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OPTIMAL L^2-CONTROL PROBLEM IN COEFFICIENTS FOR A LINEAR ELLIPTIC EQUATION. I. EXISTENCE RESULT

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Abstract. In this paper we study an optimal control problem (OCP) associated to a linear elliptic equation on a bounded domain Ω. The matrix-valued coefficients A of such systems is our control in Ω and will be taken in $L^2(\Omega;\mathbb{R}^{N\times N})$ which in particular may comprises the case of unboundedness. Concerning the boundary value problems associated to the equations of this type, one may exhibit non-uniqueness of weak solutions— namely, approximable solutions as well as another type of weak solutions that can not be obtained through the L^∞-approximation of matrix A. Following the direct method in the calculus of variations, we show that the given OCP is well-possed and admits at least one solution. At the same time, optimal solutions to such problem may have a singular character in the above sense. In view of this we indicate two types of optimal solutions to the above problem: the so-called variational and non-variational solutions, and show that some of that optimal solutions can not be attainable through the L^∞-approximation of the original problem.

In this paper we deal with the following optimal control problem (OCP) in coefficients for a linear elliptic equation

$$\left\{\begin{array}{c} \text{Minimize } I(A,y)=\|y-y_d\|^2_{L^2(\Omega)}+\displaystyle\int_\Omega (\nabla y, A^{sym}\nabla y)_{\mathbb{R}^N}\,dx \\ \text{subject to the constraints} \\ -\operatorname{div}\left(A^{sym}\nabla y+A^{skew}\nabla y\right)=f \quad \text{in } \Omega, \\ y=0 \quad \text{on } \partial\Omega \\ A\in\mathfrak{A}_{ad}, \end{array}\right. \tag{1}$$

where $(A^{sym},\, A^{skew})\in L^\infty(\Omega;R^{N\times N})\times L^2(\Omega;R^{N\times N})$ are respectively the symmetric and antisymmetric part of the control A, $y_d\in L^2(\Omega)$ and $f\in H^{-1}(\Omega)$ are

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given distributions, and $\mathfrak{A}_{Ad}$ denotes the class of admissible controls which will be precised later.

The characteristic feature of this problem is the fact that the skew-symmetric part of matrix $A(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$ belongs to L^2-space (rather than L^∞). The existence, uniqueness, and variational properties of a weak solution to (1) are usually drastically different from the corresponding properties of solutions to the elliptic equations with L^∞-matrices in coefficients. In most of the cases, the situation can deeply change for the matrices A with unremovable singularity. Typically, in such cases, the above boundary value problem may admit infinitely many weak solutions which can be divided into two classes: approximable and non-approximable solutions (see [5], [17] and [19]). A function $y = y(A)$ is called an approximable solution to the boundary value problem in (1) if it can be attained by weak solutions to the similar boundary value problems with L^∞-approximated matrix A. However, this type does not exhaust all weak solutions to the above problem. There is another type of weak solutions, which cannot be approximated by weak solutions of such regularized problems. Usually, these are called non-variational (see [17] and [19]), singular (see [1], [9], [10] and [16]), pathological (see [14] and [15]) and others.

The aim of this work is to study the existence of optimal controls to the problem (1) and discuss the scheme of their approximations coming from the truncation procedure. Using the direct method in the Calculus of Variations, we show in Section 2 that the set of optimal pairs to the above problem is nonempty even if the corresponding boundary value problem is ill-possed. This problem is thus another example of the difference between well-posedness of optimal control problems for systems with distributed parameters and ill-posedness of boundary value problems for partial differential equations.

In Section 3 we show that there are two types of optimal solutions: the so-called variational and non-variational solutions (see [6] and [7]). By the first type we mean those optimal solutions which can be attained through the sequence of optimal solutions to regularized OCP for boundary value problem (1) with skew-symmetric parts of admissible controls $A_k^{skew} \in L^\infty(\Omega; \mathbb{S}^N)$ such that $A_k^{skew} \to A^{skew}$ strongly in $L^2(\Omega; \mathbb{S}^N)$. We give the sufficient conditions which guarantee that the solutions to OCP (1) have a variational character. The second type of optimal solutions is related to those which cannot be attained by the above procedure. We discuss in Section 4 the example of an optimal control problem in coefficients with non-variational optimal solution. This stimulates us to develop another approach of approximation for the considered optimal control problems. This approach will be the object of a subsequent paper.

Let us point out that situations where the non uniqueness of some problems occurs can lead to serious numerical difficulties. A good numerical scheme is assumed to construct a desired solution. In the context of this paper, due to limited capacities of computers, any kind of representation of matrices with L^2-coefficients will lead to a truncated version of it. Naturally, thus, any attempt to treat numerically some problem of the type (1), will probably force the algorithm to obtain an optimal variational solution. However, as it is shown in Section 4, where we give the example of an optimal control problem in coefficients with non-variational optimal solution, some optimal solutions cannot be numerically attained in this way. Examples of numerics are also postponed to a subsequent paper.

Of course, one may wonder if situations of non uniqueness and moreover of lack of procedure to obtain some uniqueness are relevant from the point of view of applications. Nematic liquid crystals, as modelled by harmonic maps between manifolds, can be, throughout this model, represented by minimizing harmonic maps or stationary harmonic maps, for which, both of them satisfy formally the same equation, but mathematically not. We refer to [3] for descriptions of this topic.

In order to indicate a practical situation more related to our problem, where the conditions $A^{skew} \in L^2(\Omega;\mathbb{S}^N_{skew})$ and $A^{skew} \notin L^\infty(\Omega;\mathbb{S}^N_{skew})$ appear in a natural way, we consider the following minimization problem for $p > 2$:

$$\text{Minimize} \quad \left\{ I_\Omega(\mathcal{A}, y) = \int_\Omega |y(x) - y_d(x)|^p \, dx \right\} \tag{2}$$

subject to the constraints

$$\mathcal{A} \in U_{ad} \subset L^\infty(\Omega;\mathbb{R}^{N\times N}), \quad y \in W_0^{1,p}(\Omega), \tag{3}$$

$$-\text{div}\,\left(\mathcal{A}[(\nabla y)^{p-2}]\nabla y\right) + |y|^{p-2} y = f \quad \text{in} \quad \Omega, \tag{4}$$

$$y = 0 \quad \text{on} \quad \partial\Omega, \tag{5}$$

where U_{ad} is a class of admissible controls and

$$[(\nabla y)^{p-2}] = \text{diag}\left\{ \left|\frac{\partial y}{\partial x_1}\right|^{p-2}, \left|\frac{\partial y}{\partial x_2}\right|^{p-2}, \dots, \left|\frac{\partial y}{\partial x_N}\right|^{p-2} \right\}.$$

In this case (see [11] and [12]) the first-order optimality conditions can be represented as follows

$$\int_\Omega \left((\mathcal{A} - \mathcal{A}_0)[(\nabla y_0)^{p-2}]\, \nabla y_0, \nabla \psi \right)_{\mathbb{R}^N} \, dx \ge 0, \quad \forall \mathcal{A} \in U_{ad},$$

$$\int_\Omega \left(\mathcal{A}_0[(\nabla y_0)^{p-2}]\nabla y_0, \nabla\varphi \right)_{\mathbb{R}^N} \, dx + \int_\Omega |y_0|^{p-2} y_0 \varphi \, dx = \\ = \langle f, \varphi \rangle_{W_0^{1,p}(\Omega)}, \ \forall \varphi \in W_0^{1,p}(\Omega),$$

$$(p-1)\int_\Omega \left([(\nabla y_0)^{p-2}]\, \mathcal{A}_0 \nabla\psi, \nabla\varphi \right)_{\mathbb{R}^N} dx + (p-1)\int_\Omega |y_0|^{p-2}\, \psi\, \varphi \, dx = \\ = p \int_\Omega |y_0 - y_d| \varphi \, dx, \ \forall \varphi \in C_0^\infty(\Omega),$$

where $(\mathcal{A}_0, y_0) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ is an optimal pair to the problem (2)–(5). As a result, for the adjoint system we have

$$A^{skew} = \frac{1}{2}\left([(\nabla y_0)^{p-2}]\, \mathcal{A}_0 - \mathcal{A}_0^t [(\nabla y_0)^{p-2}] \right).$$

Since

$$\|[(\nabla y_0)^{p-2}]\mathcal{A}_0\|^2_{L^2(\Omega;\mathbb{R}^{N\times N})} \le \|\mathcal{A}_0\|^2_{L^\infty(\Omega;\mathbb{R}^{N\times N})} \sum_{i=1}^N \int_\Omega \left| \frac{\partial y_0}{\partial x_i} \right|^{2p-4} dx$$

$$\le \|\mathcal{A}_0\|^2_{L^\infty(\Omega;\mathbb{R}^{N\times N})} |\Omega|^{\frac{4-p}{p}} \sum_{i=1}^N \left\| \frac{\partial y_0}{\partial x_i} \right\|^{(2p-4)/p}_{L^p(\Omega)} \le c_1 \|y_0\|^{(2p-4)/p}_{W_0^{1,p}(\Omega)} < +\infty,$$

it follows that $A^{skew} \in L^2(\Omega;\mathbb{S}^N_{skew})$ provided $p \in (2,4)$.

