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ON HOMOGENIZATION OF AN OPTIMAL CONTROL PROBLEM FOR QUASI-LINEAR ELLIPTIC EQUATION IN ONE-DIMENSIONAL SETTING

P. I. Kogut*, E. G. Shcherbytska**

* *Department of Differential Equations, Oles Honchar Dnipro National University, Gagarin av., 72, Dnipro, 49010, e-mail: p.kogut@i.ua*

** *Department of Differential Equations, Oles Honchar Dnipro National University, Kazakova, 18, build. 14, Dnipro, 49050, e-mail: e.scherbitsky@gmail.com*

Abstract

We are concerned with the asymptotic analysis of an optimal control problem for 1-D partial differential equation with ε-periodic coefficients both in the principle part of generalized p-Laplace elliptic operator and in the cost functional, as the period ε tends to zero. We focus on the optimal control problem for quasi-linear elliptic equation with mixed (Neumann and Dirichlet) boundary conditions, L^1-bounded distributed control, and a Radon measure in the right hand side of the original equation. Using approaches of the homogenization theory and utilizing the Sobolev embedding theorems, we show that the original problem tends to the optimal control problem with clearly defined structure for a one-dimensional homogenized elliptic equation containing the standard p-Laplace operator, and its solution can be approximated by the optimal solution to the original problem in an appropriate topology as small parameter of periodicity ε tends to zero.

Keywords: quasi-linear elliptic equation, optimal control, homogenization, variational convergence

1. Introduction

In this paper we deal with the following optimal control problem for 1-D quasi-linear elliptic equation with mixed (Neumann and Dirichlet) boundary conditions

$$J_\varepsilon(u, y) = \int_I d\left(\frac{x}{\varepsilon}\right) |y'(x)|^p \, dx + \int_I |u(x)| \, dx \to \inf \tag{1.1}$$

subject to the constraints

$$-\left[c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'\right]' = f+u \quad \text{in} \quad I, \tag{1.2}$$

$$c\left(\frac{a}{\varepsilon}\right)|y'(a)|^{p-2}y'(a) = 0, \tag{1.3}$$

$$y(b) = 0, \tag{1.4}$$

$$u \in U_\partial = \left\{v \in L^1(I) : \int_I \Phi\left(|u|\right)\, dx \le C\right\}. \tag{1.5}$$

Here, $I=(a,b)$ is a bounded interval, $c,d \in L^\infty(I)$ are given 1-periodic strictly positive functions, $\Phi : [0,\infty) \to [0,\infty)$ is an increasing convex continuous function, $f \in \mathcal{M}(I)$ is a given Radon measure, $\varepsilon = 1/n$ is a small parameter, $p \in (1,+\infty)$ is a given value, and $u \in L^1(I)$ is a control function.

We study the asymptotic behavior of this problem when the ε-period tends to zero, and look for the limiting homogenized optimal control problem. In particular, we require that an optimal solution and the minimum of the cost functional for the homogenized problem are the limit values (in some reasonable sense) of the corresponding quantities of the original problem. It should be stressed that if the small parameter ε is changed, then all components of the original control problem, including the quasi-linear elliptic equation, the boundary conditions, the cost functional, and the set, where we seek its infimum, are changed as well. So, the most important point is to recovery the homogenized optimal control problem with clearly defined structure of the principle equation, boundary conditions, and the corresponding cost functional.

For the asymptotic analysis of optimal control problems in general we refer to e.g. [2, 4, 14]. The most typical procedure of homogenization consists of the following steps: at first, we write down the necessary optimality conditions for the initial problem; next we find the corresponding limiting relations as $\varepsilon \to 0$ and interpret them as necessary optimality conditions for some control problem; then, using the limiting necessary optimality conditions, we recover an optimal control problem which is called the homogenized control problem (see e.g. [5, 10, 12]). Thus, if we denote by OCP_ε, NOC_ε, HOCP, HNOC the original optimal control problem on the ε-level, the corresponding necessary optimality conditions on the ε-level, the homogenized optimal control problem and the homogenized necessary optimality system, respectively, then the above mentioned procedure can be represented in the following diagram:

$$\begin{array}{ccc} \mathrm{OCP}_\varepsilon & \stackrel{???}{\Longrightarrow} & \mathrm{HOCP} \\ \downarrow & & \updownarrow \\ \mathrm{NOC}_\varepsilon & \stackrel{\varepsilon\to 0}{\longrightarrow} & \mathrm{HNOC} \end{array}$$

However, this diagram may not commute. Moreover, it should be stressed that the approach above is suitable only for simple enough (from the point of view of control theory) optimal control problems for which there are no restrictions on admissible pairs and their optimality conditions satisfy some regularity property [11]. An attempt to extend this approach to wider class of optimal control problems was realized in [5], where it was shown that the recovery of the homogenized optimal control problem is possible only under some additional assumptions on the structure of the state equation and the dependence on the small parameter.

We consider another approach to the homogenization of optimal boundary control problems, which is based on ideas in Γ-convergence and the concept of variational convergence of constrained minimization problems. To investigate the asymptotic behavior of the considered optimal boundary control problem we apply the scheme of direct homogenization, which was recently developed in [11]. Such approach allows to reduce the procedure of the homogenization to the consecutive identification of the set of admissible solutions for the homogenized optimal control problem and then its cost functional. In particular, in this paper we show that the homogenized optimal control problem to the original one can be explicitly recovered and it takes the form

$$J_{hom}(u,y) = \int_I |u(x)|\,dx + \left\langle d c^{\frac{p}{1-p}} \right\rangle \left\langle c^{\frac{1}{1-p}} \right\rangle^{-p} \int_I |y'(x)|^p\,dx \to \inf \tag{1.6}$$

subject to the constrains

$$-\left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} \left(|y'|^{p-2}y'\right)' = u + f \quad \text{on} \quad I, \tag{1.7}$$

$$|y'(a)|^{p-2}y'(a) = 0, \quad y(b) = 0, \tag{1.8}$$

$$u \in U_\partial = \left\{ v \in L^1(I) : \int_I \Phi(|v(x)|)\,dx \le \gamma \right\}. \tag{1.9}$$

2. Preliminaries and Some Auxiliary Results

2.1. The Sobolev space $W^{1,p}(I)$

Let $I = (a,b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ be such that $1 \le p \le \infty$.

Definition 2.1. The Sobolev space $W^{1,p}(I)$ is defined to be

$$W^{1,p}(I) = \Big\{ u \in L^p(I) : \text{ there exist } \; g \in L^p(I) \text{ such that } \int_I u\varphi'\,dx = -\int_I g\varphi\,dx \quad \forall \varphi \in C_0^1(I) \Big\}.$$

Here, $C_0^1(I)$ is the set of all C^1-functions with compact support in I and, in what follows, we consider $C_0^1(I)$ as the space of test functions.

Remark 2.1. For $u \in W^{1,p}(I)$ we denote $u' = g$. We note that g is well defined (in the sense of almost everywhere) by the following well-known result: if $y \in L^1_{loc}(\Omega)$ is such that

$$\int_\Omega yf\,dx = 0 \quad \forall f \in C_0^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is an open domain, then $y = 0$ almost everywhere on Ω.

Remark 2.2. It is clear that if $u \in C^1(I) \cap L^p(I)$ and if $u' \in L^p(I)$ (here, u' is the usual derivative of u) then $u \in W^{1,p}(I)$. Moreover, the usual derivative of u coincides with its derivative in the $W^{1,p}(I)$-sense

$$\int_I u\varphi'\,dx = -\int_I g\varphi\,dx \quad \forall \varphi \in C_0^1(I),$$

that is the notion $u' = g$ is consistent.

The space $W^{1,p}(I)$ is equipped with the norm

$$\|u\|_{W^{1,p}(I)} = \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

or, if $1 < p < +\infty$, with the equivalent norm

$$\|u\|_{W^{1,p}(I)} = \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{\frac{1}{p}}$$

The following properties of $W^{1,p}$ space is well-known (see H. Brezis [3] for details):

1. The space $W^{1,p}(I)$ is a Banach space for $1 \le p \le +\infty$;
2. $W^{1,p}(I)$ is reflexive for $1 < p < +\infty$;
3. $W^{1,p}(I)$ is separable for $1 \le p < +\infty$;
4. If I is the bounded interval, then

$$C(\overline{I}) \subset W^{1,p}(I) \quad \text{for all } 1 \le p \le +\infty;$$

5. There exists a constant C (depending only on $|I| \le \infty$) such that

$$\|u\|_{L^\infty(I)} \le C\|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I) \quad \text{and for all} \;\; 1 \le p \le \infty.$$

   In other words, $W^{1,p}(I) \subset L^\infty(I)$ with continuous injection for all $1 \le p \le \infty$. Further, if I is bounded then

   (a) The injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact for all $1 < p \le +\infty$;
   (aa) The injection $W^{1,p}(I) \hookrightarrow L^p(I)$ is compact for all $1 \le p \le +\infty$;
   (aaa) The injection $W^{1,1}(I) \hookrightarrow C(\overline{I})$ is continuous, but it is never compact, even if I is a bounded interval.

6. For every $u \in W^{1,p}(I)$ with $1 \le p \le +\infty$, there exist a function $\tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ almost everywhere on I and

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\,dt \quad \forall x, y \in \overline{I} \in [a, b].$$

   In other words, every element of the space $W^{1,p}(I)$ admits one (and only one) continuous representative on $\overline{I}$, i.e. there exist an absolutely continuous function $\tilde{u}$ on $\overline{I}$ that belongs to the equivalence class of $u \in W^{1,p}(I)$;

7. Let $1 \leq p \leq +\infty$. Then there exist a bounded linear operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$, called an extension operator, satisfying the following properties:

$$(i)\ \ Pu = u \ \text{ on } \ I, \quad \forall u \in W^{1,p}(I);$$
$$(ii)\ \ \|Pu\|_{L^p(\mathbb{R})} \leq 4\|u\|_{L^p(I)} \quad \forall u \in W^{1,p}(I);$$
$$(iii)\ \ \|Pu\|_{W^{1,p}(\mathbb{R})} \leq 4\left(1 + \frac{1}{|I|}\right)\|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I);$$

8. Let $u \in W^{1,p}(I)$ with $1 \leq p < +\infty$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$ such that $u_n\big|_I \to u$ in $W^{1,p}(I)$, i.e.

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(I)} = 0.$$

However, in general, we cannot assert the existence of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(I)$ such that $u_n \to u$ in $W^{1,p}(I)$;

2.2. The Sobolev space $W_0^{1,p}(I,b)$

Let us consider the following set of smooth functions

$$C_0^\infty(\mathbb{R}, b) = \{\varphi \in C_0^\infty(\mathbb{R}) : \varphi(b) = 0\}.$$

We define, for a given $1 \leq p < \infty$, the space $W_0^{1,p}(I,b)$ as the closure of $C_0^\infty(\mathbb{R}, b)$ with respect to the norm $\|\cdot\|_{W^{1,p}(I)}$.

In view of the properties of the space $W^{1,p}(I)$ indicated above, it is clear that $W_0^{1,p}(I,b)$ is a separable Banach space provided it is equipped with the norm of $W^{1,p}(I)$. Moreover, $W_0^{1,p}(I,b)$ is a reflexive space for $p > 1$. For the reader's convenience, let us recall the following well-known Lebesgue's Dominated Convergence Theorem that we make use of later on.

Theorem 2.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued measureble function on I. Suppose that this sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leq g(x) \quad \forall n \in \mathbb{N} \quad \textit{and for a.e. } x \textit{ in } I.$$

Then f is integrable and

$$\lim_{n\to\infty} \int_I |f_n(x) - f(x)|\, dx = 0.$$

The following result provides a basic characterization of functions in $W_0^{1,p}(I,b)$.

