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ON SOME PROPERTIES OF BILINEAR FORMS ASSOCIATED WITH SKEW-SYMMETRIC $L^2(\Omega)$-MATRICES

P. I. Kogut

Днепропетровский национальный университет им. Олеся Гончара, пр. Гагаріна, 72, Днепропетровск 49010. E-mail: p.kogut@i.ua

We study the bilinear forms on the space of measurable square-integrable functions which are generated by skew-symmetric matrices with unbounded coefficients. We show that in the case when a skew-symmetric matrix contains L^2-elements, the corresponding quadratic forms can be alternative. Since these questions are closely related with the existence of a unique solution for linear elliptic equations with unbounded coefficients, we show that the energy identities for weak solutions can be studied in the framework of the corresponding alternative quadratic forms. To this end, we discuss the problems of integration by parts for measurable functions and give a generalization of some formulae for the non-Lipschitz case.

Key words: skew-symmetric matrices, positive defined matrices, unbounded bilinear form, formula integration by parts, Lipschitz functions.

1. Introduction

We denote by $\mathbb{S}^N$ the set of all skew-symmetric matrices $A=[a_{ij}]_{i,j=1}^N$, i.e., A is a square matrix whose transpose is also its negative. Thus, if $A\in\mathbb{S}^N$ then $a_{ij}=-a_{ji}$ and, hence, $a_{ii}=0$. Therefore, the set $\mathbb{S}^N$ can be identified with the Euclidean space $\mathbb{R}^{\frac{N(N-1)}{2}}$. Let $L^2(\Omega)^{\frac{N(N-1)}{2}}=L^2\big(\Omega;\mathbb{S}^N\big)$ be the space of measurable square-integrable functions whose values are skew-symmetric matrices and it is endowed with the norm

$$\|A\|_{L^2(\Omega;\mathbb{S}^N)}=\left(\max_{1\le i<j\le N}\int_\Omega\left(|a_{ij}(x)|\right)^2\,dx\right)^{1/2}.$$

Let $A\in L^2\big(\Omega;\mathbb{S}^N\big)$ be an arbitrary skew-symmetric matrix. In what follows, we always associate A with the alternating form $B(\cdot,\cdot)_A: L^2(\Omega;\mathbb{R}^N)\times L^2(\Omega;\mathbb{R}^N)\to\mathbb{R}$ following the rule

$$B(\xi,\eta)_A=\int_\Omega\big(\eta(x),A(x)\xi(x)\big)_{\mathbb{R}^N}\,dx,\quad \forall\,\xi,\eta\in L^2(\Omega;\mathbb{R}^N). \tag{1.1}$$

Here, $L^2(\Omega;\mathbb{R}^N)=L^2(\Omega)^N$ stands for the space of measurable vector-valued functions such that

$$\|v\|_{L^2(\Omega;\mathbb{R}^N)}:=\left(\int_\Omega\|v(x)\|^2_{\mathbb{R}^N}\,dx\right)^{1/2}<+\infty.$$

It is easy to see that this form is unbounded on $L^2(\Omega;\mathbb{R}^N)$, since, in general, the 'integrand' $\big(\xi(x), A(x)\eta(x)\big)_{\mathbb{R}^N}$ is not integrable on Ω. This motivates to introduce of the following set. We say that an element $\xi \in L^2(\Omega;\mathbb{R}^N)$ belongs to the set $D(A) \subset L^2(\Omega;\mathbb{R}^N)$ if

$$|B(\xi,\eta)_A| \le c(y,A)\left(\int_\Omega |\eta|^2_{\mathbb{R}^N}\,dx\right)^{1/2}, \ \forall\,\xi \in C_0^\infty(\Omega;\mathbb{R}^N) \tag{1.2}$$

with some constant c depending on y and A. As a result, having set

$$B(\xi,\eta)_A = \int_\Omega \big(\eta, A\xi\big)_{\mathbb{R}^N}\,dx, \quad \forall\,\xi \in D(A),\ \forall\,\eta \in C_0^\infty(\Omega;\mathbb{R}^N),$$

we observe that the bilinear form $B(\xi,\eta)_A$ can be defined for all $\eta \in L^2(\Omega;\mathbb{R}^N)$ using the standard rule

$$B(\xi,\eta)_A = \lim_{\varepsilon\to 0} B(\xi,\eta_\varepsilon)_A, \tag{1.3}$$

where $\{\eta_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(\Omega;\mathbb{R}^N)$ and $\eta_\varepsilon \to \eta$ strongly in $L^2(\Omega;\mathbb{R}^N)$. In this case the value $B(\xi,\eta)_A$ is finite for every $\xi \in D(A)$, although the 'integrand' $\big(\eta, A\xi\big)_{\mathbb{R}^N}$ need not be integrable, in general. This fact leads us to the conclusion

$$|B(\xi,\xi)_A| < +\infty, \quad \forall\,\xi \in D(A). \tag{1.4}$$

At the same time, if we temporary assume that $A \in L^\infty(\Omega;\mathbb{S}^N)$, then the corresponding bilinear form $B(\xi,\eta)_A$ is obviously bounded on $L^2(\Omega;\mathbb{R}^N)$, i.e. in this case $D(A) \equiv L^2(\Omega;\mathbb{R}^N)$. Indeed, in view of the Bunjakowski inequality, we get

$$\begin{aligned}|B(\xi,\eta)_A| \le& \|A\|_{L^\infty(\Omega;\mathbb{S}^N)}\int_\Omega \|\xi\|_{\mathbb{R}^N}\|\eta\|_{\mathbb{R}^N}\,dx\\ \le& \|A\|_{L^\infty(\Omega;\mathbb{S}^N)}\|\xi\|_{L^2(\Omega;\mathbb{R}^N)}\|\eta\|_{L^2(\Omega;\mathbb{R}^N)}\end{aligned}$$

Moreover, if $\xi = \eta$ then

$$B(\xi,\xi)_A := \int_\Omega \big(\xi, A\xi\big)_{\mathbb{R}^N}\,dx = -\int_\Omega \big(A\xi,\xi\big)_{\mathbb{R}^N}\,dx = -B(\xi,\xi)_A,$$

and, therefore, $B(\xi,\xi)_A = 0$ for all $\xi \in L^2(\Omega;\mathbb{R}^N)$. However, as it is shown in Section 3, there exist skew-symmetric L^2-matrices A such that the equality $|B(\xi,\xi)_A| = B(\xi,\xi)_A$ does not hold true for some $\xi \in D(A)$. Thus, the main goal of this paper is to discuss the extra conditions on matrix $A \in L^2\big(\Omega;\mathbb{S}^N\big)$ that would suffice to assert the validity of relation

$$\int_\Omega \big(\xi(x), C(x)\xi(x)\big)_{\mathbb{R}^N}\,dx + B(\xi,\xi)_A \ne 0, \quad \forall\,\xi \in D(A),\ \xi \not\equiv 0$$

for any symmetric positive defined matrix $C \in L^\infty(\Omega; R^{N(N+1)/2}$. It is worth to notice that this problem is closely related with solvability and uniqueness of solutions to the boundary value problem

$$-\operatorname{div}\big(C(x)\nabla y + A(x)\nabla y\big) = f \quad \text{in} \ \ \Omega, \tag{1.5}$$

$$y = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where $f \in H^{-1}(\Omega)$ is a given distribution.

2. Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(E)$.

Let $C_0^\infty(\mathbb{R}^N)$ be the set of all infinitely differentiable functions $\varphi : \mathbb{R}^N \to \mathbb{R}$ with compact supports in $\mathbb{R}^N$. We define the Banach space $H_0^1(\Omega)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm (see [1])

$$\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, dx\right)^{1/2}.$$

Let $H^{-1}(\Omega)$ be the dual space to $H_0^1(\Omega)$.

We define the divergence $\operatorname{div} A$ of a skew-symmetric matrix $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ as a vector-valued distribution $d \in H^{-1}(\Omega; \mathbb{R}^N)$ by the following rule

$$\langle d_i, \varphi\rangle_{H^{-1}(\Omega); H_0^1(\Omega)} = -\int_\Omega (a_i, \nabla\varphi)_{\mathbb{R}^N}\, dx, \ \forall\, \varphi \in C_0^\infty(\Omega), \tag{2.1}$$

where a_i stands for the i-th row of the matrix A. We say that a matrix $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ belongs to the space $H(\Omega, \operatorname{div}; \mathbb{S}^N)$ if $d := \operatorname{div} A \in L^1(\Omega; \mathbb{R}^N)$, that is, $H(\Omega, \operatorname{div}; \mathbb{S}^N) = \big\{A \ \big| A \in L^2\big(\Omega; \mathbb{S}^N\big),\ \operatorname{div} A \in L^1(\Omega; \mathbb{R}^N)\big\}$.

For given subset $Q \subset \mathbb{R}^N$ and $s \in [0, +\infty)$, we denote $\mathcal{H}^s(Q)$ the s-dimensional Hausdorff measure of Q on $\mathbb{R}^N$. The idea is that Q is an 's-dimensional subset' of $\mathbb{R}^N$, if $0 < \mathcal{H}^s(Q) < +\infty$, even if Q is very complicated geometrically. In particular, if $s = k$ is an integer, then $\mathcal{H}^k$-measure agrees with ordinary 'k-dimensional surface area' on nice sets. For the details we refer to [4, p.60].

Let $g : \Omega \to \mathbb{R}$ be a Lipschitz function, i.e. there exists a constant C such that $|g(x) - g(y)| \leq C\|x - y\|_{\mathbb{R}^N}$ for all $x, y \in \Omega$. By a well-know theorem [4, p.80], this function can be extended to $\mathbb{R}^N$ with the same Lipschitz constant. Moreover, by Rademacher's Theorem, the extended function g is differentiable $\mathcal{L}^N$-almost everywhere in $\mathbb{R}^N$. This is surprising fact since the inequality $|g(x) - g(y)| \leq C\|x-y\|_{\mathbb{R}^N}$ apparently says nothing about the possibility of locally approximating g by a linear map. So, the gradient $\nabla g(x) = \left[\frac{\partial g(x)}{\partial x_1}, \dots, \frac{\partial g(x)}{\partial x_N}\right]^t \in \mathbb{R}^N$ is well defined for a.e. $x \in \mathbb{R}^N$. Moreover, following classical definition of the Jacobian

$$\mathbf{J}g(x) := \det\ \big([\nabla g(x)]^t \cdot \nabla g(x)\big)^{1/2},$$

in our particular case we have

$$\mathbf{J}g(x) = (\nabla g(x), \nabla g(x))_{\mathbb{R}^N}^{1/2} = \|\nabla g(x)\|_{\mathbb{R}^N}. \tag{2.2}$$

In what follows, we make use of the well-known result (see the change of variables formula in [4, p.117]).

Theorem 2.1. *Let $g : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function. Then for each $\mathcal{L}^N$-integrable function $f : \mathbb{R}^N \to \mathbb{R}$, we have*

$$f\Big|_{g^{-1}\{y\}} \quad \text{is } \mathcal{H}^{N-1}\text{-integrable for } \mathcal{L}^1\text{-almost all } y, \tag{2.3}$$

$$\int_{\mathbb{R}^N} f(x)\,\mathbf{J}g(x)\,dx = \int_{-\infty}^{+\infty} \left[\int_{g^{-1}\{y\}} f(s)\,d\mathcal{H}^{N-1} \right] dy, \tag{2.4}$$

where $g^{-1}\{y\}$ stands for the following set

$$g^{-1}\{y\} := \left\{ x \in \mathbb{R}^N \ : \ g(x) = y \right\}. \tag{2.5}$$

Remark 2.1. Hereinafter, for each measurable function $f \in L^1_{loc}(\mathbb{R})$, by value $f(x)$ we mean the following one

$$f(x) = \lim_{r\to+0} \fint_{B(x,r)} f(y)\,dy := \lim_{r\to+0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy,$$

provided x is a Lebesgue point of f. Here, $B(x,r)$ is a ball of radius r centered at x, and x is called to be a Lebesgue point of f if

$$\lim_{r\to+0} \fint_{B(x,r)} |f(y) - f(x)|\,dy = 0.$$

Remark 2.2. We note that in view of the Lipschitz property of g, for each $y \in \mathbb{R}$, the set $g^{-1}\{y\}$, given by (2.5), is closed. Hence, this set is $\mathcal{H}^{N-1}$-measurable and, therefore, the second integral in (2.4) is well defined.