1. **Notation and Preliminaries.** Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. The spaces $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_0^\infty(\Omega)$. As usual by $H_0^1(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$-functions in the Sobolev space $H^1(\Omega)$, while $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$. The usual norm in $H_0^1(\Omega)$ will be replaced by the equivalent one defined by

$$\|y\| = \left(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\, dx \right)^{1/2}. \tag{6}$$

Let Γ be a part of the boundary $\partial\Omega$ with positive $(N-1)$-dimensional measures. We consider $C_0^\infty(\mathbb{R}^N;\Gamma) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N) \ : \ \varphi = 0 \text{ on } \Gamma\right\}$, and denote $H_0^1(\Omega;\Gamma)$ its closure with respect to the norm (6).

For any vector field $v \in L^2(\Omega;\mathbb{R}^N)$, the divergence of v is an element of the space $H^{-1}(\Omega)$ defined by the formula

$$\langle \operatorname{div} v, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = -\int_\Omega (v, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega), \tag{7}$$

where $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the scalar product in $\mathbb{R}^N$.

Symmetric and skew-symmetric matrices. Let $\mathbb{M}^N$ be the set of all $N \times N$ real matrices. We denote by $\mathbb{S}^N_{skew}$ the set of all skew-symmetric matrices $C = [c_{ij}]_{i,j=1}^N$, i.e., $c_{ij} = -c_{ji}$ and, hence, $c_{ii} = 0$. Therefore, the set $\mathbb{S}^N_{skew}$ can be identified with the Euclidean space $\mathbb{R}^{\frac{N(N-1)}{2}}$. Let $\mathbb{S}^N_{sym}$ be the set of all $N\times N$ symmetric matrices, which are obviously determined by $N(N+1)/2$ scalars. Then $\mathbb{M}^N = \mathbb{S}^N_{sym} \oplus \mathbb{S}^N_{skew}$. It is clear that for each matrix $B \in \mathbb{M}^N$, we have a unique representation

$$B = B^{sym} + B^{skew}, \tag{8}$$

where $B^{sym} := \frac{1}{2}\left(B + B^t\right) \in \mathbb{S}^N_{sym}$ and $B^{skew} := \frac{1}{2}\left(B - B^t\right) \in \mathbb{S}^N_{skew}$. In the sequel, we will always identify each matrix $B \in \mathbb{M}^N$ with its decomposition in the form (8).

Let $L^2(\Omega)^{\frac{N(N-1)}{2}} = L^2\big(\Omega;\mathbb{S}^N_{skew}\big)$ be the normed space of measurable square-integrable functions whose values are skew-symmetric matrices. By analogy, we can define the spaces $L^2(\Omega)^{\frac{N(N+1)}{2}} = L^2\big(\Omega;\mathbb{S}^N_{sym}\big)$ and $L^2(\Omega)^{N\times N} = L^2\big(\Omega;\mathbb{M}^N\big)$.

Let $A(x)$ and $B(x)$ be given matrices such that $A, B \in L^2(\Omega;\mathbb{S}^N_{skew})$. We say that these matrices are related by the binary relation $\preceq$ on the set $L^2(\Omega;\mathbb{S}^N_{skew})$ (in symbols, $A(x) \preceq B(x)$ a.e. in Ω), if

$$\mathcal{L}^N \left\{ \bigcup_{i=1}^N \bigcup_{j=i+1}^N \{x\in\Omega \ : \ |a_{ij}(x)| > |b_{ij}(x)|\} \right\} = 0. \tag{9}$$

Here, $\mathcal{L}^N(E)$ denotes the N-dimensional Lebesgue measure of $E \subset \mathbb{R}^N$ defined on the completed borelian σ-algebra.

We define the divergence $\operatorname{div} A$ of a matrix $A \in L^2\big(\Omega;\mathbb{M}^N\big)$ as a vector-valued distribution $d \in H^{-1}(\Omega;\mathbb{R}^N)$ by the following rule

$$\langle d_i, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = -\int_\Omega (\mathbf{a}_i^t, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega), \quad \forall\, i \in \{1,\dots,N\}, \tag{10}$$

where $\mathbf{a}_i$ stands for the i-th row of the matrix A.

For fixed two constants α and β such that $0 < \alpha \leq \beta < +\infty$, we define $\mathfrak{M}_\alpha^\beta(\Omega)$ as a set of all matrices $A = [a_{i\,j}]$ in $L^\infty(\Omega; \mathbb{S}^N_{sym})$ such that

$$\alpha I \leq A(x) \leq \beta I, \quad \text{a.e. in } \Omega. \tag{11}$$

Here, I is the identity matrix in $\mathbb{M}^N$, and (11) should be considered in the sense of quadratic forms defined by $(A\xi, \xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$.

Unbounded bilinear forms on $H_0^1(\Omega)$. Let $A \in L^2\big(\Omega; \mathbb{M}^N\big)$ be an arbitrary matrix. In view of the representation $A = A^{sym} + A^{skew}$, we can associate with A the form $\varphi(\cdot,\cdot)_A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ following the rule

$$\varphi(y, v)_A = \int_\Omega \big(\nabla v, A^{skew}(x)\nabla y\big)_{\mathbb{R}^N}\, dx, \quad \forall\, y, v \in H_0^1(\Omega).$$

It is easy to see that, in general, this form is unbounded on $H_0^1(\Omega)$, however, it is expected some kind of alternating and antisymmetric properties of it. In order to deal with these concepts, we introduce of the following set.

Definition 1.1. Let $A = A^{sym} + A^{skew} \in L^2\big(\Omega; \mathbb{M}^N\big)$ be a given matrix. We say that an element $y \in H_0^1(\Omega)$ belongs to the set $D(A)$ if

$$\left| \int_\Omega \big(\nabla \varphi, A^{skew}\nabla y\big)_{\mathbb{R}^N}\, dx \right| \leq c(y, A^{skew}) \left(\int_\Omega |\nabla \varphi|^2_{\mathbb{R}^N}\, dx \right)^{1/2}, \ \forall\, \varphi \in C_0^\infty(\Omega) \tag{12}$$

with some constant c depending only on y and A^{skew}.

Consequently, having set

$$[y, \varphi]_A = \int_\Omega \big(\nabla \varphi, A^{skew}(x)\nabla y\big)_{\mathbb{R}^N}\, dx, \quad \forall\, y \in D(A),\ \forall\, \varphi \in C_0^\infty(\Omega),$$

we see that $[y, \varphi]_A$ can be defined for all $\varphi \in H_0^1(\Omega)$ using (12) and the standard rule

$$[y, \varphi]_A = \lim_{\varepsilon \to 0} [y, \varphi_\varepsilon]_A, \tag{13}$$

where $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset C_0^\infty(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_0^1(\Omega)$. In this case the value $[v, v]_A$ is finite for every $v \in D(A)$, although the "integrand" $\big(\nabla v, A^{skew}\nabla v\big)_{\mathbb{R}^N}$ need not be integrable, in general.

Functions with bounded variations. Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$TV(f) := \int_\Omega |Df| = \sup \Big\{ \int_\Omega f\,(\nabla, \varphi)_{\mathbb{R}^N}\, dx \ : \ \varphi = (\varphi_1, \dots, \varphi_N) \in C_0^1(\Omega; \mathbb{R}^N),\ |\varphi(x)| \leq 1 \text{ for } x \in \Omega \Big\},$$

where $(\nabla, \varphi)_{\mathbb{R}^N} = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$.

Definition 1.2. A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $TV(f) < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation, i.e.

$$BV(\Omega) = \big\{ f \in L^1(\Omega) \ : \ TV(f) < +\infty \big\}.$$

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + TV(f)$, $BV(\Omega)$ is a Banach space. For our further analysis, we need the following properties of BV-functions (see [4]):

Proposition 1. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying condition $\sup_{k\in\mathbb{N}} TV(f_k) < +\infty$. Then*

$$f \in BV(\Omega) \quad \text{and} \quad TV(f) \le \liminf_{k\to\infty} TV(f_k)$$

and for every bounded sequence $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ there exists a subsequence, still denoted by f_k, and a function $f \in BV(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$.