Theorem 2.2. *Let $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I,b)$ if and only if $u(b) = 0$.*

Proof. To begin with, it is worth to emphasize that the expression $u(b)$ makes sense for every $u \in W^{1,p}(I)$ because of the following injection (see Property 5(aaa))

$$W_0^{1,p}(I,b) \hookrightarrow W^{1,1}(I) \hookrightarrow C(\overline{I}).$$

So, if $u \in W_0^{1,p}(I,b)$ then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $C_0^1(I,b)$ such that

$$u_n \to u \text{ in } W^{1,p}(I). \tag{2.1}$$

Since $u \in W^{1,p}(I)$ and the injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is continuous for $1 \le p < \infty$, it follows from (2.1) that

$$u_n \to u \text{ in } C(\overline{I}) \quad \text{as} \quad n \to \infty,$$

i.e.

$$\lim_{n\to\infty} \max_{a\le x\le b} |u_n(x) - u(x)| = 0. \tag{2.2}$$

As a result, the required property $u(b) = 0$ immediately follows from (2.2) and the fact that, by definition,

$$u_n(b) = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Conversely, let $u \in W^{1,p}(I)$ be such that $u(b) = 0$. Fix any function $G \in C^1(\mathbb{R})$ such that

$$G(t) = \begin{cases} 0, & |t| \le 1, \\ t, & |t| \ge 2, \end{cases}$$

and

$$|G(t)| \le |t| \quad \text{(see Fig. 1)}.$$

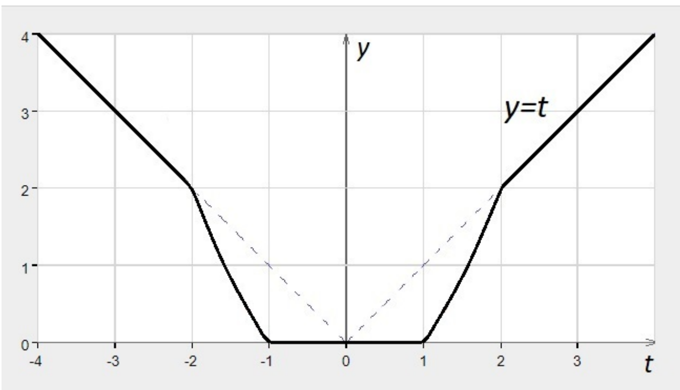

Fig. 1. Graph of G(t)

For a given $u \in W^{1,p}(I)$, we set

$$u_n = \frac{1}{n} G(nu) \quad \forall\, n \in \mathbb{N}.$$

Then $u_n \in W^{1,p}(I)$ for every $n \in \mathbb{N}$. Indeed, by definition of G, we have

$$|u_n| = \frac{1}{n} |G(nu)| \leq \frac{1}{n} n|u| = |u| \in L^p(I).$$

Hence, $u_n \in L^p(I)$. On the other hand, it is easy to check that

$$\begin{aligned} \int_I u_n \varphi' \, dx &= \frac{1}{n} \int_I G(nu)\varphi' \, dx = -\frac{1}{n} \int_I G'(nu) n u' \varphi \, dx \\ &= - \int_I G'(nu) u' \varphi \, dx, \quad \forall \varphi \in C_0^1(I). \end{aligned}$$

Since $G' \in C(\mathbb{R})$ and $G'(0) = 0$, it follows that there exists a constant $C > 0$ such that

$$\left|G'(s)\right| \leq C|s| \quad \forall\, s \in \left[-\|u\|_{L^\infty(I)}, \|u\|_{L^\infty(I)}\right].$$

Therefore,

$$\begin{aligned} |G'(nu)u'| &= |G'(nu)| \cdot |u'| \leq Cn|u| \cdot |u'| \\ &\quad \big(\text{by injection } W^{1,p}(I) \hookrightarrow L^\infty(I)\big) \\ &\leq Cn\|u\|_{L^\infty(I)} |u'| = \text{const}\, |u'|. \end{aligned}$$

Since $u' \in L^p(I)$, it follows that

$$g_n = u_n' = \frac{1}{n} G'(nu) u' \in L^p(I).$$

Thus, $u_n \in W^{1,p}(I)$ for every $n \in \mathbb{N}$. Further, we note that

$$\operatorname{supp} u_n \subset \left\{ x \in [a,b) : |u(x)| \geq \frac{1}{n} \right\},$$

that is $\operatorname{supp} u_n$ is in a compact subset of $[a,b)$ (using the fact that $u(-b) = 0$ and $u(x) \to 0$ as $|x| \to \infty, x \in I$). Thus, $u_n \in C_0(I,b)$. Since the condition $w \in W^{1,p}(I) \cap C_0(I,b)$ implies $w \in W_0^{1,p}(I,b)$, it follows that $u_n \in W_0^{1,p}(I,b) \quad \forall n \in \mathbb{N}$. It remains to note that

$$u_n \to u \quad \text{in} \quad W^{1,p}(I)$$

by the Dominated Convergence Theorem. Indeed, taking into account the definition of u_n, we have

$$\begin{aligned} u_n(x) &\to u(x) \quad \text{as} \quad n \to \infty \quad \forall x \in I, \\ u_n'(x) &\to u'(x) \quad \text{as} \quad n \to \infty \text{ for a.e. } \; x \in I. \end{aligned} \tag{2.3}$$

Moreover, in view of estimates given above, we see that

$$|u_n(x)|^p \le |u(x)|^p \quad \forall n \in \mathbb{N}, \quad \forall x \in I,$$
$$|u_n'(x)|^p \le C^p \|u\|^p_{L^\infty(I)} |u'(x)|^p \quad \forall n \in \mathbb{N} \text{ and a.e. } x \in I.$$

Hence, the Lebesgue's Dominated Convergence Theorem implies:

$$\begin{aligned} \int_I |u_n|^p\,dx &\to \int_I |u|^p\,dx, \\ \int_I |u_n'|^p\,dx &\to \int_I |u'|^p\,dx. \end{aligned} \tag{2.4}$$

Since the pointwise convergence (2.3) and properties (2.4) are sufficient to guarantee the strong convergence

$$u_n \to u \quad \text{and} \quad u_n' \to u' \quad \text{in} \quad L^p(I),$$

it follows that

$$u_n \to u \quad \text{in} \quad W^{1,p}(I).$$

Thus, $u \in W_0^{1,p}(I,b)$ and this concludes the proof. □

An important property, that takes place in $W_0^{1,p}(I)$, is expressed by the well-known Poincaré inequality.

Proposition 2.1. Let $I \in (a,b)$ be a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that

$$\|u\|_{W^{1,p}(I)} \le C\|u'\|_{L^p(I)} \quad \text{for all } u \in W_0^{1,p}(I,b). \tag{2.5}$$

Proof. Let $u \in W_0^{1,p}(I,b)$ be an arbitrary function. Since $u(b) = 0$, we have

$$|u(x)| = |u(b) - u(x)| = \left| \int_x^b u'(s)\,ds \right| \le \|u'\|_{L^1(I)}.$$

Then $\|u\|_{L^\infty(I)} \le \|u'\|_{L^1(I)}$ and, therefore, making use of the Hölder inequality, we obtain

$$\begin{aligned} \|u\|^p_{L^p(I)} &= \int_I |u|^p\,dx \le \|u\|^p_{L^\infty(I)}|I| \le \|u'\|^p_{L^1(I)}|I| = \left(\int_I |u'|\,dx\right)^p |I| \le \\ &\le \left(\int_I |u'|^p\,dx\right)\left(\int_I 1^q\,dx\right)^{\frac{p}{q}} |I| = \|u'\|^p_{L^p(I)}|I|^{\frac{p}{q}+1} = |I|^p\|u'\|^p_{L^p(I)}, \end{aligned}$$

where $\dfrac{1}{q} + \dfrac{1}{p} = 1$.
As a result, we have

$$\|u\|^p_{W^{1,p}(I)} = \|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)} \le (1 + |I|^p)\|u'\|^p_{L^p(I)},$$

that is, in (2.5) we can set $C = (1 + |I|^p)^{\frac{1}{p}}$. □

Remark 2.3. As an obvious consequence of the Poincaré inequality (2.5), we see that the norm $\|u\|_{W^{1,p}(I)}$ in $W_0^{1,p}(I,b)$ is equivalent to $\|u'\|_{L^p(I)}$.

2.3. The dual space of $W_0^{1,p}(I,b)$

In this subsection we first introduce a few notation and recall some well-known results on measures. Then we give a precise description of the structure of the dual space of $W_0^{1,p}(I,b)$.

By a nonnegative Borel measure on $I=(a,b)$ we mean a countably additive set function defined on the Borel subsets of I with values in $[0,+\infty]$. By a nonnegative Radon measure on I we mean a nonnegative Borel measure which is finite on every compact subset of I.

It is clear that for each (signed) Radon measure μ we have the representation

$$\mu=\mu^+-\mu^-,$$

where μ^+ and μ^- are nonnegative Radon measures and they stand for positive and negative parts of μ, respectively. The space of all Radon (signed) measures on I will be denoted by $\mathcal{M}(I)$.

If I is a bounded interval, then the Riesz Representation Theorem implies that the space of all bounded linear functionals on $C(I)$ (i.e. $C(I)^*$) is exactly the space of Radon measures $\mathcal{M}(I)$.

Taking into account the property 5 and Theorem 2.2, we see that the embedding

$$W_0^{1,p}(I,b)\hookrightarrow C_0(I,b)$$

is compact for $1<p<+\infty$. Let $T:W_0^{1,p}(I,b)\to C_0(I,b)$ be the corresponding compact embedding operator. Then for an arbitrary

$$\mu\in[C_o(I,b)]^*\subset[C(I)]^*=\mathcal{M}(I),$$

we have

$$\langle\mu,Tu\rangle_{\mathcal{M}(I),C(I)}:=\int_I u\,d\mu=\int_I Tu\,d\mu=\langle T^*\mu,u\rangle_{[W_0^{1,p}(I,b)]^*;W_0^{1,p}(I,b)}.$$

So, having denoted the dual space of $W_0^{1,p}(I,b)$ by $W^{-1,q}(I,b)$, where $q=p/(p-1)$, we can conclude that the mapping $T^*:\mathcal{M}(I)\to W^{-1,q}(I,b)$ is the natural embedding operator and T^* is compact for $q=p/(p-1)$. Since $p>1$, it follows that $q=p/(p-1)<+\infty$. Thus, every Radon measure $\mu\in\mathcal{M}(I)$ can be identified with some element of the space $W^{-1,q}(I,b)$. Moreover, the injection

$$\mathcal{M}(I)\hookrightarrow W^{-1,q}(I,b) \tag{2.6}$$

is compact for all $q\in(1,+\infty)$.

To characterize the set $W^{-1,q}(I,b)$, we note that this is a Banach space with respect to the norm

$$\|F\|_{W^{-1,q}(I,b)}=\sup\left\{\begin{array}{l}|F(u)|:u\in W_0^{1,p}(I,b),\\ F\text{ is a linear continuous functional,}\\ \|u\|_{W_0^{1,p}(I,b)}=\|u'\|_{L^p(I)}\le 1\end{array}\right\}$$

Moreover, since $C_0^\infty(I)$ is dense in $W_0^{1,p}(I,b)$, it follows that $W^{-1,q}(I,b)$ can be identified with a subspace of the space of distributions $\mathcal{D}'(I) = [C_0^\infty(I)]^*$ and in this sense we can write down

$$W^{-1,q}(I,b) \subset \mathcal{D}'(I).$$

Closely following H. Brezis [3] (see Proposition (2.1)), it can be proven the following result:

Theorem 2.3. *Let $F \in W^{-1,q}(I,b)$, where $q = p/(p-1)$, $p \in (1,+\infty)$, and $I = (a,b)$ is a bounded interval. Then there exist functions $f_0, f_1 \in L^q(I)$ such that*

$$\langle F, u\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)} = \int_I f_1 u'\, dx + \int_I f_0 u\, dx \quad \forall u \in W_0^{1,p}(I,b). \tag{2.7}$$

Basically, this theorem says that the elements of $W^{-1,q}(I,b)$ can be represented by a linear combination of functions in $L^q(I)$ and their first derivatives (in the sense of distribution). In particular, this implies (for the case with $f_1 \equiv 0$) that

$$L^q(I) \hookrightarrow W^{-1,q}(I,b).$$

Moreover, in view of the property 5, this embedding is compact. However, it should be emphasized that the functions f_0 and f_1 in (2.7) are not uniquely determined by $F \in W^{-1.q}(I,b)$.