3. Motivating Example

Let Ω be the unit ball in $\mathbb{R}^3$, $\Omega = \left\{ x \in \mathbb{R}^3 \ : \ \|x\|_{\mathbb{R}^3} < 1 \right\}$. Our main intention in this section is to show that for a given positive scalar value $\alpha \in \mathbb{R}$ there exist a skew-symmetric matrix $A \in L^2\big(\Omega; \mathbb{S}^3\big)$ and a function $y_d \in H^1_0(\Omega)$ such that

$$\nabla y_d \in D(A) \quad \text{and} \quad B(\nabla y_d, \nabla y_d)_A = -\frac{\alpha}{2} < 0, \tag{3.1}$$

where the bilinear form $B(\xi, \eta)_A$ is defined by (1.1).

We divide our analysis into several steps. At the first step we define a skew-symmetric matrix A as follows

$$A(x) = \begin{pmatrix} 0 & a(x) & 0 \\ -a(x) & 0 & -b(x) \\ 0 & b(x) & 0 \end{pmatrix}, \tag{3.2}$$

where $a(x) = \dfrac{x_1}{2\|x\|^2_{\mathbb{R}^3}}$ and $b(x) = \dfrac{x_3}{2\|x\|^2_{\mathbb{R}^3}}$. Since

$$\begin{aligned} \|a\|^2_{L^2(\Omega)} &= \int_\Omega \left(\frac{x_1}{2\|x\|^2_{\mathbb{R}^3}}\right)^2 dx \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{\rho^2 \cos^2\varphi \sin^2\psi}{\rho^4} \rho^2 \sin\psi \, d\psi \, d\varphi \, d\rho < +\infty, \end{aligned}$$

it follows that $a \in L^2(\Omega)$. By analogy, it can be shown that $b \in L^2(\Omega)$. Moreover, it is easy to see that the skew-symmetric matrix A, define by (3.2), satisfies the property $A \in H(\Omega, \mathrm{div}; \mathbb{S}^3)$, i.e. $A \in L^2(\Omega; \mathbb{S}^3)$ and $\mathrm{div}\, A \in L^1(\Omega; \mathbb{R}^3)$. Indeed, in view of the definition of the divergence $\mathrm{div}\, A$ of a skew-symmetric matrix, we have $\mathrm{div}\, A = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, where $d_i = \mathrm{div}\, a_i = \dfrac{x_i x_2}{\|x\|^4_{\mathbb{R}^3}}$ and a_i is i-th column of A. As a result, we get

$$\|\mathrm{div}\, a_i\|_{L^1(\Omega)} = \int_0^1 \int_0^{2\pi} \int_0^\pi \left| \frac{\rho^2 f_i(\varphi, \psi) \sin\varphi \sin\psi}{\rho^4} \right| \rho^2 \sin\psi \, d\psi \, d\varphi \, d\rho < +\infty,$$

for the corresponding $f_i = f_i(\varphi, \psi)$ $(i = 1, 2, 3)$. Therefore, $\mathrm{div}\, A \in L^1(\Omega; \mathbb{R}^3)$.

Step 2 deals with the choice of the function $y_d \in H^1_0(\Omega)$. We define it by the rule

$$y_d = \frac{\sqrt{\alpha}}{\pi} \left(1 - \|x\|^5_{\mathbb{R}^3}\right) \sqrt{4\pi - \mathrm{atan2}\left(\frac{x_2}{\|x\|_{\mathbb{R}^3}}, \frac{x_1}{\|x\|_{\mathbb{R}^3}}\right)} \quad \text{in } \Omega, \tag{3.3}$$

where the two-argument function $\mathrm{atan2}(y, x)$ is defined as follows

$$\mathrm{atan2}\,(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) + \pi, & x < 0, \\ \arctan\left(\frac{y}{x}\right) + 2\pi, & y < 0, x > 0, \\ \arctan\left(\frac{y}{x}\right), & y \geq 0, x > 0, \\ \pi/2, & y > 0, x = 0, \\ 3\pi/2, & y < 0, x = 0, \\ 0, & y = 0, x = 0. \end{cases}$$

It is easy to see that the range of $\mathrm{atan2}(y, x)$ is $[0, 2\pi]$ and

$$v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) := \frac{\alpha}{\pi^2} \left(4\pi - \mathrm{atan2}\left(\frac{x_2}{\|x\|_{\mathbb{R}^3}}, \frac{x_1}{\|x\|_{\mathbb{R}^3}}\right)\right) = \frac{\alpha}{\pi^2}\,(4\pi - \varphi), \quad \forall\, \varphi \in [0, 2\pi]$$

with respect to the spherical coordinates. Hence, $v_0 \in C^\infty(\partial\Omega)$, and, as immediately follows from (3.3), it provides that

$$y_d \in L^2(\Omega) \quad \text{and} \quad y_d = 0 \ \text{ on } \ \partial\Omega.$$

By direct computations, we get

$$\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) = \frac{1}{\|x\|^3_{\mathbb{R}^3}} \begin{bmatrix} \frac{\partial v_0}{\partial z_1}\left(\|x\|^2_{\mathbb{R}^3} - x_1^2\right) - \frac{\partial v_0}{\partial z_2} x_1 x_2 \\ \frac{\partial v_0}{\partial z_2}\left(\|x\|^2_{\mathbb{R}^3} - x_2^2\right) - \frac{\partial v_0}{\partial z_1} x_1 x_2 \\ -\frac{\partial v_0}{\partial z_1} x_1 x_3 - \frac{\partial v_0}{\partial z_2} x_2 x_3 \end{bmatrix}, \ \forall\, x \neq 0. \qquad (3.4)$$

Hence, there exists a constant $C^* > 0$ such that

$$\left\|\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right\|_{\mathbb{R}^3} \le \frac{C^*}{\|x\|_{\mathbb{R}^3}}.$$

Thus,

$$\begin{aligned} \|\nabla y_d\|_{\mathbb{R}^3} \le & \left|v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right| \left\|\nabla\left(1 - \|x\|^5_{\mathbb{R}^3}\right)\right\|_{\mathbb{R}^3} \\ & + \left(1 - \|x\|^5_{\mathbb{R}^3}\right) \left\|\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right\|_{\mathbb{R}^3} \le C_1 + \frac{C_2}{\|x\|_{\mathbb{R}^3}}. \end{aligned}$$

As a result, we infer that $\nabla y_d \in L^2(\Omega; \mathbb{R}^3)$, i.e. we finally have $y_d \in H^1_0(\Omega)$.