Variational convergence of optimal control problems. Let $k \in \mathbb{N}$ be an arbitrary positive integer. Let $I_k : \mathbb{U}_k \times \mathbb{Y}_k \to \overline{\mathbb{R}}$ be a cost functional, $\mathbb{Y}_k$ be a space of states, and $\mathbb{U}_k$ be a space of controls. Let $\min\{I_k(u,y) : (u,y) \in \Xi_k\}$ be a parameterized OCP, where

$$\Xi_k \subset \{(u_k, y_k) \in \mathbb{U}_k \times \mathbb{Y}_k \ : \ u_k \in U_k,\ I_k(u_k, y_k) < +\infty\}$$

is a set of all admissible pairs linked by some state equation. Hereinafter we always associate to such OCP the corresponding constrained minimization problem:

$$(\mathrm{CMP_k}) \ : \qquad \left\langle \inf_{(u,y)\in\,\Xi_k} I_k(u,y) \right\rangle. \tag{14}$$

Since the constrained minimization problem (14) lives in variable spaces $\mathbb{U}_k \times \mathbb{Y}_k$, we assume that there exists a Banach space $\mathbb{U} \times \mathbb{Y}$ with respect to which a convergence in the scale of spaces $\{\mathbb{U}_k \times \mathbb{Y}_k\}_{k\in\mathbb{N}}$ is well defined (for the details, we refer to [8] and [18]). In the sequel, we use the following notation for this convergence $(u_k, y_k) \xrightarrow{\mu} (u,y)$ in $\mathbb{U}_k \times \mathbb{Y}_k$. Moreover, we assume that every bounded sequence in variable space $\mathbb{U}_k \times \mathbb{Y}_k$ is sequentially compact with respect to the μ-convergence.

In order to study the asymptotic behavior of a family of $(\mathrm{CMP_k})$, the passage to the limit in (14) as the parameter k tends to $+\infty$ has to be realized. The expression "passing to the limit" means that we have to find a kind of "limit cost functional" I and "limit set of constraints" Ξ with a clearly defined structure such that the limit object $\langle \inf_{(u,y)\in\Xi} I(u,y)\rangle$ may be interpreted as some OCP.

Following the scheme of the direct variational convergence [8], we adopt the following definition for the convergence of minimization problems in variable spaces.

Definition 1.3. A problem $\langle \inf_{(u,y)\in\Xi} I(u,y)\rangle$ is the variational μ-limit of the sequence (14) as $k \to \infty$, if and only if the following conditions are satisfied:

(d) If sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{(u_n, y_n)\}_{n\in\mathbb{N}}$ are such that $k_n \to \infty$ as $n \to \infty$, $(u_n, y_n) \in \Xi_{k_n}\ \forall n \in \mathbb{N}$, and $(u_n, y_n) \xrightarrow{\mu} (u,y)$ in $\mathbb{U}_{k_n} \times \mathbb{Y}_{k_n}$, then

$$(u,y) \in \Xi; \quad I(u,y) \le \liminf_{n\to\infty} I_{k_n}(u_n, y_n); \tag{15}$$

(dd) For every $(u,y) \in \Xi \subset \mathbb{U} \times \mathbb{Y}$, there are an integer $k^0 \in \mathbb{N}$ and a sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ (called a Γ-realizing sequence) such that

$$(u_k, y_k) \in \Xi_k,\ \forall\, k \ge k^0, \quad (u_k, y_k) \xrightarrow{\ \mu\ } (u,y) \ \text{ in } \ \mathbb{U}_k \times \mathbb{Y}_k, \tag{16}$$

$$I(u,y) \ge \limsup_{k\to\infty} I_k(u_k, y_k). \tag{17}$$

Then the following result takes place [8].

Theorem 1.4. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y)\in\Xi_0} I_0(u,y) \right\rangle \tag{18}$$

is the variational μ-limit of sequence (14) in the sense of Definition 1.3 and this problem has a nonempty set of solutions

$$\Xi_0^{opt} := \left\{ (u^0, y^0) \in \Xi_0 \ : \ I_0(u^0, y^0) = \inf_{(u,y)\in\Xi_0} I_0(u,y) \right\}.$$

For every $k \in \mathbb{N}$, let $(u_k^0, y_k^0) \in \Xi_k$ be a minimizer of I_k on the corresponding set Ξ_k. If the sequence $\{(u_k^0, y_k^0)\}_{k\in\mathbb{N}}$ is relatively compact with respect to the μ-convergence in variable spaces $\mathbb{U}_k \times \mathbb{Y}_k$, then there exists a pair $(u^0, y^0) \in \Xi_0^{opt}$ such that

$$(u_k^0, y_k^0) \xrightarrow{\ \mu\ } (u^0, y^0) \quad \text{in} \ \ \mathbb{U}_k \times \mathbb{Y}_k, \tag{19}$$

$$\inf_{(u,y)\in\Xi_0} I_0(u,y) = I_0\left(u^0, y^0\right) = \lim_{k\to\infty} I_k(u_k^0, y_k^0) = \lim_{k\to\infty} \inf_{(u_k,y_k)\in\Xi_k} I_k(u_k, y_k). \tag{20}$$

2. **Setting of the Optimal Control Problem.** Let $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$ be given distributions. The optimal control problem we consider in this paper is to minimize the discrepancy (tracking error) between y_d and a solution y of the Dirichlet boundary value problem for the linear elliptic equation

$$-\operatorname{div}\left(A(x)\nabla y\right) = f \quad \text{in} \ \ \Omega, \tag{21}$$

$$y = 0 \text{ on } \partial\Omega \tag{22}$$

by choosing an appropriate control $A \in L^2(\Omega; \mathbb{M}^N)$.

More precisely, we are concerned with the following OCP

$$\text{Minimize } I(A,y) = \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y, A^{sym}\nabla y)_{\mathbb{R}^N}\, dx \tag{23}$$

$$\text{subject to the constraints (21)–(22) with } A \in \mathfrak{A}_{ad} \subset L^2(\Omega; \mathbb{M}^N). \tag{24}$$

In order to define the class of admissible controls $\mathfrak{A}_{ad}$, we begin with some preliminaries. Let $A^* \in L^2(\Omega; \mathbb{S}^N_{skew})$ be a given nonzero matrix, let c be a given positive constant, and let Q be a nonempty convex compact subset of $L^2(\Omega; \mathbb{S}^N_{skew})$ such that the null matrix $A \equiv [0]$ belongs to Q. We introduce the following sets

$$U_{a,1} = \left\{ A = [a_{i\,j}] \in L^1(\Omega; \mathbb{S}^N_{sym}) \ \middle|\ TV(a_{ij}) \le c, \ 1 \le i \le j \le N \right\}, \tag{25}$$

$$U_{b,1} = \left\{ A = [a_{i\,j}] \in L^\infty(\Omega; \mathbb{S}^N_{sym}) \ \middle|\ A \in \mathfrak{M}_\alpha^\beta(\Omega) \right\}, \tag{26}$$

$$U_{a,2} = \left\{ A = [a_{i\,j}] \in L^2(\Omega; \mathbb{S}^N_{skew}) \ \middle|\ A(x) \preceq A^*(x) \text{ a.e. in } \ \Omega \right\}, \tag{27}$$

$$U_{b,2} = \left\{ A = [a_{i\,j}] \in L^2(\Omega; \mathbb{S}^N_{skew}) \ \middle|\ A \in Q \right\}. \tag{28}$$

It is worth to note that

$$\mathfrak{A}_{ad,1} := U_{a,1} \cap U_{b,1} \neq \emptyset \quad \text{ and } \quad \mathfrak{A}_{ad,2} := U_{a,2} \cap U_{b,2} \neq \emptyset.$$

The validity of these relations immediately follows from (25)–(26), definition of the binary relation $\preceq$, and properties of the matrix A^*.

Definition 2.1. We say that a matrix $A = A^{sym} + A^{skew}$ is an admissible control to the Dirichlet boundary value problem (21)–(22) (in symbols, $A \in \mathfrak{A}_{ad} \subset L^2(\Omega; \mathbb{M}^N)$) if $A^{sym} \in \mathfrak{A}_{ad,1}$ and $A^{skew} \in \mathfrak{A}_{ad,2}$.

For our further analysis, we use of the following results.