2.4. On the compensated compactness approach

The Method of Compensated Compactness has been developed by L. Tartar in 70^{th}. Originally, this approach helps when one needs to find the limit of (u_n, v_n), where the sequences of vector fields u_n and v_n converge weakly in L^2 only, i.e. $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ as $n \to \infty$.

If none of the sequences converges strongly in L^2, the vector fields can still possess some additional properties which could compensate for the lack of strong convergence. For instance, the following result and its versions are often used in the homogenization theory (see Jhikov [9]).

Lemma 2.1. *Let $u_n, v_n \in L^2(\Omega)$ $\forall n \in \mathbb{N}$, where Ω is a bounded domain in $\mathbb{R}^N$ and let $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ in $L^2(\Omega)$. Assume that*

$$curl\ v_n = 0 \quad and \quad div\ u_n \to w \quad in \quad H^{-1}(\Omega).$$

Then $(u_n, v_n) \to (u, v)$ in the weak-$*$ topology of $L^1(\Omega)$.

Now we turn to the one-dimensional case and prove the following variant of Compensated Compactness Lemma.

Lemma 2.2. *Let I be a bounded interval, $1 < p < +\infty$, and $q = p/(p-1)$. Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be given sequences such that*

$$f_n \rightharpoonup f \quad weakly\ in \quad L^q(I) \quad and$$
$$g_n \rightharpoonup g \quad weakly\ in \quad L^p(I) \quad as \quad n \to \infty.$$

Assume that $f_n' \in \mathcal{M}(I)$ *for each* $n \in \mathbb{N}$ *and there exists a measure* $w \in \mathcal{M}(I)$ *such that*

$$f_n' \to w \quad \text{strongly in} \quad W^{-1,q}(I,b).$$

Then $f_n g_n \overset{*}{\rightharpoonup} fg$ *in* $L^1(I)$, *i.e.*

$$\lim_{n\to\infty} \int_I f_n g_n \varphi\, dx = \int_I fg\varphi\, dx \quad \forall\, \varphi \in C_0^\infty(I).$$

Proof. To begin with, we note that

$$f_n g_n = (f_n - f)(g_n - g) + f_n g - fg + fg_n,$$

where for the last three terms there exists an evident weak-$*$ limit in $L^1(I)$, namely,

$$f_n g - fg + fg_n \overset{*}{\rightharpoonup} fg \quad \text{in} \quad L^1(I).$$

So, in what follows, we may suppose that $f = g = 0 \in L^2(I)$. Moreover, let us define the elements u_n as follows

$$u_n(x) = -\int_x^b g_n(s)\, ds, \quad \forall\, x \in I,\ \forall\, n \in \mathbb{N}.$$

Then it is clear that $u_n(b) = 0$,

$$\begin{aligned}
\|u_n\|^2_{L^p(I)} &= \int_I \left(\int_x^b g(s)\, ds \right)^p dx \le \text{(by the Hölder inequality)}\\
&\le \int_I \left(\int_x^b 1\, ds \right)^{\frac{p}{q}} \left(\int_x^b |g(s)|^p\, ds \right) dx \le |I|^{\frac{p}{q}} \|g\|^p_{L^p(I)} \int_I 1\, dx\\
&= |I|^{\frac{p+q}{q}} \|g\|^p_{L^p(I)} = |I|^p \|g\|^p_{L^p(I)} < +\infty,
\end{aligned}$$

and

$$u_n' = g_n \quad \text{(in the sense of distributions)}.$$

Hence, $u_n \in W_0^{1,p}(I,b)$ $\forall\, n \in \mathbb{N}$ and because of the inequality (2.5), we have (see also Remark 2.2 and Theorem 2.2)

$$\begin{aligned}
\lim_{n\to\infty} \langle \varphi, u_n \rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)} &= \lim_{n\to\infty} \int_I \varphi' u_n'\, dx\\
&= \lim_{n\to\infty} \int_I \varphi' g_n\, dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(I).
\end{aligned}$$

Thus, $u_n \rightharpoonup 0$ in $W_0^{1,p}(I,b)$ as $n \to \infty$ and, therefore, $u_n \to 0$ in $L^2(I)$ by the compactness of injection $W_0^{1,p}(I,b) \subset L^p(I)$. As a result, for any $\varphi \in C_0^\infty(I)$, we have

$$\begin{aligned}
\int_I f_n g_n \varphi\, dx &= \int_I f_n (u_n \varphi)'\, dx - \int_I u_n f_n \varphi'\, dx\\
&= -\langle f_n', u_n \varphi \rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)} - \int_I u_n f_n \varphi'\, dx.
\end{aligned} \tag{2.8}$$

Here, we used the fact that $u_n \in C(I)$ (by property 5) and, therefore, $u_n\varphi \in C_0(I)$. Hence, f_n' is a Radon measure and $f_n' \in W^{-1,q}(I,b)$ by (2.6). It is clear now that the last integral in (2.8) tends to zero as $n \to \infty$ as a product of strongly convergent sequence ($u_n \to 0$ in $L^p(I)$) and weakly convergent sequence ($f_n\varphi \rightharpoonup 0$ in $L^q(I)$). As for the first term in (2.8), we have:

$$\{f_n'\}_{n\in\mathbb{N}} \quad \text{is compact in} \quad W^{-1,q}(I,b)$$
$$\text{and} \quad u_n\varphi \rightharpoonup 0 \quad \text{in} \quad W_0^{1,p}(I,b).$$

Thus, this term tends to zero as $n \to \infty$. The proof is complete. □

2.5. Operator equations with monotone mappings

In this subsection we recall some results concerning the abstract theory of operator equations. For the details we refer to T. Roubiček [13].

Let V be a separable reflexive Banach space and let V^* be its dual space.

Definition 2.2. Let $A : V \to V^*$ be a given mapping. We say that:

(i) $A : V \to V^*$ is monotone if and only if $\forall\, u, v \in V$ we have

$$\langle A(u) - A(v), u - v\rangle_{V^*;V} \ge 0;$$

(ii) If A is monotone and $u \neq v$ implies $\langle A(u) - A(v), u - v\rangle_{V^*;V} > 0$. Then A is called strictly monotone;

(iii) $A : V \to V^*$ is a bounded operator if $A(\{u \in V : \|u\|_V \le \rho\})$ is bounded in V^* for any $\rho > 0$;

(iv) $A : V \to V^*$ is radially continuous if and only if $\forall\, u, v \in V$ the function

$$t \mapsto \langle A(u + tv), v\rangle_{V^*;V}$$

is continuous;

(v) $A : V \to V^*$ is coercive if and only if

$$\lim_{\|u\|_V\to\infty} \frac{\langle A(u), u\rangle_{V^*;V}}{\|u\|_V} = +\infty.$$

The following result plays an important role in the theory of nonlinear operator equations.

Theorem 2.4. *Let $A : V \to V^*$ be a bounded, radially continuous, monotone and coercive operator. Then*

(i) *A is surjective; this means, for any $f \in V^*$, there is $u \in V$ such that*

$$A(u) = f; \tag{2.9}$$

(ii) *If in addition, $A : V \to V^*$ is strictly monotone, then the equation (2.9) has a unique solution.*

2.6. Nonlinear extremal problems

Let X be a linear normed space, X_∂ be a convex closed subset of X, and let $J : X_\partial \to \mathbb{R}$ be a functional defined and bounded from below on X_∂:

$$-\infty < C \leq J(x) < +\infty \quad \forall\, x \in X_\partial.$$

We assume that J is lower semicontinuous on X_∂ with respect to the weak convergence in X, i.e. for an element $\hat{x} \in X_\partial$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_\partial$ converging weakly to $\hat{x}$, we have

$$J(\hat{x}) \leq \lim_{n\to\infty} \inf J(x_n).$$

Let X_1 be a reflexive Banach space continuously embedded in X, and let V be a linear normed space. Let $F : X_1 \to V$ be a nonlinear operator. Let us consider the following extremal problem

$$\begin{aligned} &J(x) \to \inf \\ &F(x) = 0, \quad x \in X_\partial. \end{aligned} \tag{2.10}$$

We say that x is an admissible element of the problem (2.10) if

$$x_1 \in X_1,\ F(x_1) = 0,\ x_1 \in X_\partial, \text{ and } \ J(x_1) < +\infty.$$

The set of all admissible (or feasible) elements of the problem (2.10) we denote by Ξ.

We assume that the following conditions hold:

(C_1) The set Ξ is nonempty;

(C_2) The set Ξ is sequentially closed with respect to the weak topology of X_1, i.e. for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \Xi$, $\forall\, n \in \mathbb{N}$, and $x_n \rightharpoonup x$ weakly in X_1. Then $x \in \Xi$;

(C_3) For each $R > 0$ the set

$$\{x \in \Xi : J(x) < R\}$$

is bounded in the space X_1.

By a solution to the problem (2.10) we mean an element $x^0 \in \Xi$ such that

$$J(x^0) = \inf_{x\in\Xi} J(x).$$

Then the following assertion holds true.

Theorem 2.5. *Under conditions $(C_1)-(C_3)$ the extremal problem (2.10) admits a non-empty set of solutions. Moreover, if Ξ is a convex set and $J : \Xi \to \mathbb{R}$ is a strictly convex functional then the extremal problem (2.10) has a unique solution.*

For the proof and other details we refer to A. Fursikov [8].

2.7. Functions of bounded variation

By $BV(I)$ we denote the space of all functions in $L^1(I)$ for which the norm

$$\begin{aligned} \|f\|_{BV(I)} &= \|f\|_{L^1(I)} + \int_I |Df|\,dx = \\ &= \|f\|_{L^1(I)} + \sup\left\{ \begin{array}{l} \int_I f\varphi'\,dx : \varphi\in C_0^1(I), \\ |\varphi(x)|\le 1 \text{ for } x\in I \end{array} \right\} \end{aligned} \tag{2.11}$$

is finite. This space is called the space of functions of bounded variation. For $f \in BV(I)$ we denote by f' the distributional derivative of f. As follows from (2.11), if $f \in BV(I)$ then f' belongs to the space of Radon measures $\mathcal{M}(I)$.

We give some embeddings result for the space $BV(I)$ taken from Ambrosio [1].

Proposition 2.2. Let $I = (a, b)$ be a bounded interval. Then the embedding $BV(I) \hookrightarrow L^\infty(I)$ is continuous and the embedding $BV(I) \hookrightarrow L^r(I)$ is compact for every $1 \le r < +\infty$.

For more details on functions of bounded variation we refer to the monograph Ambrosio, Chapter 3 [1].

2.8. On variational convergence of constrained minimization problems

Let $J_k : U \times Y \to \mathbb{R}$ be a cost functional, Y be a space of states, and U be a space of controls. Let

$$\min\{J_k(u,y) : (u,y)\in\Xi_k\} \tag{2.12}$$

be a parameterized minimization problem, where Ξ_k stands for the set of feasible solutions in $U \times Y$ such that $J_k(u, y) < +\infty$ for all $(u, y) \in \Xi_k$, and these solutions are linked by some state equation. Hereinafter, we always associate to each OCP the corresponding constrained minimization problem

$$\left\langle \inf_{(u,y)\in\Xi_k} J_k(u,y) \right\rangle, \quad k\in\mathbb{N}. \tag{2.13}$$

Let σ be the product of weak topologies for the normed spaces U and Y. Moreover, for the simplicity, we assume that every bounded sequence in $U \times Y$ is sequentially compact with respect to the σ-convergence.