Step 3. We show that the function y_d, which was introduced before, belongs to the set $D(A)$. To do so, we have to prove the estimate

$$\left|\int_\Omega \left(\nabla\varphi, A(x)\nabla y_d\right)_{\mathbb{R}^3} dx\right| \le \widetilde{C}(y_d) \left(\int_\Omega |\nabla\varphi|^2_{\mathbb{R}^3}\right)^{1/2} \quad \forall\, \varphi \in C_0^\infty(\Omega). \qquad (3.5)$$

To this end, we make use of the following transformations

$$\begin{aligned} \int_\Omega \left(\nabla\varphi, A\nabla\psi\right)_{\mathbb{R}^3} dx &= -\left\langle \operatorname{div}\left(A\nabla\psi\right), \varphi\right\rangle_{H^{-1}(\Omega); H^1_0(\Omega)} \\ &= \left\langle \operatorname{div} \begin{bmatrix} (a_1)^t \nabla\psi \\ (a_2)^t \nabla\psi \\ (a_3)^t \nabla\psi \end{bmatrix}, \varphi \right\rangle_{H^{-1}(\Omega); H^1_0(\Omega)} \\ &= \sum_{i=1}^3 \left\langle \operatorname{div} a_i, \varphi \frac{\partial\psi}{\partial x_i}\right\rangle_{H^{-1}(\Omega); H^1_0(\Omega)} + \underbrace{\int_\Omega \sum_{i=1}^3 \sum_{j=1}^3 \left(a_{ij} \frac{\partial^2\psi}{\partial x_i \partial x_j}\right) \varphi\, dx}_{\substack{=0 \\ \text{since } A \in L^2(\Omega; \mathbb{S}^3)}} \\ &\qquad \text{(due to the fact that } \operatorname{div} A \in L^1(\Omega; \mathbb{R}^3)) \\ &= \int_\Omega \left(\operatorname{div} A, \nabla\psi\right)_{\mathbb{R}^3} \varphi\, dx, \end{aligned}$$

which are obviously true for all $\psi, \varphi \in C_0^\infty(\Omega)$. Since

$$\left| \int_\Omega (\operatorname{div} A, \nabla \psi)_{\mathbb{R}^3} \varphi \, dx \right| = \left| \int_\Omega \left(\nabla \varphi, \; A \nabla \psi\right)_{\mathbb{R}^3} dx \right| \le C \|A\|_{L^2(\Omega; \mathbb{S}^3_{skew})} \|\psi\|_{H^1_0(\Omega)},$$

it follows that, using the continuation principle, we can extend the previous equality with respect to ψ to the following one

$$\int_\Omega (\nabla \varphi, A \nabla y_d)_{\mathbb{R}^3} \; dx = \int_\Omega \varphi \, (\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3} \; dx \quad \forall \, \varphi \in C_0^\infty(\Omega). \tag{3.6}$$

Let us show that $(\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3} \in L^\infty(\Omega)$. In this case, relation (3.6) implies the estimate

$$\begin{aligned} \left| \int_\Omega (\nabla \varphi, A \nabla y_d)_{\mathbb{R}^3} \; dx \right| &\le \|(\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3}\|_{L^\infty(\Omega)} \int_\Omega |\varphi| \, dx \\ &\le \widetilde{C}(y_d) \left(\int_\Omega |\nabla \varphi|^2_{\mathbb{R}^N} \right)^{1/2} \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^N), \end{aligned}$$

which means that the element y_d belongs to the set $D(A)$.

Indeed, as follows from (3.4), we have the equality

$$\left(\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big), \frac{x}{\|x\|^3_{\mathbb{R}^3}} \right)_{\mathbb{R}^3} = 0. \tag{3.7}$$

Thus, the gradient of the function $\nabla v_0(\frac{x}{\|x\|_{\mathbb{R}^3}})$ is orthogonal to the vector field $Q = x/\|x\|^3_{\mathbb{R}^3}$ outside the origin. Therefore,

$$\begin{aligned} (\nabla y_d, \operatorname{div} A)_{\mathbb{R}^3} &:= \left(\nabla \left[\left(1 - \|x\|^5_{\mathbb{R}^3}\right) v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \right], \frac{x}{\|x\|^3_{\mathbb{R}^3}} \frac{x_2}{\|x\|_{\mathbb{R}^3}} \right)_{\mathbb{R}^3} \\ &= \left(\nabla \left(1 - \|x\|^5_{\mathbb{R}^3}\right), \frac{x}{\|x\|^3_{\mathbb{R}^3}} \right)_{\mathbb{R}^3} v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \frac{x_2}{\|x\|_{\mathbb{R}^3}} \\ &\quad + \left(1 - \|x\|^5_{\mathbb{R}^3}\right) \left(\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big), \frac{x}{\|x\|^3_{\mathbb{R}^3}} \right)_{\mathbb{R}^3} \frac{x_2}{\|x\|_{\mathbb{R}^3}} = I_1 + I_2, \end{aligned} \tag{3.8}$$

where $I_2 = 0$ by (3.7). Since $\nabla \left(1 - \|x\|^5_{\mathbb{R}^3}\right) = -5\|x\|^3_{\mathbb{R}^3} x$, $\frac{x_2}{\|x\|_{\mathbb{R}^3}} = \sin \varphi \sin \psi$ with respect to the spherical coordinates, and function v_0 is smooth, it follows that there exists a constant $C_0 > 0$ such that $|(\nabla y_d, \operatorname{div} A)_{\mathbb{R}^3}| \le C_0$ almost everywhere in Ω. Thus, $(\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3} \in L^\infty(\Omega)$ and we have obtained the required property.