Proposition 2. *If $\{A_k^{sym}\}_{k\in\mathbb{N}} \subset \mathfrak{A}_{ad,1}$ and $A_k^{sym} \to A_0^{sym}$ in $L^1(\Omega; \mathbb{S}^N_{sym})$ as $k \to \infty$, then $A_0^{sym} \in \mathfrak{A}_{ad,1}$ and*

$$A_k^{sym} \to A_0^{sym} \quad \text{in} \ \ L^p(\Omega; \mathbb{S}^N_{sym}), \ \forall\, p \in [1, +\infty). \tag{29}$$

Proof. Since the sequence $\{A_k^{sym}\}_{k\in\mathbb{N}}$ converges strongly to A_0^{sym} in $L^1(\Omega;\mathbb{S}^N_{sym})$ and $A_k^{sym}\in\mathfrak{M}_\alpha^\beta(\Omega)$ for every $k\in\mathbb{N}$, it follows that $\alpha I\le A_0^{sym}\le\beta I$ a.e. in Ω. Hence, $A_0^{sym}\in U_{b,1}$. At the same time, following assertion (i) of Proposition 1, we have $TV(a_{ij})\le c$ for each entry of matrix A_0^{sym}. As a result, we conclude $A_0^{sym}\in U_{a,1}$, and, therefore, $A_0^{sym}\in\mathfrak{A}_{ad,1}$. Concerning the property (29), it immediately follows from the following estimate

$$\|A_k^{sym}-A_0^{sym}\|^p_{L^p(\Omega;\mathbb{S}^N_{sym})}=\int_\Omega\left(\max_{\substack{i,j=1,\dots,N\\ j\ge i}}|a^k_{ij}(x)-a^0_{ij}(x)|\right)^p dx$$

$$=\int_\Omega\left(\max_{\substack{i,j=1,\dots,N\\ j\ge i}}|(a^k_{ij}(x)-\alpha)-(a^0_{ij}(x)-\alpha)|\right)^{p-1}\max_{\substack{i,j=1,\dots,N\\ j\ge i}}|a^k_{ij}(x)-a^0_{ij}(x)|\,dx$$

$$\le 2^{p-1}(\beta-\alpha)^{p-1}\|A_k^{sym}-A_0^{sym}\|_{L^1(\Omega;\mathbb{S}^N_{sym})},\quad\forall\,p\in[1,+\infty).$$

□

Proposition 3. *$\mathfrak{A}_{ad,1}$ is a sequentially compact subset of $L^p(\Omega;\mathbb{S}^N_{sym})$ for every $p\in[1,+\infty)$.*

Proof. Let $\{A_k^{sym}\}_{k\in\mathbb{N}}$ be a sequence of $\mathfrak{A}_{ad,1}$. In view of definition of the set $U_{a,1}$, we see that $\{A_k^{sym}\}_{k\in\mathbb{N}}$ is a bounded sequence in $BV(\Omega;\mathbb{S}^N_{sym})$. Hence, to conclude the proof, it is enough to apply Propositions 1 and 2. □

Proposition 4. *The set $\mathfrak{A}_{ad}$ is nonempty, convex, and sequentially compact with respect to the strong topology of $L^2(\Omega;\mathbb{M}^N)$.*

Proof. Let $\{A_k=A_k^{sym}+A_k^{skew}\}_{k\in\mathbb{N}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls. Since

$$\mathfrak{A}_{ad}=\mathfrak{A}_{ad,1}\oplus\mathfrak{A}_{ad,2},\quad\mathfrak{A}_{ad,1}\subset BV(\Omega;\mathbb{S}^N_{sym}),$$
$$\mathfrak{A}_{ad,2}\subset U_{b,2},\quad\text{and}\quad U_{b,2}\text{ is a compact in }\ L^2(\Omega;\mathbb{S}^N_{skew}),$$

we may suppose that there exist matrices $A_0^{sym}\in BV(\Omega;\mathbb{S}^N_{sym})\cap L^\infty(\Omega;\mathbb{S}^N_{sym})$ (see Propositions 2–3) and $A_0^{skew}\in U_{b,2}\subset L^2(\Omega;\mathbb{S}^N_{skew})$ such that within a subsequence

$$A_k^{sym}\to A_0^{sym}\quad\text{in}\quad L^p(\Omega;\mathbb{S}^N_{sym}),\ \forall\,p\in[1,+\infty),\tag{30}$$
$$A_k^{sym}\overset{*}{\rightharpoonup}A_0^{sym}\quad\text{in}\quad L^\infty(\Omega;\mathbb{S}^N_{sym}),\tag{31}$$
$$A_k^{skew}\to A_0^{skew}\quad\text{in}\quad L^2(\Omega;\mathbb{S}^N_{skew}),\tag{32}$$
$$\text{and}\quad A_k^{skew}\to A_0^{skew}\quad\text{almost everywhere in}\ \ \Omega.\tag{33}$$

Combining these facts with (27) and the definition of the binary relation $\preceq$ (see (9)), we arrive at the conclusion: $A_0^{skew}\in U_{a,2}$, and hence

$$A_k:=A_k^{sym}+A_k^{skew}\to A_0^{sym}+A_0^{skew}=:A_0\quad\text{in}\quad L^2(\Omega;\mathbb{M}^N).$$

Thus, $A_0\in\mathfrak{A}_{ad}$. Since the convexity of $\mathfrak{A}_{ad}$ is obviously valid, this concludes the proof. □

The distinguishing feature of optimal control problem (23)–(24) is the fact that the matrix-valued control $A\in\mathfrak{A}_{ad}$ is merely measurable and belongs to the space $L^2\big(\Omega;\mathbb{M}^N\big)$ (rather than the space of bounded matrices $L^\infty\big(\Omega;\mathbb{M}^N\big)$). As we will see later, this entails a number of pathologies with respect to the standard properties of optimal control problems for the classical elliptic equations, even with 'a good' right-hand f. In particular, the unboundedness of the skew-symmetric part of matrix $A \in \mathfrak{A}_{ad}$ can have a reflection in non-uniqueness of weak solutions to the corresponding boundary value problem.

Definition 2.2. We say that a function $y = y(A, f)$ is a weak solution to boundary value problem (21)–(22) for a fixed control $A = A^{sym} + A^{skew} \in \mathfrak{A}_{ad}$ and a given distribution $f \in H^{-1}(\Omega)$, if $y \in H_0^1(\Omega)$ and the integral identity

$$\int_\Omega \left(\nabla \varphi, A^{sym}\nabla y + A^{skew}\nabla y\right)_{\mathbb{R}^N} dx = \langle f, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \tag{34}$$

holds true for any $\varphi \in C_0^\infty(\Omega)$.

Note that by Hölder's inequality this definition makes sense for any matrix $A \in L^2\big(\Omega;\mathbb{M}^N\big)$. At the same time, in view of Definition 1.1, the following result gives another motivation to introduce the set $D(A)$.

Proposition 5. *Let $y \in H_0^1(\Omega)$ be a weak solution to the boundary value problem (21)–(22) for a given control $A = A^{sym}+A^{skew} \in \mathfrak{A}_{ad}$ in the sense of Definition 2.2. Then $y \in D(A)$.*

Proof. In order to verify the validity of this assertion it is enough to rewrite the integral identity (34) in the form

$$[y, \varphi]_A = -\int_\Omega \left(A^{sym}\nabla y, \nabla \varphi\right)_{\mathbb{R}^N} dx + \langle f, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \tag{35}$$

and apply Hölder's inequality to the right-hand side of (35). As a result, we have

$$\begin{aligned}\Big|[y,\varphi]_A\Big| &\le \Big(\|A^{sym}\|_{L^\infty(\Omega;\mathbb{S}^N_{sym})}\|\nabla y\|_{L^2(\Omega;\mathbb{R}^N)} + \|f\|_{H^{-1}(\Omega)}\Big)\|\varphi\|_{H_0^1(\Omega)}\\ &\le \Big(\beta\|y\|_{H_0^1(\Omega)} + \|f\|_{H^{-1}(\Omega)}\Big)\|\varphi\|_{H_0^1(\Omega)}.\end{aligned}$$

□

Remark 1. Due to Proposition 5, Definition 2.2 can be reformulated as follows: y is a weak solution to the problem (21)–(22) for a given control $A = A^{sym} + A^{skew} \in \mathfrak{A}_{ad}$, if and only if $y \in D(A)$ and

$$\int_\Omega \left(A^{sym}\nabla y, \nabla \varphi\right)_{\mathbb{R}^N} dx + [y,\varphi]_A = \langle f, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \quad \forall\, \varphi \in H_0^1(\Omega). \tag{36}$$

Moreover, as immediately follows from (13) and (36), every weak solution $y \in D(A)$ to the problem (21)–(22) satisfies the energy equality

$$\int_\Omega \left(A^{sym}\nabla y, \nabla y\right)_{\mathbb{R}^N} dx + [y,y]_A = \langle f, y\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}. \tag{37}$$

It is well known that boundary value problem (21)–(22) is ill-posed, in general (see, for instance, [5], [14], [15], [17] and [19]). It means that there exists a matrix $A \in L^2\big(\Omega;\mathbb{M}^N\big)$ such that the corresponding state $y \in H_0^1(\Omega)$ may be not unique. It is clear that in this case, it would not be correct to write down $y = y(A, f)$. To avoid this situation, we adopt the following notion.

Definition 2.3. We say that (A, y) is an admissible pair to the OCP (23)–(24) if $A \in \mathfrak{A}_{ad} \subset L^2\big(\Omega;\mathbb{M}^N\big)$, $y \in D(A) \subset H_0^1(\Omega)$, and the pair (A, y) is related by the integral identity (36).