The main question we are going to discuss in this subsection is: how to pass to the limit in (2.12) as $k \to \infty$? Moreover, the concept of this limit passage has to guarantee the following property: a "limit cost functional" J and a "limit set of constrains" Ξ must have a clearly defined structure such that the limit object $\langle \inf_{(u,y)\in\Xi} J(u,y)\rangle$ can be interpreted as some optimal control problem.

With that in mind, we will follow the scheme of the direct variational convergence (see Kogut & Leugeriny I. [11]). As a result, we adopt the following definition for the convergence of minimization problems in normed spaces:

Definition 2.3. A problem $\left\langle \inf_{(u,y)\in\Xi} J(u,y)\right\rangle$ is variational σ-limit of the sequence (2.13) as $k\to\infty$ if and only if the following conditions are satisfied:

(d) If sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{(u_n,y_n)\}_{n\in\mathbb{N}}$ are such that

$$\begin{gathered} k_n\to\infty \quad \text{as} \quad u\to\infty, \\ (u_n,y_n)\in\Xi_{k_n} \quad \forall\, n\in\mathbb{N}, \text{ and} \\ (u_n,y_n)\xrightarrow{\sigma}(u,y) \quad \text{in} \quad U\times Y, \end{gathered}$$

then

$$(u,y)\in\Xi \quad \text{and} \quad \lim_{n\to\infty}\inf J_{k_n}(u_n,y_n)\geq J(u,y);$$

(dd) For every $(u,y)\in\Xi$ there are an integer $k_0>0$ and a sequence $\{(u_k,y_k)\}_{k\in\mathbb{N}}$ (called a realizing sequence) such that

$$\begin{gathered} (u_k,y_k)\in\Xi_k, \quad \forall\, k\geq k_0, \\ (u_n,y_n)\xrightarrow{\sigma}(u,y) \quad \text{in} \quad U\times Y, \quad \text{and} \\ \lim_{k\to\infty}\sup J(u_k,y_k)\leq J(u,y). \end{gathered}$$

Then the following result takes place (for the details we refer to Kogut & Leugeriny [11]).

Theorem 2.6. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y)\in\Xi} J(u,y)\right\rangle$$

is the variational σ-limit of (2.13) in the sense of Definition 2.3, and this problem admits a unique solution $(u^0,y^0)\in\Xi$, i.e.

$$J(u^0,y^0)=\inf_{(u,y)\in\Xi} J(u,y)$$

For every $k\in\mathbb{N}$, let $(u_k^0,y_k^0)\in\Xi_k$ be a minimizer of J_k on the corresponding set Ξ_k. If the sequence $\left\{(u_k^0,y_k^0)\right\}_{k\in\mathbb{N}}$ is relatively σ-compact in $U\times Y$ then

$$\begin{gathered} (u_k^0,y_k^0)\xrightarrow{\sigma}(u^0,y^0) \quad \textit{in} \quad U\times Y \\ \inf_{(u,y)\in\Xi} J(u,y)=J(u^0,y^0)=\lim_{k\to\infty} J_k(u_k^0,y_k^0). \end{gathered}$$

3. Setting of the Optimal Control Problem and its Previous Analysis

Let $\beta>\alpha>0,\gamma>0$, and a and b ($a<b$) be given real values. Let $c,d\in L^\infty(I)$, where $I=(a,b)$, be positive 1-periodic functions such that

$$0<\alpha\leq c(x),\, d(x)\leq\beta<+\infty \quad \text{for almost all} \quad x\in(a,b).$$

We consider the following optimal problem:
Minimize

$$J_\varepsilon(u,y)=\int_I d\left(\frac{x}{\varepsilon}\right)|y'(x)|^p\,dx+\int_I |u(x)|\,dx \tag{3.1}$$

subject to the constraints

$$-\left[c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'\right]'=f+u \quad \text{on} \quad I, \tag{3.2}$$

$$c\left(\frac{a}{\varepsilon}\right)|y'(a)|^{p-2}y'(a)=0, \tag{3.3}$$

$$y(b)=0, \tag{3.4}$$

$$u\in U_\partial=\left\{v\in L^1(I):\int_I \Phi\left(|u|\right)\,dx\le C\right\}. \tag{3.5}$$

Here, $\Phi:[0,\infty)\to[0,\infty)$ is an increasing convex continuous function such that

$$\lim_{\xi\to\infty}\frac{\Phi(\xi)}{\xi}\to\infty, \tag{3.6}$$

$f\in\mathcal{M}(I)$ is a given Radon measure, $\varepsilon=1/n$ is a small parameter, $p\in(1,+\infty)$ is a given value, and $u\in L^1(I)$ is a control function.

It is clear that, due to the initial assumptions, the boundary value problem (3.2)–(3.4) may not have a classical solution under some $u\in U_\partial$ and a given $f\in\mathcal{M}(I)$. That's why we define the set of feasible solutions Ξ to the OCP (3.1)–(3.5) as follows: we say that (u,y) is a feasible pair if $u\in L^1(I)$ is an admissible control, i.e. $u\in U_\partial$, $y\in W_0^{1,p}(I,b)$, and y is a weak solution of (3.2)–(3.4) in following sense

$$\int_I c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'\varphi'\,dx=\int_I\varphi\,df+\int_I u\varphi\,dx \quad \forall\,\varphi\in C_0^\infty(I,b). \tag{3.7}$$

Having denoted the set of feasible pairs by Ξ_ε, the optimal control problem (3.1)–(3.5) can be written as follows

$$\inf_{(u,y)\in\Xi_\varepsilon} J_\varepsilon(u,y),$$

where by a solution of this problem we mean a pair $(u_\varepsilon^0,y_\varepsilon^0)\in\Xi_\varepsilon$ such that

$$J_\varepsilon(u_\varepsilon^0,y_\varepsilon^0)=\inf_{(u,y)\in\Xi_\varepsilon} J_\varepsilon(u,y).$$

4. On the solvability of parameterized OCP (3.1)–(3.5)

To begin with, let us show that the Hypothesis (C_1) (see subsection 2.6) is satisfied. To do so, we define on operator $A_\varepsilon: W_0^{1,p}(I,b)\to W^{-1,q}(I,b)$, related to the boundary value problem (3.2)–(3.4), as follows (see the integral identity (3.5))

$$\langle A_\varepsilon(y),z\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}=\int_I c\left(\frac{x}{\varepsilon}\right)|y'(x)|^{p-2}y'(x)z'(x)\,dx \quad \forall\,z,y\in W_0^{1,p}(I,b). \tag{4.1}$$

Lemma 4.1. *Let $I=(a,b)$ be a bounded interval and $p\in(1,+\infty)$. Then $A_\varepsilon : W_0^{1,p}(I,b)\to W^{-1,q}(I,b)$, given by (4.1), is a bounded operator for every $\varepsilon>0$.*

Proof. We prove that the set

$$A_\varepsilon\left(\left\{y\in W_0^{1,p}(I,b) : \|y\|_{W_0^{1,p}(I,b)}\le\rho\right\}\right)$$

is bounded in $W^{-1,q}(I,b)$ for any $\rho>0$. Indeed, in this case we have the estimate

$$\begin{aligned}\sup_{\|y\|_{W_0^{1,p}(I,b)}\le\rho}\|A_\varepsilon(y)\|_{W^{-1,q}(I,b)} &= \sup_{\|y\|_{W_0^{1,p}(I,b)}\le\rho}\ \sup_{\|z\|_{W_0^{1,p}(I,b)}\le 1}\langle A_\varepsilon(y),z\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}\\ &=\sup_{\|y\|\le\rho}\ \sup_{\|z\|\le 1}\int_I c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'z'\,dx\le\|c\|_{L^\infty(0,1)}\sup_{\|y\|\le\rho}\ \sup_{\|z\|\le 1}\int_I|y'|^{p-1}|z'|\,dx\\ &\text{(by Hölder inequality)}\\ &\le\|c\|_{L^\infty(0,1)}\sup_{\|y\|\le\rho}\ \sup_{\|z\|\le 1}\left(\int_I|y'|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_I|z'|^p\,dx\right)^{\frac{1}{p}}\\ &=\|c\|_{L^\infty(0,1)}\sup_{\|y\|\le\rho}\ \sup_{\|z\|\le 1}\|y\|^{p-1}_{W_0^{1,p}(I,b)}\|z\|_{W_0^{1,p}(I,b)}\\ &=\|c\|_{L^\infty(0,1)}\rho^{\rho-1}<+\infty.\end{aligned}$$

□

Lemma 4.2. *Under assumptions of Lemma 4.1, the operator $A_\varepsilon : W_0^{1,p}(I,b)\to W^{-1,q}(I,b)$ is coercive.*

Proof. It is enough to observe that

$$\langle A_\varepsilon(y),y\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}:=\int_I c\left(\frac{x}{\varepsilon}\right)|y'|^p\,dx\ge\alpha\|y\|^p_{W_0^{1,p}(I,b)}.$$

Hence,

$$\frac{\langle A_\varepsilon(y),y\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}}{\|y\|_{W_0^{1,p}(I,b)}}\to\infty\quad\text{as}\quad\|y\|_{W_0^{1,p}(I,b)}\to\infty.$$

□

Lemma 4.3. *If the suppositions of Lemma 4.1 hold true, then $A_\varepsilon : W_0^{1,p}(I,b)\to W^{-1,q}(I,b)$ is radially continuous.*

Proof. Let $y,z\in W_0^{1,p}(I,b)$ be arbitrary elements. Then

$$c\left(\frac{x}{\varepsilon}\right)|y'+tz'|^{p-2}(y'+tz')z'\to c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'z'\quad\text{as}\quad t\to 0 \tag{4.2}$$

for almost all $x\in I$ because of the continuity of the mapping $g\mapsto|g|^{p-2}g$.

Let us show that the integrant

$$c\left(\frac{x}{\varepsilon}\right)|y'+tz'|^{p-2}(y'+tz')z'$$

is dominated on I for all $t\in[0,1]$ by some integrable function $g(x)$. With that in mind, we make use of the inequality

$$(\xi+\eta)^{p-1}\le 2^{p-1}\left(\xi^{p-1}+\eta^{p-1}\right) \tag{4.3}$$

which is valid for all $\xi,\eta>0$ and $1\le p<+\infty$, and the following estimate

$$\begin{aligned}
&\langle A_\varepsilon(y+tz),z\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}=\int_I c\left(\frac{x}{\varepsilon}\right)|y'+tz'|^{p-2}(y'+tz')z'\,dx\\
&\le \|c\|_{L^\infty(0,1)}\int_I |y'+tz'|^{p-1}|z'|\,dx\\
&\overset{(\text{by }(4.3))}{\le} 2^{p-1}\|c\|_{L^\infty(0,1)}\int_I\left(|y'|^{p-1}|z'|+t^{p-1}|z'|^p\right)dx\\
&(\text{by Hölder inequality})\\
&\le 2^{p-1}\|c\|_{L^\infty(0,1)}\left(\int_I|y'|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_I|z'|^p\,dx\right)^{\frac{1}{p}}+2^{p-1}\|c\|_{L^\infty(0,1)}\|z\|^p_{W_0^{1,p}(I,b)}\\
&=2^{p-1}\|c\|_{L^\infty(0,1)}\|z\|_{W_0^{1,p}(I,b)}\left(\|y\|^{p-1}_{W_0^{1,p}(I,b)}+\|z\|^{p-1}_{W_0^{1,p}(I,b)}\right)=\text{const}<+\infty.
\end{aligned} \tag{4.4}$$

Thus, the existence of dominating integrable function g obviously follows from (4.4). To conclude the proof, it is enough to apply the Lebesgue Dominated Convergence Theorem. □

In order to prove the monotonicity of operator A_ε, let us first recall the following well-known inequalities

$$\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\ge\begin{cases}2^{2-p}|\xi-\eta|^p, & \text{if } p\ge 2\\ \dfrac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, & \text{if } p\in[1,2)\end{cases}\quad \forall\,\xi,\eta\in\mathbb{R} \tag{4.5}$$

Lemma 4.4. *Under assumptions of Lemma 4.1 the operator $A_\varepsilon: W_0^{1,p}(I,b)\to W^{-1,q}(I,b)$ is strictly monotone for every $\varepsilon>0$.*

Proof. We begin with the case $p\ge 2$. Then we have the following chain of inequalities

$$\begin{aligned}
&\langle A_\varepsilon(y)-A_\varepsilon(z),y-z\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}\\
&=\int_I c\left(\frac{x}{\varepsilon}\right)(|y'|^{p-2}y'-|z'|^{p-2}z')(y'-z')\,dx\\
&\overset{(by (4.5))}{\ge}\alpha\int_I 2^{2-p}|y'-z'|^p\,dx=\alpha 2^{2-p}\|y-z\|^p_{W_0^{1,p}(I,b)}>0
\end{aligned}$$

provided $y\ne z$. Hence, A_ε is strictly monotone.