Step 4. Using results of the previous steps, we show that the function y_d satisfies the condition $B(\nabla y_d, \nabla y_d)_A = -\frac{\alpha}{2} < 0$. Indeed, let $\{\varphi_\varepsilon\}_{\varepsilon \to 0} \subset C_0^\infty(\Omega)$ be a sequence such that $\varphi_\varepsilon \to y_d$ strongly in $H^1_0(\Omega)$. Then by continuity, we have

$$\begin{aligned} B(\nabla y_d, \nabla y_d)_A &= \lim_{\varepsilon \to 0} \int_\Omega (\nabla \varphi_\varepsilon, A \nabla y_d)_{\mathbb{R}^3} \; dx \\ &\overset{\text{by (3.6)}}{=} \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon \, (\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3} \; dx \end{aligned}$$

Since $(\operatorname{div} A, \nabla y_d)_{\mathbb{R}^3} \in L^\infty(\Omega)$ and $\varphi_\varepsilon \to y_d$ strongly in $H_0^1(\Omega)$, we can pass to the limit in the right-hand side of this relation. As a result, we get

$$B(\nabla y_d, \nabla y_d)_A = \int_\Omega y_d \left(\operatorname{div} A, \nabla y_d\right)_{\mathbb{R}^3} \, dx = \frac{1}{2} \int_\Omega \left(\operatorname{div} A, \nabla y_d^2\right)_{\mathbb{R}^3} \, dx. \tag{3.9}$$

Let $\Omega_\varepsilon = \left\{x \in \mathbb{R}^3 \mid \varepsilon < \|x\|_{\mathbb{R}^3} < 1\right\}$ and let $\Gamma_\varepsilon = \{\|x\|_{\mathbb{R}^3} = \varepsilon\}$ be the sphere of radius ε centered at the origin. Then

$$\begin{aligned}
\int_{\Omega_\varepsilon} \left(\operatorname{div} A, \nabla y_d^2\right)_{\mathbb{R}^3} dx &\overset{\text{since } \ y_d \in H_0^1(\Omega)}{=} \int_{\Gamma_\varepsilon} (\operatorname{div} A, \nu)_{\mathbb{R}^3}\, y_d^2 \, d\mathcal{H}^2 \\
&= \int_{\Gamma_\varepsilon} (\operatorname{div} A, \nu)_{\mathbb{R}^3} \left(1 - \|x\|_{\mathbb{R}^3}^5\right)^2 v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 \\
&= \int_{\Gamma_\varepsilon} (\operatorname{div} A, \nu)_{\mathbb{R}^3}\, v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) \\
&= \int_{\Gamma_\varepsilon} \left(\frac{x}{\|x\|_{\mathbb{R}^3}^3}, \Big(-\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right)_{\mathbb{R}^3} \frac{x_2}{\|x\|_{\mathbb{R}^3}} v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) \\
&= -\varepsilon^{-2} \int_{\Gamma_\varepsilon} \frac{x_2}{\|x\|_{\mathbb{R}^3}} v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) \\
&= -\int_\Gamma b_0(x) v_0^2(x) \, d\mathcal{H}^2 + o(1),
\end{aligned}$$

where $b_0 = \sin\varphi \sin\psi$ and $v_0^2 = \frac{\alpha}{\pi^2}\,(4\pi - \varphi)$. Since

$$\int_{\partial\Omega} b_0 v_0^2 \, d\mathcal{H}^2 = \frac{\alpha}{\pi^2} \int_0^{2\pi} \sin\varphi \,(4\pi - \varphi)\, d\varphi \int_0^\pi \sin^2\psi \, d\psi = \alpha > 0,$$

it remains to combine this result with (3.9) and relation

$$\int_\Omega \left(\operatorname{div} A, \nabla y_d^2\right)_{\mathbb{R}^3} \, dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \left(\operatorname{div} A, \nabla y_d^2\right)_{\mathbb{R}^3} \, dx.$$

As a result, we infer $B(\nabla y_d, \nabla y_d)_A = -\alpha/2 < 0$. This concludes our analysis.

4. On Formula of Integration by Parts for Measurable Functions

Let $u \in L^1(\Omega)$, $g \in L^1(\Omega)$, and $f \in C_0^\infty(\mathbb{R})$ be given functions. We assume that $g(x) \geq 1$ almost everywhere in Ω. The main goal of this section is to proof of the following formula

$$\int_\Omega u(x) f(g(x)) \, dx = -\int_0^\infty f'(\lambda) \int_{F_\lambda} u(x) \, dx \, d\lambda, \tag{4.1}$$

which can be viewed as a particular case of the integration by parts in the Lebesgue theory. Here, the set F_λ is defined as follows

$$F_\lambda = \left\{ x \in \Omega \ : \ \lim_{r \to +0} \fint_{B(x,r)} g(y) \, dy \leq \lambda \right\}. \tag{4.2}$$

Hereafter in this section we assume that $g : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function. Then, g is continuous and, hence,

$$F_\lambda = \{x \in \Omega \ : \ g(x) \le \lambda\}\,. \tag{4.3}$$

Following the standard formula of integration by parts [5, p.375], we get

$$\begin{aligned}\int_0^\infty f'(\lambda) \int_{F_\lambda} u(x)\,dx\,d\lambda &= \lim_{K\to\infty} \int_0^K f'(\lambda) \int_{F_\lambda} u(x)\,dx\,d\lambda \\ &= \lim_{K\to\infty} \left[\left(f(\lambda) \int_{F_\lambda} u(x)\,dx \right)\Big|_0^K - \int_0^K f(\lambda) \frac{d}{d\lambda} \left(\int_{F_\lambda} u(x)\,dx \right) d\lambda \right]. \end{aligned} \tag{4.4}$$

In view of our initial assumptions on g, we obviously have

$$\left(f(\lambda) \int_{F_\lambda} u(x)\,dx \right)\Big|_{\lambda=0} = 0.$$

Moreover, since f has a compact support in $\mathbb{R}$ and the function $d(\lambda) := \int_{F_\lambda} u(x)\,dx$ is monotonic, it follows that

$$\lim_{K\to\infty} \left(f(\lambda) \int_{F_\lambda} u(x)\,dx \right)\Big|_{\lambda=K} = 0.$$

As a result, combining formulae (4.1) and (4.4), we obtain

$$-\int_0^\infty f'(\lambda) \int_{F_\lambda} u(x)\,dx\,d\lambda = \int_0^\infty f(\lambda) \frac{d}{d\lambda} \left(\int_{F_\lambda} u(x)\,dx \right) d\lambda. \tag{4.5}$$