We denote by Ξ the set of all admissible pairs for the OCP (23)–(24). We define the topology τ on $\Xi \subset L^2\big(\Omega;\mathbb{M}^N\big) \times H^1_0(\Omega)$ as the product of the strong topology of $L^2\big(\Omega;\mathbb{M}^N\big)$ and the weak topology of $H^1_0(\Omega)$. We say that a pair $(A^0, y^0) \in L^2\big(\Omega;\mathbb{M}^N\big) \times D(A^0)$ is optimal for problem (23)–(24) if

$$(A^0, y^0) \in \Xi \quad\text{and}\quad I(A^0, y^0) = \inf_{(A,y)\in\,\Xi} I(A,y).$$

As follows from the definition of the form $[y,\varphi]_A$, the value $[y,y]_A$ may not of constant sign for all $y \in D(A)$. Hence, the energy equality (37) does not allow us to derive a reasonable a priory estimate in H^1_0-norm for the weak solutions. In spite of this, we show that the OCP (23)–(24) is well-posed. This problem is, thus, yet another example for the difference between well-posedness for optimal control problems for systems with distributed parameters and partial differential equations (see [8] for a discussion and further examples).

Theorem 2.4. *Assume that OCP (23)–(24) is regular, i.e. $\Xi \neq \emptyset$. Then, for each $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$, this problem admits at least one solution.*

Proof. Since the original problem is regular and the cost functional for the given problem is bounded below on Ξ, it follows that there exists a minimizing sequence $\{(A_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ such that

$$I(A_k, y_k) \xrightarrow[k\to\infty]{} I_{\min} \equiv \inf_{(A,y)\in\,\Xi} I(A,y) \geq 0.$$

Hence, $\sup_{k\in\mathbb{N}} I(A_k, y_k) \leq C$, where the constant C is independent of k. Since

$$\sup_{k\in\mathbb{N}} \|y_k\|^2_{H^1_0(\Omega)} \leq \alpha^{-1} \sup_{k\in\mathbb{N}} \int_\Omega (\nabla y_k, A^{sym}_k \nabla y_k)_{\mathbb{R}^N}\, dx \leq \alpha^{-1} \sup_{k\in\mathbb{N}} I(A_k, y_k) \leq \alpha^{-1} C,$$

in view of Proposition 4, it follows that passing to a subsequence if necessary, we may assume the existence of a pair $(A_0, y_0) \in \mathfrak{A}_{ad} \times H^1_0(\Omega)$ such that

$$A_k := A^{sym}_k + A^{skew}_k \to A^{sym}_0 + A^{skew}_0 =: A_0 \quad\text{in}\quad L^2(\Omega;\mathbb{M}^N), \tag{38}$$

$$A^{sym}_k \to A^{sym}_0 \quad\text{in}\quad L^p(\Omega;\mathbb{S}^N_{sym}),\ \forall\, p \in [1,+\infty), \tag{39}$$

$$A^{skew}_k \to A^{skew}_0 \quad\text{in}\quad L^2(\Omega;\mathbb{S}^N_{skew}), \tag{40}$$

$$y_k \rightharpoonup y_0 \quad\text{in}\quad H^1_0(\Omega), \quad I(A_0, y_0) < +\infty. \tag{41}$$

Since $(A_k, y_k) \in \Xi$ for every $k \in \mathbb{N}$, it follows that the integral identity

$$\int_\Omega \big(\nabla\varphi, A^{sym}_k \nabla y_k\big)_{\mathbb{R}^N}\, dx + \int_\Omega \big(\nabla\varphi, A^{skew}_k \nabla y_k\big)_{\mathbb{R}^N}\, dx = \langle f, \varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} \tag{42}$$

holds true for all $\varphi \in C^\infty_0(\Omega)$.

In order to pass to the limit in (42), we note that

$$\int_\Omega \big(\nabla\varphi, A^{skew}_k \nabla y_k\big)_{\mathbb{R}^N}\, dx = -\int_\Omega \big((A^{skew}_k - A^{skew}_0)\nabla\varphi, \nabla y_k\big)_{\mathbb{R}^N}\, dx - \int_\Omega \big(A^{skew}_0 \nabla\varphi, \nabla y_k\big)_{\mathbb{R}^N}\, dx = I_{1,k} + I_{2,k}$$

by the skew-symmetry property of A_k^{skew} and A_0^{skew}. Hence, in view of (40)–(41), we have

$$\lim_{k\to\infty} |I_{1,k}| \le \|\varphi\|_{C^1(\Omega)} \sup_{k\in\mathbb{N}} \|\nabla y_k\|_{L^2(\Omega;\mathbb{R}^N)} \lim_{k\to\infty} \left\|A_k^{skew} - A_0^{skew}\right\|_{L^2(\Omega;\mathbb{S}^N_{skew})} = 0,$$

$$\lim_{k\to\infty} I_{2,k} \overset{\text{by (41)}}{=} -\int_\Omega \left(A_0^{skew}\nabla\varphi, \nabla y_0\right)_{\mathbb{R}^N} dx = \int_\Omega \left(\nabla\varphi, A_0^{skew}\nabla y_0\right)_{\mathbb{R}^N} dx$$
$$\text{since } \; A_0^{skew}\nabla\varphi \in L^2(\Omega;\mathbb{R}^N) \quad \forall\,\varphi\in C_0^\infty(\Omega).$$

Having applied the same arguments to the first term in (42), as a result of the limit passage in (42), we finally obtain: the pair (A_0, y_0) is related by identity (34). Hence, $y_0 \in D(A_0)$ by Proposition 5. Thus, (A_0, y_0) is an admissible pair to problem (23)–(24).

It remains to show that (A_0, y_0) is an optimal pair. Indeed, in view of the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, one gets

$$\begin{aligned} I_{\min} &= \lim_{k\to\infty} I(A_k, y_k) = \lim_{k\to\infty} \left[\|y_k - y_d\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N} \, dx \right] \\ &= \|y_0 - y_d\|^2_{L^2(\Omega)} + \lim_{k\to\infty} \int_\Omega \left\| (A_k^{sym})^{1/2} \nabla y_k \right\|^2_{\mathbb{R}^N} \, dx. \end{aligned}$$

At the same time, due to (39), we obviously have $(A_k^{sym})^{1/2} \to (A_0^{sym})^{1/2}$ in $L^2(\Omega;\mathbb{S}^N_{sym})$. Hence, taking into account the condition (41), we get $(A_k^{sym})^{1/2}\nabla y_k \rightharpoonup (A_0^{sym})^{1/2}\nabla y_0$ in $L^2(\Omega;\mathbb{R}^N)$. So, using the lower semicontinuity of $\|\cdot\|_{L^2(\Omega;\mathbb{R}^N)}$ with respect to the the weak topology of $L^2(\Omega;\mathbb{R}^N)$, we finally obtain

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \left\| (A_k^{sym})^{1/2} \nabla y_k \right\|^2_{\mathbb{R}^N} \, dx &\ge \int_\Omega \left\| (A_0^{sym})^{1/2} \nabla y_0 \right\|^2_{\mathbb{R}^N} \, dx \\ &= \int_\Omega \left(\nabla y_0, A_0^{sym}\nabla y_0\right)_{\mathbb{R}^N} \, dx. \end{aligned} \tag{43}$$

Thus, $I_{\min} \ge \|y_0 - y_d\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y_0, A_0^{sym}\nabla y_0\right)_{\mathbb{R}^N} \, dx = I(A_0, y_0)$, and hence, the pair (A_0, y_0) is optimal for problem (23)–(24). The proof is complete. □

3. **On variational solutions to OCP (23)–(24) and their approximation.** The question we are going to discuss in this section is about some pathological properties that can be inherited by optimal pair to the problem (23)–(24) and other unexpected surprises concerning the approximation of the original OCP and its solutions.

To begin with, we show that the main assumption on the regularity property of OCP (23)–(24) in Theorem 2.4 can be eliminated due to the approximation approach. It is clear that the condition $A^* \in L^2(\Omega;\mathbb{S}^N_{skew})$ ensures the existence of the sequence of skew-symmetric matrices $\{A_k^*\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{skew})$ such that $A_k^* \to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$. This leads us to the idea to consider the following sequence of constrained minimization problems associated with matrices A_k^*

$$\left\{ \left\langle \inf_{(A,y)\in\Xi_k} I_k(A,y) \right\rangle, \quad k\to\infty \right\}. \tag{44}$$

Here,

$$I_k(A,y) := I(A,y) \quad \forall\,(A,y) \in L^2(\Omega;\mathbb{M}^N)\times H_0^1(\Omega), \quad \forall\, k\in\mathbb{N}, \tag{45}$$

$$\Xi_k = \left\{ (A,y) \;\middle|\; \begin{array}{c} -\operatorname{div}\left(A^{sym}\nabla y + A^{skew}\nabla y\right) = f \quad \text{in} \ \ \Omega,\\ y=0 \text{ on } \partial\Omega,\\ A = A^{sym}+A^{skew} \in \mathfrak{A}^k_{ad} = \mathfrak{A}_{ad,1}\oplus\mathfrak{A}^k_{ad,2},\ y\in H_0^1(\Omega),\\ \mathfrak{A}^k_{ad,2} = U_{a,2}\cap U^k_{b,2},\\ U^k_{b,2} = \left\{ B=[b_{i\,j}]\in L^2(\Omega;\mathbb{S}^N_{skew}) \;\middle|\; B(x)\preceq A^*_k(x) \text{ a.e. in } \ \Omega\right\}. \end{array} \right\} \tag{46}$$

Before we will provide an accurate analysis of the optimal control problems (44), we make use of the following auxiliary result.

