\, dxdt\right| \\ &\le \int_0^T \left(\left|\langle \mu_t^\varepsilon, \varphi\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})}\right| + \sqrt{\varepsilon}\int_{\mathcal{O}} \left|\sqrt{\varepsilon}\mu_x^\varepsilon\varphi_x\right| dx + \left|\langle u, \varphi\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})}\right|\right) dt \\ &\overset{\text{by (4.13), (4.15)}}{\le} \left(C + \|u\|_{L^2(0,T;H^{-1}(\mathcal{O}))}\right)\|\varphi\|_{L^2(0,T;H_0^1(\mathcal{O}))}.\end{aligned}$$

Hence the sequence $\{\mu_x^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^{-1}(\mathcal{O})$. Therefore, in view of (5.9), we can suppose that $\mu_x^* \in L^2(0,T;H^{-1}(\mathcal{O}))$ and

$$\mu_x^\varepsilon \overset{*}{\rightharpoonup} \mu_x^* \ \text{ in } \ L^2(0,T;H^{-1}(\mathcal{O})), \tag{5.11}$$

As a result, applying the arguments of Remark 4.5, we come to the required conclusion (5.10).

Let $\{(\widehat{v}^\varepsilon, \widehat{\rho}^\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence with property (5.8). Then there exist a value $\varepsilon_0 > 0$ and a constant $c > 0$ independent of ε such that the following inequality holds true

$$\|v_0^\varepsilon\|^2_{L^2(0,T;\mathcal{M}(\mathcal{O}))} + \varepsilon^{-1}\left\|(f_2(\mu^\varepsilon))_x - v_0^\varepsilon\right\|_{L^2(0,T;H^{-1}(\mathcal{O}))} \le I_\varepsilon(\widehat{v}^\varepsilon, \widehat{\rho}^\varepsilon) \le c \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{5.12}$$

Hence the sequence of optimal fictitious controls $\{v_0^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(0,T;\mathcal{M}(\mathcal{O}))$. Therefore, by Lemma 5.1 the sequence of optimal pairs $\{(v_0^\varepsilon, \rho_0^\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the σ-topology of $L^2(0,T;\mathcal{M}(\mathcal{O})) \times L^2(0,T;H^{-1}(\mathcal{O}))$. Moreover, every its σ-cluster point $(v^*, \rho^*) \in \mathcal{Y}_1$ possesses the properties (a)–(c) of Lemma 5.1.

Further we note that the inequality (5.12) leads to the estimate

$$0 \le \left\|(f_2(\mu^\varepsilon))_x - v_0^\varepsilon\right\|_{L^2(0,T;H^{-1}(\mathcal{O}))} \le \varepsilon c \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{5.13}$$

Since $(f_2(\mu^\varepsilon))_x - v_0^\varepsilon \rightharpoonup (f_2(\mu^*))_x - v^*$ in $L^2(0,T;H^{-1}(\mathcal{O}))$ (see (5.10)), $v_0^\varepsilon \overset{*}{\rightharpoonup} v^*$ in $L^2(0,T;\mathcal{M}(\mathcal{O}))$, and $\mathcal{M}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$, we can pass to the limit in (5.13) as $\varepsilon \to 0$. Then, in view of the lower semicontinuity property, we obtain

$$0 \le \|(f_2(\mu^*))_x - v^*\|_{L^2(0,T;H^{-1}(\mathcal{O}))} \le \liminf_{\varepsilon\to 0} \|(f_2(\mu^\varepsilon))_x - v_0^\varepsilon\|_{L^2(0,T;H^{-1}(\mathcal{O}))} \le 0.$$

Since this is equivalent to the equality $v^* = (f_2(\mu^*))_x$ a.e. in $(0,T)\times\Omega$, by Lemma 4.3 it follows that the pair (ρ^*,μ^*) is an approximately entropy solution to the initial-boundary value problem (4.2)–(4.8). This concludes the proof. $\square$

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