Further, we see that

$$\frac{d}{d\lambda} \left(\int_{F_\lambda} u(x)\,dx \right) = \int_{g^{-1}\{\lambda\}} \frac{u(s)}{\|\nabla g(s)\|_{\mathbb{R}^N}}\ d\mathcal{H}^{N-1} \tag{4.6}$$

for $\mathcal{L}^1$-almost all $\lambda \in (0,\infty)$. Indeed, taking into account the definition of the set F_λ and the fact that $\mathbf{J}g(x) = \|\nabla g(s)\|_{\mathbb{R}^N}$, we get

$$\int_{F_\lambda} u(x)\,dx = \int_{\mathbb{R}^N} \chi_\lambda(x) \frac{u(x)}{\|\nabla g(x)\|_{\mathbb{R}^N}}\, \mathbf{J}g(x)\,dx, \tag{4.7}$$

where χ_λ stands for the characteristic function of the set F_λ (here we used the fact that functions $u/\|\nabla g(s)\|_{\mathbb{R}^N}$ and $\mathbf{J}g$ are extended by zero to $\Omega^c = \mathbb{R}^N \setminus \Omega$. As a result, having applied Theorem 2.1 to the right-hand side of (4.7), we arrive at the relation

$$\begin{aligned}\int_{F_\lambda} u(x)\,dx &= \int_{-\infty}^{+\infty} \left[\int_{g^{-1}\{t\}} \chi_\lambda(s) \frac{u(s)}{\|\nabla g(s)\|_{\mathbb{R}^N}}\, d\mathcal{H}^{N-1}(s) \right] dt \\ &= \int_{-\infty}^{\lambda} \left[\int_{g^{-1}\{t\}} \frac{u(s)}{\|\nabla g(s)\|_{\mathbb{R}^N}}\, d\mathcal{H}^{N-1}(s) \right] dt, \end{aligned} \tag{4.8}$$

which immediately leads us to the formula (4.6).

In view of (4.8), formula (4.5) can be justified as follows

$$
\begin{aligned}
-\int_0^\infty f'(\lambda)\left(\int_{F_\lambda} u(x)\,dx\right)d\lambda
&= \int_0^\infty f(\lambda)\left[\int_{g^{-1}\{\lambda\}} \frac{u(s)}{\|\nabla g(s)\|_{\mathbb{R}^N}}\,d\mathcal{H}^{N-1}(s)\right]d\lambda\\
&= \int_0^\infty \left[\int_{g^{-1}\{\lambda\}} f(\lambda)\chi_\Omega(s)\left(\frac{u(s)}{\|\nabla g(s)\|_{\mathbb{R}^N}}\right)d\mathcal{H}^{N-1}(s)\right]d\lambda\\
&\overset{\text{by }(2.4)}{=} \int_{\mathbb{R}^N}\left(f(\lambda)\chi_\Omega(x)\frac{u(x)}{\|\nabla g(x)\|_{\mathbb{R}^N}}\right)\Bigg|_{\lambda=g(x)} \mathbf{J}g(x)\,dx\\
&\overset{\text{by }(2.2)}{=} \int_\Omega f(g(x))u(x)\,dx.
\end{aligned}
$$

Thus, relation (4.1) is definitely true provided $g \in L^1(\Omega)$ is a Lipschitz function. Our goal is to extend of this formula to a wider class of integrable functions $g : \Omega \to \mathbb{R}$.

5. On substantiation of formula (4.1) for a non-Lipschitz case

To begin with, we introduce the following notion.

Definition 5.1. We say that a function $g \in L^1(\Omega)$ is singular if there exists a subset $\Lambda(g) \in \mathbb{R}$ such that $\mathcal{L}^1(\Lambda(g)) \neq \emptyset$ and, for each $\lambda \in \Lambda(g)$, the level set $\left\{x \in \Omega \ : \ \lim_{r\to+0} \fint_{B(x,r)} g(y)\,dy = \lambda\right\}$ has a nonzero $\mathcal{L}^N$-measure.

Following Titchmarsh [5, p.366], singular functions in $L^1(\Omega)$ can be defined by means of an appropriate Cantor's subsets of Ω.

We are now in a position to prove the first result.

Theorem 5.1. *Equality* (4.1) *holds true for each non-singular function* $g \in L^1(\Omega)$.

Proof. Let $g \in L^1(\Omega)$ be a given non-singular function. Since the space $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, we can suppose that there exists a sequence of smooth functions $\{\varphi_\varepsilon\}_{\varepsilon\to 0}$ such that $\varphi_\varepsilon \to g$ strongly in $L^1(\Omega)$ and, hence, $\varphi_\varepsilon(x) \to g(x)$ $\mathcal{L}^N$-almost everywhere in Ω. As the sets $\{x \in \Omega \ : \ \varphi_\varepsilon(x) = \lambda\}$ have zero measure for L^1-almost all $\lambda \in \mathbb{R}$, we get that

$$\chi_{F_{\varepsilon,\lambda}}(x) \to \chi_{F_\lambda}(x) \quad \mathcal{L}^N\text{-almost everywhere in } \Omega,$$

where $F_{\varepsilon,\lambda} = \{x \in \Omega \ : \ \varphi_\varepsilon(x) \leq \lambda\}$. Hence,

$$
\begin{aligned}
\chi_{F_{\varepsilon,\lambda}}(x)u(x) &\to \chi_{F_\lambda}(x)u(x) \quad \mathcal{L}^N\text{-almost everywhere in } \Omega,\\
\chi_{F_{\varepsilon,\lambda}}(x)u(x) &\leq \chi_\Omega(x)u(x) \quad \mathcal{L}^N\text{-almost everywhere in } \Omega.
\end{aligned}
$$

Therefore, $\chi_{F_{\varepsilon,\lambda}} u \to \chi_{F_\lambda}(x) u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ by Dominated Convergence Theorem. Taking this fact into account, we can pass to the limit as $\varepsilon \to 0$ in relation

$$\int_\Omega u(x) f(\varphi_\varepsilon(x))\, dx = -\int_0^\infty f'(\lambda) \int_{F_{\varepsilon,\lambda}} u(x)\, dx\, d\lambda,$$

which is obviously true for each $\varepsilon > 0$ by arguments of the previous section. As a result, we arrive at the desired formula (4.1). The proof is complete. □

Our next intention is to discuss of the formula (4.1) for the case when the function $g \in L^1(\Omega)$ is closely related to function $u \in L^1(\Omega)$. Namely, let us suppose that $u = \|y\|^2_{\mathbb{R}^N}$, where y is an element of $H^1_0(\Omega)$. Having assumed that y and its gradient ∇y are extended by zero to $\mathbb{R}^N \setminus \Omega$, we define function g as follows

$$g(x) = \max\left\{ \sup_{r>0} \fint_{B(x,r)} \|\nabla y(s)\|_{\mathbb{R}^N}\, ds, \frac{|y(x)|}{d(x)} \right\}, \quad \mathcal{L}^N\text{-a. e. in } \Omega, \tag{5.1}$$

where $d(x)$ is the distance from x to the boundary of Ω.