Lemma 3.1. *The sequence of sets* $\left\{U^k_{b,2}\right\}_{k\in\mathbb{N}}$ *converges to* $U_{b,2}$ *as* $k\to\infty$ *in the sense of Kuratowski with respect to the strong topology of* $L^2(\Omega;\mathbb{S}^N_{skew})$.

Proof. We recall here that a sequence $\left\{U^k_{b,2}\right\}_{k\in\mathbb{N}}$ of the subsets of $L^2(\Omega;\mathbb{S}^N_{skew})$ is said to be convergent to a closed set S in the sense of Kuratowski with respect to the strong topology of $L^2(\Omega;\mathbb{S}^N_{skew})$, if the following two properties hold:

(K_1) for every $B\in S$, there exists a sequence of matrices $\left\{B_k\in U^k_{b,2}\right\}_{k\in\mathbb{N}}$ such that $B_k\to B$ in $L^2(\Omega;\mathbb{S}^N_{skew})$ as $k\to\infty$;

(K_2) if $\{k_n\}_{n\in\mathbb{N}}$ is a sequence of indices converging to $+\infty$, $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of skew-symmetric matrices such that $B_n\in U^{k_n}_{b,2}$ for each $n\in\mathbb{N}$, and $\{B_n\}_{n\in\mathbb{N}}$ strongly converges in $L^2(\Omega;\mathbb{S}^N_{skew})$ to some matrix B, then $B\in S$.

For the details we refer to [8].

In order to show that $S=U_{b,2}$, we begin with the verification of (K_2)-item. Let $\{k_n\}_{n\in\mathbb{N}}$ be a given sequence of indices such that $k_n\to\infty$, and let $\left\{B_n\in U^{k_n}_{b,2}\right\}_{n\in\mathbb{N}}$ be a sequence satisfying the property $B_n\to B$ in $L^2(\Omega;\mathbb{S}^N_{skew})$ and, hence, $B_n(x)\to B(x)$ almost everywhere in Ω as $n\to\infty$. By definition of $U^k_{b,2}$, we have

$$B_n(x)\preceq A^*_{k_n}(x) \quad \text{a.e. in } \ \Omega, \tag{47}$$

where $A^*_k\to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$. Taking into account the fact that the binary relation $\preceq$ is reflexive and transitive, we can pass to the limit in relation (47) as $n\to\infty$ (in the sense of almost everywhere) and get $B(x)\preceq A^*(x)$ almost everywhere in Ω, hence, $B\in U_{b,2}$.

It remains to verify the (K_1)-item. To this end, we fix an arbitrary skew-symmetric matrix $B\in U_{b,2}$ and make use of the concept of the Lebesgue set $\mathfrak{W}(B)$. We say that $x\in\Omega$ is of the Lebesgue set $\mathfrak{W}(B)$ for the matrix $B\in U_{b,2}\subset L^2(\Omega;\mathbb{S}^N_{skew})$, if x is a Lebesgue point of B. In other words, at this point matrix $B(x)$ must be approximately continuous and, hence, it does not oscillate too much, in an average sense. It is well known that almost each point in Ω is a Lebesgue point for an absolutely locally integrable function [4]. Hence, $\mathcal{L}^N(\Omega\setminus\mathfrak{W}(B))=0$. Moreover, since $A^*_k\in L^\infty(\Omega;\mathbb{S}^N_{skew})$, it follows that any point of approximate continuity of A^*_k is its Lebesgue point [4]. As a result, we construct a strong convergent sequence $\left\{B_k \in U_{b,2}^k\right\}_{k\in\mathbb{N}}$ to $B \in U_{b,2}$ as follows: $B_k(x) = [b_{ij}^k(x)]_{i,j=1}^N$, where

$$b_{ij}^k(x) = \begin{cases} b_{ij}(x), & \text{if } \ |b_{ij}(x)| \le \left|a_{ij}^{*,k}(x)\right| \ \text{ and } \ x \in \mathfrak{W}(B), \\ a_{ij}^{*,k}(x), & \text{if } \ |b_{ij}(x)| > \left|a_{ij}^{*,k}(x)\right| \ \text{ and } \ x \in \mathfrak{W}(B), \\ 0, & \text{otherwise}, \end{cases} \tag{48}$$

for all $i, j \in \{1, \dots, N\}$ and $k \in \mathbb{N}$.

Since the strong convergence $A_k^* \to A^*$ in $L^2(\Omega; \mathbb{S}^N_{skew})$ implies (up to a subsequence) the pointwise convergence a.e. in Ω, and $B \preceq A^*$, it follows that the sequence $\left\{B_k \in U_{b,2}^k\right\}_{k\in\mathbb{N}}$, given by (48), satisfies all properties of (K_1)-item. This concludes the proof. □

We are now in a position to study the optimal control problems (44).

Theorem 3.2. *Let $y_d \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$ be given distributions. Then for every $k \in \mathbb{N}$ there exists a minimizer $(A_k^0, y_k^0) \in \Xi_k$ to the corresponding minimization problems (44) such that the sequence of pairs $\left\{(A_k^0, y_k^0) \in \Xi_k\right\}_{k\in\mathbb{N}}$ is relatively compact with respect to the τ-topology on $L^2(\Omega; \mathbb{M}^N) \times H_0^1(\Omega)$ and each of its τ-cluster pairs $(\widehat{A}, \widehat{y})$ possesses the properties:*

$$(\widehat{A}, \widehat{y}) \in \Xi, \quad [\widehat{y}, \widehat{y}]_{\widehat{A}} \ge 0. \tag{49}$$

Proof. To begin with, we show that the sequence of minimal values for the problems (44) is uniformly bounded, i.e.

$$\sup_{k\in\mathbb{N}} \inf_{(A,y)\in\Xi_k} I_k(u, y) \le C \quad \text{for some } \ C > 0. \tag{50}$$

Indeed, for each $k \in \mathbb{N}$, we obviously have $\mathfrak{A}^k_{ad,2} \ne \emptyset$ and $\mathfrak{A}^k_{ad,2} \subset L^\infty(\Omega; \mathbb{S}^N_{skew})$. Hence, for any admissible control $A_k = A_k^{sym} + A_k^{skew} \in \mathfrak{A}^k_{ad}$, we can claim that $A_k^{skew} \in L^\infty(\Omega; \mathbb{S}^N_{skew})$ and, therefore, the corresponding bilinear form

$$[y, \varphi]_{A_k} = \int_\Omega \left(\nabla\varphi, A_k^{skew}\nabla y\right)_{\mathbb{R}^N} dx$$

is bounded on $H_0^1(\Omega)$ and satisfies the identity

$$\int_\Omega \left(\nabla\varphi, A_k^{skew}\nabla y\right)_{\mathbb{R}^N} dx = -\int_\Omega \left(\nabla y, A_k^{skew}\nabla\varphi\right)_{\mathbb{R}^N} dx.$$

Therefore,

$$\int_\Omega \left(\nabla v, A_k^{skew}(x)\nabla v\right)_{\mathbb{R}^N} dx = 0 \quad \forall\, v \in H_0^1(\Omega) \tag{51}$$

and, hence, the boundary value problem (46) has a unique solution $y_k \in H_0^1(\Omega)$ for each $A_k \in \mathfrak{A}^k_{ad} \subset L^\infty(\Omega; \mathbb{M}^N)$ by the Lax-Milgram lemma. As obvious consequence of this observation and the property of τ-lower semicontinuity of the cost functional I_k, we conclude (see for comparison Theorem 2.4): the corresponding minimization problem (44) admits at least one solution [13]

$$I_k(A_k^0, y_k^0) = \inf_{(A,y)\in\Xi_k} I_k(A, y), \quad (A_k^0, y_k^0) \in \Xi_k.$$

Moreover, having fixed a control $A_k \in \mathfrak{A}_{ad}^k$, condition (51) implies the fulfilment of the following identities for every $k \in \mathbb{N}$

$$\int_\Omega \left(\nabla\varphi, A_k^{sym}\nabla y_k + A_k^{skew}\nabla y_k\right)_{\mathbb{R}^N} dx = \langle f, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}, \ \forall\,\varphi \in C_0^\infty(\Omega), \quad (52)$$

$$\int_\Omega \left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N}\, dx = \langle f, y_k\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}, \quad (53)$$

where $y_k = y_k(A_k, f) \in H_0^1(\Omega)$ are the corresponding solutions to the boundary value problems (46). Hence, the sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and due to the a priori estimate

$$\|y_k\|_{H_0^1(\Omega)} \le \alpha^{-1}\|f\|_{H^{-1}(\Omega)}, \quad (54)$$

we arrive at the relation

$$\begin{aligned} I_k(A_k^0, y_k^0) &= \inf_{(A,y)\in\Xi_k} I_k(A,y) \le I_k(A_k, y_k) \\ &\le 2\|y_d\|^2_{L^2(\Omega)} + 2\|y_k\|^2_{L^2(\Omega)} + \beta\|y_k\|^2_{H_0^1(\Omega)} \quad (55)\\ &\le 2\|y_d\|^2_{L^2(\Omega)} + (2C_1+\beta)\alpha^{-2}\|f\|^2_{H^{-1}(\Omega)} \le C \quad \forall\, k \in \mathbb{N}. \quad (56)\end{aligned}$$