Now, let $p \in (1,2)$. We begin with the following auxiliary estimate

$$\begin{aligned}\int_I |y'-z'|^p\,dx &= \int_I |y'-z'|^p\left(\frac{1}{|y'|+|z'|}\right)^{\frac{p(2-p)}{2}}\left(|y'|+|z'|\right)^{\frac{p(2-p)}{2}}\,dx\\ &\left(\text{by Hölder inequality with exponents } r=\frac{2}{p}, r'=\frac{2}{2-p}\right)\\ &\le \left(\int_I \frac{|y'-z'|^2}{(|y'|+|z'|)^{2-p}}\,dx\right)^{\frac{p}{2}}\left(\int_I (|y'|+|z'|)^p\,dx\right)^{\frac{2-p}{2}}.\end{aligned}\tag{4.6}$$

Hence,

$$\begin{aligned}&\langle A_\varepsilon(y)-A_\varepsilon(z), y-z\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)}\\ &\quad= \int_I c\left(\frac{x}{\varepsilon}\right)\left(|y'|^{p-2}y'-|z'|^{p-2}z'\right)(y'-z')\,dx\\ &\quad\overset{(\text{by }(4.5))}{\ge} \alpha\int_I \frac{|y'-z'|^2}{(|y'|+|z'|)^{2-p}}\,dx\\ &\quad\overset{(\text{by }(4.6))}{\ge} \alpha\left(\int_I |y'-z'|^p\,dx\right)^{\frac{2}{p}}\left(\int_I (|y'|+|z'|)^p\,dx\right)^{\frac{p-2}{p}}\\ &\quad\ge \alpha 2^{p-2}\|y-z\|^2_{W_0^{1,p}(I,b)}\left(\int_I (|y'|^p+|z'|^p)\,dx\right)^{\frac{p-2}{p}}\\ &\quad= \alpha 2^{p-2}\left(\|y'\|^p_{W_0^{1,p}(I,b)}+\|z'\|^p_{W_0^{1,p}(I,b)}\right)^{\frac{p-2}{p}}\|y-z\|^2_{W_0^{1,p}(I,b)} > 0 \quad \text{provided} \quad y\neq z.\end{aligned}$$

The proof is complete. □

As a result, in view of Lemmas 4.1–4.4 and the fact that $L^1(I)\subset\mathcal{M}(I)\subset W^{-1,q}(I,b)$ (see (2.6)), Theorem 2.5 leads us to the following conclusion:

Theorem 4.1. *If $I=(a,b)$ is a bounded interval and $p\in(1,+\infty)$, then for any admissible control $u\in U_\partial$ and $\varepsilon>0$ there exists a unique weak solution y_ε to the problem (3.2)–(3.4) such that $y_\varepsilon\in W_0^{1,p}(I,b)$ and the following integral identity*

$$\int_I c\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon' z'\,dx = \int_I zu\,dx + \int_I z\,df \tag{4.7}$$

holds true for each $z\in W_0^{1,p}(I,b)$.

As an obvious consequence of this result, we have

Corollary 4.1. *Under assumptions of Theorem 4.1 the set of feasible pairs Ξ_ε is nonempty for every $\varepsilon>0$.*

Thus the Hypothesis (C_1) is satisfied.

The next step is to verify the Hypothesis (C_2). Let $\{(u_k, y_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ be a sequence of feasible pairs such that

$$\begin{aligned} u_k &\rightharpoonup u^* \quad \text{in} \quad L^1(I), \\ y_k &\rightharpoonup y^* \quad \text{in} \quad W_0^{1,p}(I,b). \end{aligned} \tag{4.8}$$

Let us show that $(u^*, y^*) \in \Xi$. Taking into account the definition of weak convergence in $L^1(I) \times W_0^{1,p}(I,b)$, we conclude from (4.8)

$$\lim_{k\to\infty} \int_I u_k \varphi \, dx = \int_I u^* \varphi \, dx \quad \forall\, \varphi \in L^\infty(I), \tag{4.9}$$

$$\lim_{k\to\infty} \int_I y_k' \psi \, dx = \int_I (y^*)' \psi \, dx \quad \forall\, \psi \in L^q(I), \tag{4.10}$$

$$y_k \to y^* \quad \text{strongly in} \quad C(\overline{I}) \tag{4.11}$$

for $1 < p < +\infty$ and bounded I.

Let us show that

$$\left\{ \xi_k = c\left(\frac{x}{\varepsilon}\right) |y_k'|^{p-2} y_k' \right\}_{k\in\mathbb{N}}$$

is bounded sequence in BV(I). Indeed,

$$\|\xi_k\|^q_{L^q(I)} = \int_I |\xi_k|^q \, dx = \int_I c^q \left(\frac{x}{\varepsilon}\right) |y_k'|^{(p-1)q} \, dx \leq \|c\|^q_{L^\infty(0,1)} \int_I |y_k'|^p \, dx < +\infty$$

because of (4.10) (every weakly converging sequence in reflexive Banach space is bounded). Thus,

$$\xi_k \in L^q(I), \quad \text{where} \quad q = \frac{p}{p-1} > 1.$$

On the other hand, (3.2) implies that

$$\xi_k' = u_k + f \quad \text{in} \quad I.$$

Since $u_k + f \in \mathcal{M}(I)$, and $\{uk\}_{k\in\mathbb{N}}$ is bounded in $L^1(I)$ and, hence, in $\mathcal{M}(I)$, it follows that $\{\xi_k'\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{M}(I)$. Thus, $\xi_k \in BV(I)$, $\forall\, k \in \mathbb{N}$, and

$$\begin{aligned} \sup_{k\in\mathbb{N}} \|\xi_k\|_{BV(I)} &= \sup_{k\in\mathbb{N}} \left(\|\xi_k\|_{L^1(I)} + \|\xi_k'\|_{\mathcal{M}(I)} \right) \leq (\text{by Hölder inequality}) \\ &\leq \sup_{k\in\mathbb{N}} \left(|I|^{\frac{1}{p}} \|\xi_k\|_{L^q(I)} + \|\xi_k'\|_{\mathcal{M}(I)} \right) < +\infty. \end{aligned}$$

Hence, by compactness embedding result for BV-space, we have: there exists an element $\xi^* \in BV(I)$ such that within a subsequence (see Proposition 2.2)

$$\xi_k \to \xi^* \quad \text{in} \quad L^r(I) \quad \text{for} \quad 1 \leq r < +\infty, \quad \xi^* \in L^\infty(I). \tag{4.12}$$

As a result, we may suppose that $c\left(\dfrac{x}{\varepsilon}\right)|y_k'|^{p-2}y_k' = \xi_k(x) \to \xi(x)$ almost everywhere in I. Since $y_k' \rightharpoonup (y^*)'$ in $L^p(I)$ and $\eta \mapsto c\left(\frac{x}{\varepsilon}\right)|\eta|^{p-2}\eta$ is a continuous mapping, it follows from (4.12) that

$$\xi^*(x) = c\left(\frac{x}{\varepsilon}\right)|(y^*)'(x)|^{p-2}(y^*)'(x) \quad \text{for almost all} \quad x \in I. \tag{4.13}$$

Taking this fact into account, we can pass to the limit as $k \to \infty$ in the integral identity

$$\int_I c\left(\frac{x}{\varepsilon}\right)|y_k'|^{p-2}y_k'\varphi'\,dx = \int_I u_k\varphi\,dx + \int_I \varphi\,df,$$

where $\varphi \in W_0^{1,p}(I,b)$ is a test function.

We get

$$\int_I c\left(\frac{x}{\varepsilon}\right)|y_k'|^{p-2}y_k'\varphi'\,dx \to \int_I c\left(\frac{x}{\varepsilon}\right)|(y^*)'|^{p-2}(y^*)'\varphi'\,dx \quad \text{by (4.12)–(4.13);}$$
$$\int_I u_k\varphi\,dx \to \int_I u^*\varphi\,dx \quad \text{by(4.8) and the fact that } W_0^{1,p}(I,b) \subset L^\infty(I).$$

Thus, we have shown that the limit pair (u^*, y^*) is a weak solution to the boundary value problem (3.2)–(3.4). To conclude the proof of Hypothesis (C_2) it remains to show that

$$u^* \in U_\partial \quad \text{and} \quad J_\varepsilon(u^*, y^*) < +\infty. \tag{4.14}$$

As for the last condition in (4.14), it immediately follows from estimate

$$J_\varepsilon(u,y) := \int_I d\left(\frac{x}{\varepsilon}\right)|y'|^p\,dx + \int_I |u|\,dx \le \beta\|y\|^p_{W_0^{1,p}(I,b)} + \|u\|_{L^1(I)}$$

and property (4.8).

To deduce $u^* \in U_\partial$, it is enough to note that

$$\|u^*\|_{L^1(I)} \le \liminf_{k\to\infty} \|u_k\|_{L^1(I)}$$

(by the lower semicontinuity of the norm $\|\cdot\|_{L^1(I)}$ with respect to the weak convergence in $L^1(I)$) and Φ in (3.5) is the convex continuous and increasing function. Hence,

$$\gamma \ge \liminf_{k\to\infty} \int_I \Phi(|u_k|)\,dx \overset{(by (3.6))}{\ge} \int_I \Phi(|u^*|)\,dx.$$

Remark 4.1. Here, we made use of the well known fact that convex functions are lower semicontinuous with respect to the weak convergence. It means that if $F : \mathbb{R} \to \mathbb{R}$ is convex and

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^1$$

then

$$\int F(u)\,dx \le \liminf_{n\to\infty} \int F(u_n)\,dx$$

For the details we refer to L. C. Evans [7].

Thus, summing up the previous reasoning, we arrive at the following conclusion.