Since $y \in H^1_0(\Omega)$, it follows that

$$\sup_{r>0} \fint_{B(x,r)} \|y(s)\|_{\mathbb{R}^N}\, ds \in L^2(\Omega).$$

Moreover, in view of our initial assumptions and the Hardy inequality

$$\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, dx > \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{y^2}{\|x - x^*\|^2_{\mathbb{R}^N}}\, dx, \quad \forall\, x^* \in \partial\Omega,$$

we obviously have $|u(x)|/d(x) \in L^2(\Omega)$ and, hence, $g \in L^2(\Omega)$. It is worth to note that due to the definition of function g (see (5.1)), we have

$$g(x) = \lim_{r\to+0} \fint_{B(x,r)} g(\gamma)\, d\gamma, \quad \text{for } \mathcal{L}^N\text{-almost all } x \in \Omega.$$

Therefore, hereafter we associate $g \in L^2(\Omega)$ and $\lambda \geq 0$ with the set F_λ which is defined by (4.4).

Theorem 5.2. *Let $y \in H^1_0(\Omega)$ and $f \in C^\infty_0(\mathbb{R})$ be given functions. Let $g \in L^2(\Omega)$ be defined by* (5.1)*. Then the relations*

$$\int_\Omega \|\nabla y(x)\|^2_{\mathbb{R}^N} f(g(x))\, dx = -\int_0^\infty f'(\lambda) \int_{F_\lambda} \|\nabla y(x)\|^2_{\mathbb{R}^N}\, dx\, d\lambda, \tag{5.2}$$

$$\int_\Omega \zeta(x) \|\nabla y(x)\|_{\mathbb{R}^N} f(g(x))\, dx = \int_0^\infty f'(\lambda) \int_{\Omega\setminus F_\lambda} \zeta(x) \|\nabla y\|_{\mathbb{R}^N}\, dx\, d\lambda \tag{5.3}$$

hold true for all $\zeta \in L^2(\Omega)$.

Proof. Since $\zeta\|\nabla y\|_{\mathbb{R}^N} \in L^1(\Omega)$, it follows that equality (5.3) is a complementary version of (5.2). Therefore, we concentrate at the proof of (5.2). To do so, it is enough to show that the function g is not singular in $L^1(\Omega)$. Indeed, the level set $\{x \in \Omega \,:\, g(x) = \lambda\}$ is a subset of Ω where the sum $\|\nabla y(x)\|_{\mathbb{R}^N} + |y(x)|/d(x)$ is strictly separated from zero. Consequently (see Brezis [3, p.195], it ensures that the sets $\{x \in \Omega \,:\, |y(x)| = Cd(x)\}$ and $\{x \in \Omega \,:\, \|\nabla y(x)\|_{\mathbb{R}^N} = C\}$ have zero $\mathcal{L}^N$-measure for $\mathcal{L}^1$-almost all $C > 0$. Thus, g is not singular in the sense of Definition 5.1 and in order to end of the proof it remains to apply Theorem 5.1. $\square$

6. Proof of the Uniqueness Result

Theorem 6.1. *Let $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ be a skew-symmetric matrix such that*

$$\lim_{p\to+\infty} \frac{1}{p}\|A\|_{L^p\left(\Omega;\mathbb{S}^N\right)} = 0. \tag{6.1}$$

Let $C \in L^\infty(\Omega; \mathbb{R}^{N(N+1)/2})$ be a symmetric positive defined matrix satisfying condition $(\zeta, C(x)\zeta)_{\mathbb{R}^N} \geq \beta\|\zeta\|^2_{\mathbb{R}^N}$ for all $\zeta \in \mathbb{R}^N$ with $\beta > 0$. Assume that a function $y \in D(A) \subset H^1_0(\Omega)$ is related with matrices C and A as follows

$$-\mathrm{div}\,\left(C(x)\nabla y\right) = \mathrm{div}\,\left(A(x)\nabla y\right) \ \text{ in } \Omega \text{ (in the sense of distributions).} \tag{6.2}$$

Then the equality

$$\int_\Omega \big(\nabla y(x), C(x)\nabla y(x)\big)_{\mathbb{R}^N}\, dx + B(\nabla y, \nabla y)_A = 0 \tag{6.3}$$

implies $y \equiv 0$.

Remark 6.1. The direct calculations show that a skew-symmetric matrix $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ with entries a_{ij} having behavior as $\ln\left(\ln \|x\|_{\mathbb{R}^N}\right)$, satisfies the property (6.1). Hence, the fulfilment of the condition (6.1) does not imply that $A \in L^\infty\big(\Omega; \mathbb{S}^N\big)$.

Proof. Let $y \in D(A)$ be defined by (6.2). Then the integral identity

$$\int_\Omega \big(\nabla \varphi, C(x)\nabla y + A(x)\nabla y\big)_{\mathbb{R}^N}\, dx = 0 \tag{6.4}$$

holds true for all $\varphi \in C^\infty_0(\Omega)$. Having assumed that y and ∇y are extended by zero to $\Omega^c = \mathbb{R}^N \setminus \Omega$, we define the function g and the set F_λ by (5.1) and (4.4), respectively. Since $y \in H^1_0(\Omega)$, it follows that (see [4, p. 255]) $|y(x) - y(s)| \leq c\lambda\|x - s\|_{\mathbb{R}^N}$ for $\mathcal{L}^N$-almost all $x, s \in F_\lambda$, where the constant $c \geq 1$ depends only on N. Moreover, $|y(x)| \leq \lambda d(x)$ on F_λ by construction. Hence, the function $y(x)\big|_{F_\lambda \cup \Omega^c}$ satisfies the Lipschitz condition with the same constant $c\lambda$, and by a well-know result [4, p.80] it can be extended to $\mathbb{R}^N$ with the Lipschitz constant preserved. As a result, we obtain a function y_λ such that

$$\begin{gathered} y_\lambda \in H_0^1(\Omega), \quad y_\lambda = y \ \ \mathcal{L}^N\text{-a.e. in } F_\lambda, \\ \nabla y_\lambda = \nabla y \ \ \mathcal{L}^N\text{-a.e. in } F_\lambda, \quad \|\nabla y_\lambda\|_{\mathbb{R}^N} \le c\lambda \ \ \mathcal{L}^N\text{-a.e. in } \Omega. \end{gathered} \tag{6.5}$$