Thus, (50) holds true and it implies that $\sup_{k\in\mathbb{N}}\|y_k^0\|^2_{H_0^1(\Omega)} < +\infty$. So, we can suppose that the sequence of optimal states $\left\{y_k^0\right\}_{k\in\mathbb{N}}$ is weakly convergent: $y_k^0 \rightharpoonup \widehat{y}$ in $H_0^1(\Omega)$. In the meantime, due to the definition of the sets $\mathfrak{A}_{ad}^k$, it is easy to see that the corresponding sequence of optimal controls $\left\{A_k^0\right\}_{k\in\mathbb{N}}$ belongs to $U_{a,1}\oplus U_{b,2}$. Hence, applying the arguments of the proof of Proposition 4, we get: there exists a matrix $\widehat{A} \in U_{a,1}\oplus U_{b,2}$ such that

$$A_k^0 := A_k^{0,sym} + A_k^{0,skew} \to \widehat{A}^{sym} + \widehat{A}^{skew} =: \widehat{A} \ \text{ in } \ L^2(\Omega;\mathbb{M}^N), \quad (57)$$

$$A_k^{0,sym} \to \widehat{A}^{sym} \quad \text{in } \ L^p(\Omega;\mathbb{S}^N_{sym}),\ \forall\, p\in[1,+\infty), \quad (58)$$

$$A_k^{0,sym} \overset{*}{\rightharpoonup} \widehat{A}^{sym} \quad \text{in } \ L^\infty(\Omega;\mathbb{S}^N_{sym}), \quad (59)$$

$$A_k^{0,skew} \to \widehat{A}^{skew} \quad \text{in } \ L^2(\Omega;\mathbb{S}^N_{skew}). \quad (60)$$

Therefore, in view of Lemma 3.1, we can conclude: $\widehat{A} \in \mathfrak{A}_{ad}$. As a result, summing up the above properties of the sequences $\left\{y_k^0\right\}_{k\in\mathbb{N}}$ and $\left\{A_k^0\right\}_{k\in\mathbb{N}}$, we obtain:

$$(A_k^0, y_k^0) \overset{\tau}{\to} (\widehat{A}, \widehat{y}) \ \text{ and } \ (\widehat{A}, \widehat{y}) \in \Xi.$$

It remains to prove the properties (49). To do so, we note that due to the strong convergence $A_k^0 \to \widehat{A}$ in $L^2(\Omega;\mathbb{M}^N)$, we get

$$\begin{aligned} &\left|\int_\Omega \left(\nabla\varphi, \widehat{A}\nabla\widehat{y} - A_k^0\nabla y_k^0\right)_{\mathbb{R}^N} dx\right| \\ &\le \int_\Omega \|A_k^0 - \widehat{A}\|_{\mathbb{M}^N}\|\nabla y_k^0\|_{\mathbb{R}^N}\|\nabla\varphi\|_{\mathbb{R}^N}\,dx + \left|\int_\Omega \left(\widehat{A}\nabla\varphi, \nabla\widehat{y} - \nabla y_k^0\right)_{\mathbb{R}^N} dx\right| \\ &\le \|\varphi\|_{C^1(\overline{\Omega})}\sup_{k\in\mathbb{N}}\|y_k^0\|_{H_0^1(\Omega)}\|A_k^0 - \widehat{A}\|_{L^2(\Omega;\mathbb{M}^N)} \\ &+ \left|\int_\Omega \left(\widehat{A}\nabla\varphi, \nabla\widehat{y}\right)_{\mathbb{R}^N} dx - \int_\Omega \left(\widehat{A}\nabla\varphi, \nabla y_k^0\right)_{\mathbb{R}^N} dx\right| \longrightarrow 0 \quad \text{as } k\to\infty\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. Hence, $A_k^0\nabla y_k^0 \overset{*}{\rightharpoonup} \widehat{A}\nabla\widehat{y}$ in $L^1(\Omega;\mathbb{R}^N)$. It means that we can pass to the limit in integral identity (52) with $A = A_k^0$. As a result, we have: the pair $(\widehat{A}, \widehat{y})$ is related by the integral identity (34), therefore, $\widehat{y}$ is a weak solution to the original boundary value problem (21)–(22) under $A = \widehat{A}$. Thus, $\widehat{y} \in D(\widehat{A})$ and, hence, $(\widehat{A}, \widehat{y}\,) \in \Xi$.

In order to proof the property $(49)_2$, we pass to the limit in the energy equality (53) using the lower semicontinuity of the norm $\|\cdot\|_{H^1(\Omega)}$ with respect to the weak convergence $\nabla y_k^0 \rightharpoonup \nabla\widehat{y}$ in $L^2(\Omega;\mathbb{R}^N)$ and the property (59). To do so, we note that due to the inclusion $\widehat{A}^{sym} \in \mathfrak{A}_{ab,1}$, we have $A \in \mathfrak{M}_\alpha^\beta(\Omega)$. Hence, the norms $\|y\|_{H^1(\Omega)}$ and $|||y||| := \left(\int_\Omega \left(\nabla y, \widehat{A}^{sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx\right)^{1/2}$ are equivalent in $H_0^1(\Omega)$. As a result, we obtain

$$\begin{aligned}\langle f, \widehat{y}\,\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} &= \lim_{k\to\infty}\int_\Omega \left(\nabla y_k^0, (A_k^{0,sym} - \widehat{A}^{sym})\nabla y_k^0\right)_{\mathbb{R}^N}\,dx \\ &+ \lim_{k\to\infty}\int_\Omega \left(\nabla y_k^0, \widehat{A}^{sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx \overset{\text{by (58)}}{=} \lim_{k\to\infty}\int_\Omega \left(\nabla y_k^0, \widehat{A}^{sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx \\ &\overset{\text{by (11)}}{\geq} \int_\Omega \left(\nabla\widehat{y}, \widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx. \qquad (61)\end{aligned}$$

Thus, the desired inequality $(49)_2$ obviously follows from (37) and (61). The proof is complete. □

Remark 2. As Theorem 3.2 proves, for any approximation $\{A_k^*\}_{k\in\mathbb{N}}$ of the matrix $A^* \in L^2\big(\Omega;\mathbb{S}^N_{skew}\big)$ with properties $\{A_k^*\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{skew})$ and $A_k^* \to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$, optimal solutions to the regularized OCPs (44)–(46) always lead us in the limit to some admissible (but not optimal in general) solution $(\widehat{A}, \widehat{y}\,)$ of the original OCP (23)–(24). Moreover, this limit pair can depend on the choice of the approximative sequence $\{A_k^*\}_{k\in\mathbb{N}}$. It is reasonably to call such pairs attainable admissible solutions to OCP. However, the entire structure of the set of all attainable solutions remain unclear; for instance, it is not known whether this set is convex and closed in Ξ. It is also unknown whether the optimal solution to OCP (23)–(24) is attainable. At the end of this section we give the conditions on the matrix A^* which ensures the attainability of optimal solutions to the original OCP.

Taking these observations into account, we make use of the following notion.

Definition 3.3. We say that a pair $(\widehat{A}, \widehat{y}\,) \in L^2(\Omega;\mathbb{M}^N) \times H_0^1(\Omega)$ is a variational solution to OCP (23)–(24) if

$$I(\widehat{A}, \widehat{y}\,) = \inf_{(A,y)\in\Xi} I(A, y), \quad (\widehat{A}, \widehat{y}\,) \in \Xi, \qquad (62)$$

and $(\widehat{A}, \widehat{y}\,)$ is related by energy equality

$$\int_\Omega \left(A^{sym}\nabla\widehat{y}, \nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx = \langle f, \widehat{y}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}\,. \qquad (63)$$

As a direct consequence of Definition 3.3, Theorem 3.2, and properties of the variational limits of constrained minimization problems (see Theorem 1.4), we have the following result.