Theorem 4.2. *If I is a bounded interval and $p \in (1,+\infty)$ then the set Ξ is sequentially closed with respect to the weak convergence in $L^1(I) \times W_0^{1,p}(I,b)$.*

Thus, the Hypothesis (C_2) is proved. To conclude the Hypothesis (C_3), we reformulate it in the following way: for each $R > 0$ the set

$$\{(u,y) \in \Xi_\varepsilon : J_\varepsilon(u,y) \le R\}$$

is bounded in $L^1(I) \times W_0^{1,p}(I,b)$, and weakly compact. Indeed, the boundedness of the set immediately follows from the estimate

$$\begin{aligned}\|u\|_{L^1(I)} + \|y\|^p_{W_0^{1,p}(I,b)} &\le \|u\|_{L^1(I)} + \frac{1}{\alpha}\int_I d\left(\frac{x}{\varepsilon}\right)|y'|^p\,dx \\ &\le \max\left\{1,\frac{1}{\alpha}\right\}\left(\|u\|_{L^1(I)} + \int_I d\left(\frac{x}{\varepsilon}\right)|y'|^p\,dx\right) \\ &= \max\left\{1,\alpha^{-1}\right\} J_\varepsilon(u,y) \le R\max\left\{1,\alpha^{-1}\right\}\end{aligned} \tag{4.15}$$

As for its compactness property with respect to the weak convergence in $L^1(I)\times W_0^{1,p}(I,b)$, we have: (4.15) implies the boundedness of states y and, therefore, in view of the reflexivity of $W_0^{1,p}(I,b)$, any bounded sequence in $W_0^{1,p}(I,b)$ is relatively weakly compact. As for the controls, we see that

$$\begin{aligned}\|u\|_{L^1(I)} \le R\max\left\{1,\alpha^{-1}\right\} < +\infty \quad \text{and} \\ u \in U_\partial \Rightarrow \int_I \Phi(|u|)\,dx \le \gamma.\end{aligned} \tag{4.16}$$

Since Φ is an increasing function, it follows that U_∂ is equi-integrable set. Hence, by the Dunford-Pettice criterion, conditions (4.16) are sufficient to guarantee the weak compactness of the set of admissible controls in $L^1(I)$ (for the details we refer to I. Ekeland, R. Temam [6]).

As a result, we can give the following conclusion:

Theorem 4.3. *Under assumptions of Theorem 4.2, the Hypothesis (C_3) is satisfied.*

Combining Theorems 4.1-4.3 with the fact that the cost functional $J_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ is lower semicontinuous with respect to the weak convergence in $L^1(I) \times W_0^{1,p}(I,b)$, we finally arrive at the following result (see Theorem 2.6)

Theorem 4.4. *Assume that $I = (a,b)$ is a bounded interval. Then, for any $p \in (1,+\infty)$, the OCP (3.1)–(3.2) has a nonempty set of solutions.*

To the end of this section we note that for any $\varepsilon > 0$ the optimal pairs $(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ are related by integral identity

$$\int_I c\left(\frac{x}{\varepsilon}\right)|(y^0_\varepsilon)'|^{p-2}(y^0_\varepsilon)'\varphi'\,dx = \int_I u^0_\varepsilon\varphi\,dx + \int_I \varphi\,df \quad \forall\,\varphi \in C_0^\infty(I,b). \tag{4.17}$$

Since $C_0^\infty(I,b)$ is dense in $W_0^{1,p}(I,b)$, we can take $\varphi = y_\varepsilon^0$ in (4.17). Hence, we have the energy equality

$$\int_I c\left(\frac{x}{\varepsilon}\right) |(y_\varepsilon^0)'|^p\, dx = \int_I u_\varepsilon^0 y_\varepsilon^0\, dx + \int_I y_\varepsilon^0\, df. \tag{4.18}$$

Using the fact that $W_0^{1,p}(I,b) \subset L^\infty(I)$, we obtain

$$\int_I u_\varepsilon^0 y_\varepsilon^0\, dx \le |u_\varepsilon^0\|_{L^1(I)} \|y_\varepsilon^0\|_{L^\infty(I)} \le C\|u_\varepsilon^0\|_{L^1(I)} \|y_\varepsilon^0\|_{W_0^{1,p}(I,b)}. \tag{4.19}$$

Moreover, the last term in (4.18) can be estimated in a similar manner (here we should use the compactness of the embedding $W_0^{1,p}(I,b) \hookrightarrow C(\overline{I})$), namely

$$\int_I y_\varepsilon^0\, df \le \|f\|_{\mathcal{M}(I)} \|y_\varepsilon^0\|_{C(\overline{I})} \le C\|f\|_{\mathcal{M}(I)} \|y_\varepsilon^0\|_{W_0^{1,p}(I,b)}. \tag{4.20}$$

As result, the estimates (4.19)–(4.20) and energy equality (4.18) leads us to the following relation

$$\alpha \|y_\varepsilon^0\|^p_{W_0^{1,p}(I,b)} \le \int_I c\left(\frac{x}{\varepsilon}\right) |(y_\varepsilon^0)'|^p\, dx \le C\left(\|u_\varepsilon^0\|_{L^1(I)} + \|f\|_{\mathcal{M}(I)}\right) \|y_\varepsilon^0\|_{W_0^{1,p}(I,b)}.$$

Hence,

$$\|y_\varepsilon^0\|^{p-1}_{W_0^{1,p}(I,b)} \le \alpha^{-1} C\left(\|u_\varepsilon^0\|_{L^1(I)} + \|f\|_{\mathcal{M}(I)}\right) \quad \text{for any } \ \varepsilon > 0. \tag{4.21}$$

Thus, if the sequence of optimal controls $\{u_\varepsilon^0\}_{\varepsilon>0} \subset L^1(I)$ is uniformly bounded, then the sequence of corresponding optimal states $\{y_\varepsilon^0\}_{\varepsilon>0}$ is bounded (with respect to $\varepsilon > 0$) in $W_0^{1,p}(I,b)$.

Let us show that there exists a constant $C^* > 0$ such that

$$\sup_{\varepsilon>0} \|u_\varepsilon^0\|_{L^1(I)} \le C^*.$$

Indeed, let $u^* \in L^1(I)$ be an arbitrary function such that $u^* \in U_\partial$, i.e. u^* is an admissible control to the problem (3.1)–(3.2). Then, by Theorem 4.1 there exists a sequence $\{y_\varepsilon^*\}_{\varepsilon>0} \subset W_0^{1,p}(I,b)$ such that $(u^*, y_\varepsilon^*) \in \Xi_\varepsilon$ for each $\varepsilon > 0$. Hence, in view of (4.21), we have the similar estimate, i.e.

$$\sup_{\varepsilon>0} \|y_\varepsilon^*\|_{W_0^{1,p}(I,b)} \le \left(\alpha^{-1} C\left[\|u^*\|_{L^1(I)} + \|f\|_{\mathcal{M}(I)}\right]\right)^{\frac{1}{p-1}}. \tag{4.22}$$

On the other hand, it is easy to see that

$$J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \le J_\varepsilon(u^*, y_\varepsilon^*) = \int_I d\left(\frac{x}{\varepsilon}\right) |(y_\varepsilon^*)'|^p\, dx + \|u^*\|_{L^1(I)},$$

and, therefore,

$$\begin{aligned}\|u_\varepsilon^0\|_{L^1(I)} &\le \int_I d\left(\frac{x}{\varepsilon}\right) |(y_\varepsilon^*)'|^p\, dx + \|u^*\|_{L^1(I)} \le \beta \|y_\varepsilon^*\|^p_{W_0^{1,p}(I,b)} + \|u^*\|_{L^1(I)} \\ &\overset{\text{by } (4.22)}{\le} \left(\alpha^{-1} C\left[\|u^*\|_{L^1(I)} + \|f\|_{\mathcal{M}(I)}\right]\right)^{\frac{p}{p-1}} \beta + \|u^*\|_{L^1(I)} =: C^*, \quad \forall\, \varepsilon > 0.\end{aligned}$$

Thus, the sequence of optimal controls and, in view of (4.21), the sequence of optimal pairs $\left\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ to the OCP (3.1)–(3.5) in uniformly bounded in $L^1(I)\times W^{1,p}_0(I,b)$. Namely, the following estimate

$$\sup_{\varepsilon>0}\left[\|u^0_\varepsilon\|_{L^1(I)} + \|y^0_\varepsilon\|_{W^{1,p}_0(I,b)}\right] \le C^* + \left(\alpha^{-1}C\left[C^* + \|f\|_{\mathcal{M}(I)}\right]\right)^{\frac{1}{p-1}} \tag{4.23}$$

holds true.

Taking into account the fact that the control sequence $\{u^0_\varepsilon\}_{\varepsilon>0}$ lies in the set U_∂, this sequence is equi-integrable. Hence, by Dunford-Pettis criterion, this sequence is relatively compact with respect the weak topology of $L^1(\Omega)$. The same compactness property for the sequence of optimal states $\{y^0_\varepsilon\}_{\varepsilon>0}$ takes a place in $W^{1,p}_0(I,b)$ (it follows from (4.23) and reflexivity of $W^{1,p}_0(I,b)$).

Thus, we may assert that there exists a pair $(u^0,y^0)\in L^1(I)\times W^{1,p}_0(I,b)$ such that, within a subsequence, the following remarkable properties hold true:

$$\begin{aligned} u^0_\varepsilon &\rightharpoonup u^0 \quad \text{in} \quad L^1(I) \quad \text{and}\\ y^0_\varepsilon &\rightharpoonup y^0 \quad \text{in} \quad W^{1,p}_0(I,b) \quad \text{as} \quad \varepsilon\to 0. \end{aligned} \tag{4.24}$$

The question we are going to discuss further is: how this pair can be characterized and is there any relation (or association) of this limit pair to the OCP (3.1)–(3.5) as $\varepsilon\to 0$?

5. Asymptotic Analysis of OCP (3.1)–(3.5)

We begin with the following technical result.

Lemma 5.1. *Let $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ be a sequence such that*

$$(u_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon \quad \forall\,\varepsilon>0, \tag{5.1}$$

$$u_\varepsilon \rightharpoonup u^* \quad \text{in} \quad L^1(I) \quad \text{as} \quad \varepsilon\to 0, \tag{5.2}$$

$$y_\varepsilon \rightharpoonup y^* \quad \text{in} \quad W^{1,p}_0(I,b) \quad \text{as} \quad \varepsilon\to 0, \tag{5.3}$$

where $p\in(1,+\infty)$ and I is assumed to be a bounded interval. Then $y^*\in W^{1,p}_0(I,b)$ is unique weak solution of the following boundary value problem

$$\begin{aligned} -\left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}\left(|y'|^{p-2}y'\right)' &= u^* + f \quad \text{in} \quad I,\\ |y'(a)|^{p-2}y'(a) = 0, &\quad y(b)=0, \end{aligned} \tag{5.4}$$

where $\langle\cdot\rangle$ stands for the average operator

$$\langle z\rangle = \int_0^1 z(x)\,dx. \tag{5.5}$$

Proof. Let us define the sequence $\{\xi_\varepsilon\}_{\varepsilon>0}$ as follows

$$\xi_\varepsilon(x) = c\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon' \quad \forall\, \varepsilon > 0.$$

Since

$$\int_I |\xi_\varepsilon|^q\,dx \le \beta^q \int_I |y_\varepsilon'|^{(p-1)q}\,dx = \beta^q \|y_\varepsilon\|^p_{W_0^{1,p}(I,b)} < +\infty$$

it follows that

$$\{\xi_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(I). \tag{5.6}$$

Here, $q = p/(p-1)$ stands for the conjugate exponent to p.