Since the function y $\mathcal{L}^N$-a.e. in F_λ coincides with a continuous function in Ω, we can take by continuity $\varphi = y_\lambda$ in (6.4). Then, using the skew-symmetry property of A and positiveness of the matrix C, we get

$$\begin{aligned} \int_\Omega \big(\nabla y_\lambda, C\,\nabla y\big)_{\mathbb{R}^N} dx &= \int_{F_\lambda} \big(\nabla y_\lambda, C\,\nabla y\big)_{\mathbb{R}^N} dx \\ &\quad + \int_{\Omega\setminus F_\lambda} \big(\nabla y_\lambda, C\,\nabla y\big)_{\mathbb{R}^N} dx, \\ \int_{F_\lambda} \big(\nabla y_\lambda, C\,\nabla y\big)_{\mathbb{R}^N} dx &= \int_{F_\lambda} \big(\nabla y, C\,\nabla y\big)_{\mathbb{R}^N} dx, \\ \beta \int_{F_\lambda} \|\nabla y\|^2_{\mathbb{R}^N}\, dx &\le \int_{F_\lambda} \big(\nabla y, C\,\nabla y\big)_{\mathbb{R}^N} dx, \\ \int_{F_\lambda} \big(\nabla y_\lambda, A(x)\nabla y\big)_{\mathbb{R}^N} dx &= \int_{F_\lambda} \big(\nabla y, A(x)\nabla y\big)_{\mathbb{R}^N} dx, \\ -\int_{F_\lambda} \big(\nabla y, A(x)\nabla y\big)_{\mathbb{R}^N} dx &= \int_{F_\lambda} \big(A(x)\nabla y, \nabla y\big)_{\mathbb{R}^N} dx = 0. \end{aligned}$$

Taking these relations into account, integral identity (6.4) with $\varphi = y_\lambda$ leads us to the inequality

$$\begin{aligned} \beta \int_{F_\lambda} \|\nabla y\|^2_{\mathbb{R}^N}\, dx &\le -\int_{\Omega\setminus F_\lambda} \big(\nabla y_\lambda, C + A\,\nabla y\big)_{\mathbb{R}^N} dx \\ &\le c\lambda \int_{\Omega\setminus F_\lambda} \left(\|A(x)\|_{\mathbb{S}^N} + \|C(x)\|_{\mathbb{R}^{N(N+1)/2}}\right) \|\nabla y\|_{\mathbb{R}^N}\, dx \\ &\le c\lambda \int_{\Omega\setminus F_\lambda} \left(\|A(x)\|_{\mathbb{S}^N} + \beta^*\right) \|\nabla y\|_{\mathbb{R}^N}\, dx, \end{aligned} \tag{6.6}$$

where $\beta^* = \|C(x)\|_{L^\infty(\Omega;\mathbb{R}^{N(N+1)/2})}$. Therefore, by analogy with [6], for each $0 < \varepsilon < 1$, we can reduce the inequality (6.6) to the following form

$$\begin{aligned} &\beta \int_0^\infty \lambda^{-(1+\varepsilon)} \int_{F_\lambda} \|\nabla y\|^2_{\mathbb{R}^N}\, dx\, d\lambda \\ &\qquad \le c \int_0^\infty \lambda^{-\varepsilon} \int_{\Omega\setminus F_\lambda} \left(\|A(x)\|_{\mathbb{S}^N} + \beta^*\right) \|\nabla y\|_{\mathbb{R}^N}\, dx\, d\lambda \end{aligned} \tag{6.7}$$

We are now in a position to apply Theorem 5.2. To this end, we set

$$f(\lambda) = -\frac{1}{\varepsilon}\lambda^{-\varepsilon} \quad \text{in (5.2)} \quad \text{and} \quad f(\lambda) = \frac{1}{1-\varepsilon}\lambda^{1-\varepsilon} \quad \text{in (5.3)},$$

and observe that

$$\int_0^\infty \lambda^{-(1+\varepsilon)} \int_{F_\lambda} \|\nabla y\|^2_{\mathbb{R}^N}\, dx\, d\lambda \overset{\text{by (5.2)}}{=} \frac{1}{\varepsilon}\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, g^{-\varepsilon}\, dx,$$

$$\int_0^\infty \lambda^{-\varepsilon} \int_{\Omega\setminus F_\lambda} \zeta(x)\, \|\nabla y\|_{\mathbb{R}^N}\, dx\, d\lambda \overset{\text{by (5.3)}}{=} \frac{1}{1-\varepsilon}\int_\Omega \zeta\, \|\nabla y\|_{\mathbb{R}^N}\, g^{1-\varepsilon}\, dx,$$

where $\zeta(x) = \|A(x)\|_{\mathbb{S}^N} + \beta^*$. As a result, we can rewrite inequality (6.7) as follows

$$\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, g^{-\varepsilon}\, dx \leq \frac{c\varepsilon}{\beta(1-\varepsilon)} \int_\Omega \left(\|A(x)\|_{\mathbb{S}^N} + \beta^*\right) \|\nabla y\|_{\mathbb{R}^N}\, g^{1-\varepsilon}\, dx. \tag{6.8}$$

Now it remains to apply the Holder inequality to the right-hand side of (6.8) with conjugates $p = 2/\varepsilon$, $r = 2$, and $q = 2/(1-\varepsilon)$. One gets

$$\int_\Omega \left(\|A(x)\|_{\mathbb{S}^N} + \beta^*\right) \|\nabla y\|_{\mathbb{R}^N}\, g^{1-\varepsilon}\, dx \leq \left(\int_\Omega \left|\|A(x)\|_{\mathbb{S}^N} + \beta^*\right|^{2/\varepsilon}\, dx\right)^{\varepsilon/2} \times \left(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, dx\right)^{1/2} \left(\int_\Omega g^2\, dx\right)^{(1-\varepsilon)/2}.$$

As a result, we finally get

$$\lim_{\varepsilon\to 0} \int_\Omega \|\nabla y\|^2_{\mathbb{R}^N} \frac{g^{-\varepsilon}}{\varepsilon^2}\, dx \overset{\text{by (6.8)}}{\leq} \frac{c}{\beta} \|\nabla y\|_{L^2(\Omega;\mathbb{R}^N)} \|g\|_{L^2(\Omega)} \times \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\int_\Omega \left|\|A(x)\|_{\mathbb{S}^N} + \beta^*\right|^{2/\varepsilon}\, dx\right)^{\varepsilon/2} \overset{\text{by (6.1)}}{=} 0.$$

Hence, $y \equiv 0$ and the proof is complete. □

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