Moreover, due to the inclusion $(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$, we have

$$\xi_\varepsilon' = -u_\varepsilon - f \quad \text{in the sense of distributions on } \ I. \tag{5.7}$$

Since $u_\varepsilon \rightharpoonup u^*$ in $L^1(I)$, the space $L^1(I)$ continuously embedded in $\mathcal{M}(I)$, and $\mathcal{M}(I) \hookrightarrow W^{-1,q}(I,b)$ compactly, it follows that

$$-u_\varepsilon - f \to -u^* - f \quad \text{strongly in } W^{-1,q}(I,b) \text{ as } \ \varepsilon \to 0.$$

On the other hand, condition (5.6) implies the existence of element $\xi \in L^q(I)$ such that

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in} \quad L^q(I) \quad \text{as} \quad \varepsilon \to 0. \tag{5.8}$$

Therefore,

$$\begin{aligned}\langle \xi_\varepsilon', \varphi\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)} &= -\int_I \xi_\varepsilon \varphi'\,dx \\ &\overset{\varepsilon\to 0}{\to} -\int_I \xi\varphi'\,dx = \langle -u^* - f, \varphi\rangle_{W^{-1,q}(I,b);W_0^{1,p}(I,b)} \quad \text{for every} \quad \varphi \in C_0^\infty(I).\end{aligned}$$

Hence,

$$\xi' = -u^* - f \quad \text{in the sense of distribution.} \tag{5.9}$$

The main point is to relative the functions ξ and y^*. With that in mind, we introduce the following auxiliary problem:

Find a 1-periodic function $v_\eta \in L^p(I)$, where $\eta \in \mathbb{R}$ is a constant, such that

$$\langle v_\eta \rangle = \int_0^1 v(x)\,dx = 0, \tag{5.10}$$

$$\left[c(x)|\eta + v_\eta|^{p-2}(\eta + v_\eta)\right]' = 0. \tag{5.11}$$

Let us show that this problem has a unique solution. Indeed, from (5.11) we deduce that

$$c(x)|\eta + v(x)|^{p-2}(\eta + v(x)) = \tau = \text{const}. \tag{5.12}$$

Let us define τ such that $v_\eta(x)$ for a given τ is a unique solution to the auxiliary problem (5.10)–(5.11). It is not difficult to see that (5.12) can be rewritten in the form

$$\eta + v(x) = \left[c^{-1}(x)|\tau|\right]^{\frac{1}{p-1}} \frac{\tau}{|\tau|}. \tag{5.13}$$

Taking into account (5.10), we have

$$\tau = \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta,$$

and, therefore (see (5.13)),

$$\eta + v_\eta(x) = [c(x)]^{\frac{1}{1-p}} \left\langle c^{\frac{1}{1-p}} \right\rangle^{-1} \eta. \tag{5.14}$$

As a result, (5.14) and (5.11) imply that

$$\begin{aligned} \left[c(x)|\eta + v_\eta(x)|^{p-2}(\eta + v_\eta(x))\right]' &= \left[c(x)c^{-1}(x) \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta\right]' \\ &= \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} \left(|\eta|^{p-2}\eta\right)' = 0. \end{aligned} \tag{5.15}$$

Thus, we arrive at the following inference: for every fixed $\eta \in \mathbb{R}$ there exist a unique solution w_η to the problem

$$-\left[c(x)|\eta + w_\eta'(x)|^{p-2}\left(\eta + w_\eta'(x)\right)\right]' = 0, \tag{5.16}$$

w_η is 1-periodic, and

$$w_\eta' = v_\eta(x) = [c(x)]^{\frac{1}{1-p}} \left\langle c^{\frac{1}{1-p}} \right\rangle^{-1} \eta - \eta.$$

Now we introduce the following test functions: for every $\eta \in \mathbb{R}$ we set

$$z_\varepsilon(x) = \eta x + \varepsilon w_\eta\left(\frac{x}{\varepsilon}\right),$$

where $w_\eta(\cdot)$ is the solution of (5.16).

Then, by monotonicity of the operator

$$A_\varepsilon(y) = \left(c\left(\frac{x}{\varepsilon}\right)|y'|^{p-2}y'\right)',$$

for every $\eta \in \mathbb{R}$ and every $\varphi \in C_0^\infty(I)$, $\varphi \geq 0$, we have

$$\int_I \varphi(x) \underbrace{\left[c\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon' - c\left(\frac{x}{\varepsilon}\right)|z_\varepsilon'|^{p-2}z_\varepsilon'\right]}_{f_\varepsilon} \underbrace{\left(y_\varepsilon' - z_\varepsilon'\right)}_{g_\varepsilon} dx \geq 0. \tag{5.17}$$

Let us show that in this case all assumptions of Lemma 2.2 (Compensated Compactness Result) are satisfied. Indeed, since

$$\begin{gathered} v_\eta \in L^p(0,1), \\ z_\varepsilon' = \eta + v_\eta\left(\frac{x}{\varepsilon}\right), \quad \int_0^1 v_\eta(x)\,dx = 0, \end{gathered}$$

and v_η is a 1-periodic function, it follows that

$$z_\varepsilon' \rightharpoonup \eta \quad \text{weakly in} \quad L^p(I) \quad \text{as} \quad \varepsilon \to 0.$$

Then, by (5.3), we deduce:

$$g_\varepsilon = y_\varepsilon' - z_\varepsilon' \rightharpoonup (y^*)' - \eta \quad \text{weakly in} \quad L^p(I). \tag{5.18}$$

On other hand

$$\xi_\varepsilon := c\left(\frac{x}{\varepsilon}\right) |y_\varepsilon'|^{p-2} y_\varepsilon' \rightharpoonup \xi \quad \text{weakly in} \quad L^q(I) \quad \text{by (5.8)},$$

$$c\left(\frac{x}{\varepsilon}\right) |z_\varepsilon'|^{p-2} z_\varepsilon' = c\left(\frac{x}{\varepsilon}\right) \left|\eta + v_\eta\left(\frac{x}{\varepsilon}\right)\right|^{p-2} \left(\eta + v_\eta\left(\frac{x}{\varepsilon}\right)\right)$$
$$\overset{\text{by (5.14)}}{=} \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta \quad \forall\, \varepsilon > 0.$$

Hence,

$$f_\varepsilon := c\left(\frac{x}{\varepsilon}\right) |y_\varepsilon'|^{p-2} y_\varepsilon' - c\left(\frac{x}{\varepsilon}\right) |z_\varepsilon'|^{p-2} z_\varepsilon' \rightharpoonup \xi - \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta \quad \text{weakly in} \quad L^q(I). \tag{5.19}$$

It remains to note that

$$f_\varepsilon' = -u_\varepsilon - f - \underbrace{\left(c\left(\frac{x}{\varepsilon}\right) |z_\varepsilon'|^{p-2} z_\varepsilon'\right)'}_{=0 \text{ by (5.16)}} = -u_\varepsilon - f.$$

Since $u_\varepsilon + f \to u^* + f$ in $W^{-1,q}(I,b)$ and $u^* \in L^1(I) \subset \mathcal{M}(I)$, it follows that $\{f_\varepsilon'\}_{\varepsilon>0}$ is a compact sequence in $W^{-1,q}(I,b)$. Hence, by Lemma 2.2, we can pass to the limit in (5.17) as $\varepsilon \to 0$. As a result, (5.18)–(5.19) lead to the relation

$$\int_0^1 \varphi(x) \left(\xi - \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta\right) \left((y^*)' - \eta\right)\, dx \geq 0$$
$$\forall\, \varphi \in C_0^\infty(I),\ \varphi(x) \geq 0.$$

This implies that, for any $\eta \in \mathbb{R}$,

$$\left(\xi(x) - \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |\eta|^{p-2}\eta\right) \left((y^*)'(x) - \eta\right) \geq 0 \tag{5.20}$$

almost everywhere on $I = (a, b)$. By the strict monotonicity of the operator

$$A_{hom}(y) = -\left(|y'|^{p-2} y'\right)' \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p},$$

the inequality (5.20) ensures that

$$\xi(x) = \left\langle c^{\frac{1}{1-p}} \right\rangle^{1-p} |(y^*)'|^{p-2} (y^*)' \quad \text{a.e. on} \quad I. \tag{5.21}$$

Taking this fact into account, we can pass to the limit in the integral identity

$$\int_I c\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon'\varphi'\,dx=\int_I u_\varepsilon\varphi\,dx+\int_I\varphi\,df,$$

as $\varepsilon\to 0$, where $\varphi\in W_0^{1,p}(I,b)$ is an arbitrary test function. We obtain

$$\left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}\int_I|(y^*)'|^{p-2}(y^*)'\varphi'\,dx=\int_I u^*\varphi\,dx+\int_I\varphi\,df\quad\forall\,\varphi\in W_0^{1,p}(I,b).$$

Thus, $y^*\in W_0^{1,p}(I,b)$ is a unique weak solution to the problem (5.4). The proof is complete. □

The following result is a direct consequence of the compactness of embedding $BV(I)\hookrightarrow L^r(I)\ \forall\,r\in[1,+\infty)$ and relation (5.21).

Corollary 5.1. *Under conditions of Lemma 5.1 the sequence*

$$\left\{\xi_\varepsilon:=c\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon'\right\}_{\varepsilon>0}\tag{5.22}$$

is bounded in $BV(I)$ and

$$c\left(\frac{\cdot}{\varepsilon}\right)|y_\varepsilon'|^{p-2}y_\varepsilon'\to\left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}|(y^*)'|^{p-2}(y^*)'\quad\textit{strongly in}\quad L^q(I).\tag{5.23}$$

Proof. Indeed, the $BV(I)$-boundedness of the sequence (5.22) immediately follows from (5.7) and (5.2). Hence, by Proposition 2.2, this sequence is compact in $L^r(I)$ for $r=q=p/(p-1)$. It remains to note that (5.23) is ensured by (5.8) and (5.21). □

Remark 5.1. It is worth to notice that the property (5.23) implies the strong convergence in $L^1(I)$ of $|\xi_\varepsilon|^q$ to $\left|\left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}|(y^*)'|^{p-2}(y^*)'\right|^q$, i.e.

$$c^q\left(\frac{\cdot}{\varepsilon}\right)|y_\varepsilon'|^p\to\left\langle c^{\frac{1}{1-p}}\right\rangle^{-p}|(y^*)'|^p\quad\text{in}\quad L^1(I).\tag{5.24}$$

Now, we are in a position to establish the main result of the article. Namely, we show that there exists a variational limit for OCP (3.1)–(3.5) as $\varepsilon\to 0$, this limit has a clearly defined structure

$$\inf_{(u,y)\in\Xi_{hom}}J_{hom}(u,y)\tag{5.25}$$

and it can be recovered in the form of some optimal control problem.

Theorem 5.1. *Let $I=(a,b)$ be a bounded interval and let $f\in\mathcal{M}(I)$ be a given Radon measure. Then, for OCP (3.1)–(3.5) there exist variational limit as $\varepsilon\to 0$, this limit can be represented as the constrained minimization problem (5.25), and this problem can be recovered in the following form:*

Minimize

$$J_{hom}(u,y)=\int_I |u(x)|\,dx+\left\langle dc^{\frac{p}{1-p}}\right\rangle\left\langle c^{\frac{1}{1-p}}\right\rangle^{-p}\int_I |y'(x)|^p\,dx \tag{5.26}$$

subject to the constrains

$$-\left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}\left(|y'|^{p-2}y'\right)'=u+f\quad on\quad I, \tag{5.27}$$

$$|y'(a)|^{p-2}y'(a)=0,\quad y(b)=0, \tag{5.28}$$

$$u\in U_\partial=\left\{v\in L^1(I):\int_I\Phi(|v(x)|)\,dx\le\gamma\right\}. \tag{5.29}$$

Proof. In order to prove this result, it enough to show that the conditions (d) and (dd) of Definition 2.3 hold true provided σ is the product of weak topologies of $L^1(I)$ and $W_0^{1,p}(I,b)$, and

$$\Xi_{hom}=\left\{(u,y)\;\middle|\;\begin{array}{c} u\in L^1(I),\quad y\in W_0^{1,p}(I,b),\\ u\in U_\partial,\\ \left\langle c^{\frac{1}{1-p}}\right\rangle^{1-p}\displaystyle\int_I |y'|^{p-2}y'\varphi'\,dx=\\ =\displaystyle\int_I u\varphi\,dx+\int_I\varphi\,df\quad\forall\,\varphi\in W_0^{1,p}(I,b).\end{array}\right\} \tag{5.30}$$

We begin with the property (d). Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{(u_n,y_n)\}_{n\in\mathbb{N}}$ be sequences such that

$$\begin{gathered}\varepsilon_n\to 0\quad\text{as}\quad n\to\infty,\\ u_n\rightharpoonup u\quad\text{in}\quad L^1(I),\\ y_n\rightharpoonup y\quad\text{in}\quad W_0^{1,p}(I,b),\\ \text{and}\quad(u_n,y_n)\in\Xi_{\varepsilon_n}\quad\forall\,n\in\mathbb{N}.\end{gathered}$$

Then $u\in U_\partial$ by the lower semicontinuity of the norm $\|\cdot\|_{L^1(I)}$ with respect to the weak convergence in $L^1(I)$ and the fact that Φ is a non-negative convex increasing function with property (3.6). As a result, we have

$$\int_I\Phi(|u|)\,dx\le\liminf_{n\to\infty}\int_I\Phi(|u_n|)\,dx\le\gamma.$$

So, in order to deduce the inclusion $(u,y)\in\Xi_{hom}$, it is enough to make use of Lemma 5.1 and representation (5.30). Moreover, in view of Remark 5.1, we have

$$c^q\left(\frac{\cdot}{\varepsilon_n}\right)|y_n'|^p\to\left\langle c^{\frac{1}{1-p}}\right\rangle^{-p}|y'|^p\quad\text{in}\quad L^1(I)$$

and

$$d\left(\frac{\cdot}{\varepsilon_n}\right)c^{-q}\left(\frac{\cdot}{\varepsilon_n}\right)\overset{*}{\rightharpoonup}\left\langle dc^{\frac{p}{1-p}}\right\rangle=\int_0^1 d(x)c^{\frac{p}{1-p}}(x)\,dx\quad\text{in}\quad L^\infty(I).$$

Hence,

$$\begin{aligned}\int_I d\left(\frac{x}{\varepsilon_n}\right)|y_n'|^p\,dx &= \int_I\left[d\left(\frac{x}{\varepsilon_n}\right)c^{-q}\left(\frac{x}{\varepsilon_n}\right)\right]c^q\left(\frac{x}{\varepsilon_n}\right)|y_n'|^p\,dx\\ &\to \left\langle dc^{\frac{p}{1-p}}\right\rangle\left\langle c^{\frac{1}{1-p}}\right\rangle^{-p}\int_I|y'|^p\,dx\end{aligned} \tag{5.31}$$

as the product of the strong and weak-∗ convergent sequences. Then the inequality

$$\liminf_{n\to\infty}\int_I|u_n(x)|\,dx \ge \int_I|u(x)|\,dx$$

(because of the weak convergence $u_n \rightharpoonup u$ in $L^1(I)$ and relation (5.31)) lead us to the conclusion:

$$\liminf_{n\to\infty} J_{\varepsilon_n}(u_n,y_n) \ge J_{hom}(u,y).$$

So, the property (d) is valid.

Now we check the property (dd). Let $(u,y)\in\Xi_{hom}$ be an arbitrary pair. We construct a Γ-realizing sequence $\{(u_\varepsilon,y_\varepsilon)\}_{\varepsilon>0}$ as follows: $u_\varepsilon\equiv u$ for all $\varepsilon>0$ and $y_\varepsilon = y_\varepsilon(u)$ is a unique weak solution to the boundary value problem (3.2)–(3.4) for a given control $u\in U_\partial$. Then $(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon$ for all $\varepsilon>0$ and the following estimate (see (4.22))

$$\sup_{\varepsilon>0}\|y_\varepsilon\|^{p-1}_{W_0^{1,p}(I,b)} \le \alpha^{-1}C\left[\|u\|_{L^1(I)}+\|f\|_{\mathcal{M}(I)}\right]$$

holds true. Hence, there is a subsequence of $\{y_\varepsilon\}_{\varepsilon>0}$ such that

$$y_{\varepsilon_k}\rightharpoonup y \quad\text{in}\quad W_0^{1,p}(I,b)\quad\text{as}\quad k\to\infty$$

and, by Lemma 5.1 $(u,y)\in\Xi_{hom}$. Since the homogenized boundary value problem (5.27)–(5.28) has a unique solution in $W_0^{1,p}(I,b)$ for a given control u, it follows that the same inference is valid for any convergent subsequence of $\{y_\varepsilon\}_{\varepsilon>0}$. Thus, we can suppose that there exists an element $y\in W_0^{1,p}(I,b)$ such that

$$y_\varepsilon\rightharpoonup y \quad\text{in}\quad W_0^{1,p}(I,b)\quad\text{as}\quad \varepsilon\to 0$$

and, hence, $(u,y)\in\Xi_{hom}$ by Lemma 5.1. It remains to notice that

$$\begin{aligned}\limsup_{\varepsilon\to0} J_\varepsilon(u_\varepsilon,y_\varepsilon) &= \lim_{\varepsilon\to0} J_\varepsilon(u,y_\varepsilon) = \lim_{\varepsilon\to0}\int_I d\left(\frac{x}{\varepsilon}\right)|y_\varepsilon'|^p\,dx + \int_I|u(x)|\,dx\\ &\overset{\text{by (5.31)}}{=} \left\langle dc^{\frac{p}{1-p}}\right\rangle\left\langle c^{\frac{1}{1-p}}\right\rangle^{-p}\|y\|^p_{W_0^{1,p}(I,b)}+\|u\|_{L^1(I)} = J_{hom}(u,y).\end{aligned}$$

This concludes the proof. □

Our next intension is to show that the homogenized OCP (5.26)–(5.29) possesses the fine variational properties (2.12). By analogy with Theorem 4.4, it can be proved that the OCP (5.26)–(5.29) has a non-empty set of solutions, i.e. there exists at least one pair $(u^0,y^0)\in\Xi_{hom}$ such that

$$J_{hom}(u^0,y^0) = \inf_{(u,y)\in\Xi_{hom}} J_{hom}(u,y).$$

Theorem 5.2. *Let* $\left\{(u^0_\varepsilon, y^0_\varepsilon)\right\}_{\varepsilon>0}$ *be a sequence of optimal pairs to the problems (3.1)–(3.5). Then this sequence is relatively weakly compact in* $L^1(I)\times W_0^{1,p}(I,b)$ *and for each weakly converging subsequence* $\{(u_{\varepsilon_k}, y_{\varepsilon_k})\}_{k\in\mathbb{N}}$ *in* $L^1(I)\times W_0^{1,p}(I,b)$ *there is a pair* (u^0,y^0) *such that* $(u^0,y^0)\in \Xi_{hom}$ *and*

$$\lim_{k\to\infty} J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k}) = \lim_{k\to\infty} \inf_{(u,y)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(u,y) = J_{\text{hom}}(u^0,y^0) = \inf_{(u,y)\in\Xi_{\varepsilon_k}} J_{hom}(u,y). \tag{5.32}$$

Proof. Due to the estimate (4.23) we see that the sequence of optimal pair is uniformly bounded in $L^1(I)\times W_0^{1,p}(I)$ and $\{u^0_\varepsilon\}_{\varepsilon>0}$ lives in the equi-integrable set U_∂. So, by Dunford-Pettis criterion, we may extract a subsequence $\{(u^0_{\varepsilon_k}, y^0_{\varepsilon_k})\}_k$ weakly converging in $L^1(I)\times W_0^{1,p}(I,b)$ to some pair (u^*,y^*). Then, by Lemma 5.1, this pair belongs to the set Ξ_{hom}. Moreover, taking the property (d) of Definition 2.3 into account, we have

$$\begin{aligned}\lim_{k\to\infty}\inf \min_{(u,y)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(u,y) &= \lim_{k\to\infty}\inf J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k})\\ &\ge J_{hom}(u^*,y^*) \ge \min_{(u,y)\in\Xi_{hom}} J_{hom}(u,y) = J_{hom}(u^0,y^0),\end{aligned} \tag{5.33}$$

where (u^0,y^0) is an optimal pair to the problem (5.26)–(5.29).

On the other hand, since $(u^0,y^0)\in\Xi_{hom}$, it follows that there exists a Γ-realizing sequence (see property (dd)) $\{(u_\varepsilon,y_\varepsilon)\}_{\varepsilon>0}$ weakly converging to (u^0,y^0) in $L^1(I)\times W_0^{1,p}(I,b)$ such that

$$(u_\varepsilon, y_\varepsilon)\in\Xi_\varepsilon, \quad \forall\,\varepsilon>0$$

and

$$J_{hom}(u^0,y^0)\ge \lim_{\varepsilon\to 0}\sup J_\varepsilon(u_\varepsilon,y_\varepsilon).$$

Consequently,

$$\begin{aligned}\min_{(u,y)\in\Xi_{hom}} J_{hom}(u,y) = J_{hom}(u^0,y^0) &\ge \lim_{k\to\infty}\sup J_\varepsilon(u_\varepsilon,y_\varepsilon)\\ &\ge \lim_{\varepsilon\to0}\sup \min_{(u,y)\in\Xi_\varepsilon} J_\varepsilon(u,y) \ge \lim_{k\to\infty}\sup \min_{(u,y)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(u,y)\\ &= \lim_{k\to\infty}\sup J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k}).\end{aligned} \tag{5.34}$$

Hence, by (5.33), we get

$$\lim_{k\to\infty}\inf J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \ge \lim_{k\to\infty}\sup J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k}).$$

As a result, combining (5.33) and (5.34), we come to the following conclusion:

$$J_{hom}(u^*,y^*) = J_{hom}(u^0,y^0) = \min_{(u,y)\in\Xi_{hom}} J_{hom}(u,y),$$

and as a consequence

$$J_{hom}(u^*,y^*) = \lim_{k\to\infty}\min_{(u,y)\in\Xi_{\varepsilon_k}} J_{\varepsilon_k}(u^0_{\varepsilon_k}, y^0_{\varepsilon_k}).$$

The proof is complete. □

References

1. *L. Ambrosio, N. Fusco, D. Pallava*, *Functions of Bounded Variation and Free Discontinuous Problems*, Oxford, New York, 2000.
2. *H. Attouch*, *Variational Convergence for Functional and Operators*, *Appl. Math. Ser.*, Pitman, Boston-London, 1984.
3. *H. Brezis*, *Functional Analysis, Sobolev Spaces, and Partial Differential Equtions*, Springer, New York, 2010.
4. *G. Buttazzo, G. Dal Maso*, *Γ-convergence and optimal control problems*, *J. Optim. Theory Appl.*, **32**(1982), 385–407.
5. *C. Conca, A. Osses, J. Saint Jean Paulin*, *A semilinear control problem involving in homogenization*, *Electr.J. Diff.Equ.*, 2001, No. 6, 109–122.
6. *I. Ekeland, R. Temam*, *Convex Analysis and Variational Problems*, North Holland, Amsterdam, 1976.
7. *L.C. Evans*, *Weak convergence methods for nonlinear PDEs*, Vol. 74 of CBMS Pogional Conference Series in Mathematics, Washington, DC, 1990.
8. *A.V. Fursikov*, Optimal Control of Distributed Systems Theory and Applications, AMS, Providence, Rhode Island, 2000.
9. *V.V. Jhikov, S.M. Kozlov, O.A. Oleinik*, *omogenization of Differential Operators and Integral Functionals*, Springer-Verlay, Berlin, 1994.
10. S. Kesavan, J. Saint Jean Paulin, *Optimal control on perforated domains*, *J. Math. Anal. Appl.*, **229**(1999), 563-586.
11. *P.I. Kogut, G. Leugering*, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains. Approximation and Asymptotic Analysis*, Series: Systems and Control, Birkhäuser Verlag, Boston, 2011.
12. *J. Saint Jean Paulin, H. Zoubairi*, *Optimal control and "strange term" for the Stokes problem in perforated domains*, Portugaliac Mathematica, **59**(2002), No. 2, 161–178.
13. *T. Roubiček*, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, New York, 2010.
14. *T. Roubiček*, *Relaxation in Optimization Theory and Variational Calculus*, Walter de Gruyter, Berlin, New York, 1997.

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