Proposition 6. *Let $\{A_k^*\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{skew})$ be an approximation of the matrix $A^* \in L^2\big(\Omega;\mathbb{S}^N_{skew}\big)$ with property $A_k^* \to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$. Assume that the corresponding sequence of minimization problems, defined by the rules (45)–(46),* *is such that*

$$\left\langle \inf_{(A,y)\in\,\Xi_k} I_k(A,y)\right\rangle \xrightarrow[k\to\infty]{Var} \left\langle \inf_{(A,y)\in\Xi} I(A,y)\right\rangle \quad \text{in the sense of Definition 1.3.} \tag{64}$$

Let $\left\{(A_k^0, y_k^0)\in\Xi_k\right\}_{k\in\mathbb{N}}$ *be a sequence of optimal solutions to the corresponding regularized OCPs (44)–(46). Then this sequence is relatively compact with respect to the* τ*-convergence, i.e. up to a subsequence we have*

$$\left.\begin{aligned} A_k^0 \to \widehat{A} \text{ strongly in } \ L^2(\Omega;\mathbb{M}^N),\\ y_k^0 \rightharpoonup \widehat{y} \text{ weakly in } \ H_0^1(\Omega) \quad \text{as } k\to\infty,\\ \lim_{k\to\infty}\int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx = \int_\Omega \left(\nabla \widehat{y}, \widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N} dx, \end{aligned}\right\} \tag{65}$$

and each its τ*-cluster pair* $(\widehat{A},\widehat{y}\,)\in L^2(\Omega;\mathbb{M}^N)\times H_0^1(\Omega)$ *is a variational solution to OCP (23)–(24) in the sense of Definition 3.3.*

Proof. Indeed, the τ-compactness of the sequence $\left\{(A_k^0, y_k^0)\in\Xi_k\right\}_{k\in\mathbb{N}}$ is a direct consequence of a priori estimates (54)–(56) and properties (57)–(60). So, within a subsequence, we have

$$y_k^0 \rightharpoonup \widehat{y} \ \text{ in } \ H_0^1(\Omega), \tag{66}$$

$$A_k^0 := A_k^{0,sym} + A_k^{0,skew} \to \widehat{A}^{sym} + \widehat{A}^{skew} =: \widehat{A} \ \text{ in } \ L^2(\Omega;\mathbb{M}^N), \tag{67}$$

$$A_k^{0,sym} \to \widehat{A}^{sym} \quad \text{in } \ L^p(\Omega;\mathbb{S}^N_{sym}),\ \forall\, p\in[1,+\infty). \tag{68}$$

On the other hand, following main properties of the variational convergence (see Theorem 1.4), we can claim that there exists an optimal pair $(A^0,y^0)\in\Xi$ for the problem (23)–(24) such that

$$\begin{aligned} \inf_{(A,y)\in\,\Xi} I(A,y) =& I\left(A^0,y^0\right) := \left\|y^0-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y^0, A^{0,sym}\nabla y^0\right)_{\mathbb{R}^N} dx\\ &= \lim_{k\to\infty}\inf_{(A_k,y_k)\in\,\Xi_k} I_k(A_k,y_k) = \lim_{k\to\infty} I_k(A_k^0,y_k^0)\\ &= \lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx\right]. \end{aligned} \tag{69}$$

However, because of condition (66)–(68), it turns out that (see the estimate (43))

$$\begin{aligned} \inf_{(A,y)\in\,\Xi} I(A,y) \ &\overset{\text{by } (69)}{=} \lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx\right]\\ &\ge \left\|\widehat{y}-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla\widehat{y}, \widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N} dx. \end{aligned}$$

Since the pair $(\widehat{A},\widehat{y}\,)$ is admissible for the problem (23)–(24) (see Theorem 3.2), it follows that $(\widehat{A},\widehat{y}\,)$ is an optimal pair, that is, in view of (69), it gives

$$\begin{aligned} \inf_{(A,y)\in\,\Xi} I(A,y) =& I\left(\widehat{A},\widehat{y}\right) := \left\|\widehat{y}-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla\widehat{y}, \widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N} dx\\ &= \lim_{k\to\infty}\inf_{(A_k,y_k)\in\,\Xi_k} I_k(A_k,y_k) = \lim_{k\to\infty} I_k(A_k^0,y_k^0)\\ &= \lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx\right]. \end{aligned} \tag{70}$$

Hence, (65) is a direct consequence of properties (66)–(70). It remains to prove the energy equality (63). To this end, it is enough to note that $[y_k^0, y_k^0]_{A_k^0} = 0$ for all $k \in \mathbb{N}$. As a result, we get

$$0 \overset{\text{by (51)}}{=} \lim_{k\to\infty} [y_k^0, y_k^0]_{A_k^0} \overset{\text{by (53)}}{=} -\lim_{k\to\infty} \int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx$$
$$+ \lim_{k\to\infty} \left\langle f, y_k^0\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \overset{\text{by (65) and (70)}}{=} -\int_\Omega \left(\nabla \widehat{y}, \widehat{A}^{sym}\nabla \widehat{y}\right)_{\mathbb{R}^N} dx$$
$$+ \left\langle f, \widehat{y}\,\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \overset{\text{by (37)}}{=} [\widehat{y}, \widehat{y}]_{\widehat{A}}.$$

□

Remark 3. Since for some matrices $A \in L^2\big(\Omega;\mathbb{M}^N\big)$ the weak solutions to the boundary value problem (21)–(22) are not unique in general, it follows from Remark 2 and Proposition 6 that even if the OCP (23)–(24) has a unique solution (A^0, y^0), it does not ensure that the pair (A^0, y^0) is the variational solution to the above problem. The matter is that the existence at least one approximation $\{A_k^*\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{skew})$ of the matrix $A^* \in L^2\big(\Omega;\mathbb{S}^N_{skew}\big)$ with property $A_k^* \to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$ such that the approximated OCPs (44)–(46) would lead to the pair (A^0, y^0) in the sense of conditions (15)–(17) is an open problem. In other words, the existence of Γ-realizing sequence for the pair $(A^0, y^0) \in \Xi$ (see Definition 1.3) is not established.

We are now in a position to discuss the existence of variational solutions to the OCP (23)–(24).

Theorem 3.4. *Assume that for every matrix $A \in \mathfrak{A}_{ad} \subset L^2\big(\Omega;\mathbb{M}^N\big)$, we have*

$$[y, y]_A = 0 \quad \forall\, y \in D(A). \tag{71}$$

Then the OCP (23)–(24) has variational solutions.

Proof. Let us consider $\{A_k^*\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{S}^N_{skew})$ and $A^* \in L^2\big(\Omega;\mathbb{S}^N_{skew}\big)$ such that $A_k^* \to A^*$ strongly in $L^2(\Omega;\mathbb{S}^N_{skew})$. With each matrix A_k^* we associate the constrained minimization problem

$$\left\langle \inf_{(A,y)\in\Xi_k} I_k(A, y)\right\rangle,$$

where the cost functional I_k and the set Ξ_k are defined by (45)–(46).

Let $\{(A_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega;\mathbb{M}^N) \times H_0^1(\Omega)$ with the following properties:

- (a) $(A_k, y_k) \in \Xi_{n_k}$ for every $k \in \mathbb{N}$, where $\{n_k\}_{k\in\mathbb{N}}$ is a subsequence converging to ∞ as k tends to ∞;
- (aa) $y_k \rightharpoonup y$ in $H_0^1(\Omega)$ and $A_k \to A$ in $L^2(\Omega;\mathbb{M}^N)$ with additional properties as in (58)–(59).

Then proceeding as in the proof of Theorem 3.2, it can be shown that the limit pair (A, y) is admissible to the original OCP (23)–(24). Hence, this problem is regular and, therefore, it is solvable by Theorem 2.4. Our aim is to show that this problem can be interpreted as the variational limit of the sequence of constrained minimization problems (44). To do so, we have to verify the fulfilment of all conditions of Definition 1.3.

Indeed, it is easy to see that property (d) immediately follows from the relation

$$\liminf_{k\to\infty} I_k(A_k,y_k) = \liminf_{k\to\infty}\left[\|y_k-y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y_k, A_k^{sym}\nabla y_k)_{\mathbb{R}^N}\,dx\right]$$
$$\overset{\text{by (43)}}{\geq} \|y-y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y, A^{sym}\nabla y)_{\mathbb{R}^N}\,dx = I(A,y),$$

which holds true for any sequence $\{(A_k,y_k)\}_{k\in\mathbb{N}} \subset \mathfrak{A}_{ad}\times H^1_0(\Omega)$ with properties (a)–(aa).

We focus now on the verification of condition (dd) of Definition 1.3. Let $(A^\sharp,y^\sharp)$ be an arbitrary admissible pair to the original problem. Since $A^{\sharp,skew}\preceq A^*$, we make use of the hint of Lemma 3.1 in order to construct a sequence of admissible controls $\left\{A_k\in\mathfrak{A}^k_{ad}\subset L^2(\Omega;\mathbb{M}^N)\right\}_{k\in\mathbb{N}}$. Namely, we proceed as follows. Let $A^*_k = [a^{*,k}_{ij}(x)]^N_{i,j=1}$ and $A^{\sharp,skew}=[a^\sharp_{ij}(x)]^N_{i,j=1}$. Then we set:

$$A_k^{sym} = A^{\sharp,sym}\ \ \forall\, k\in\mathbb{N}\quad \text{for the symmetric parts of}\ \ A_k, \tag{72}$$

and for the skew symmetric parts $A_k^{skew}(x)=[a^k_{ij}(x)]^N_{i,j=1}$, we put

$$a^k_{ij}(x) = \begin{cases} a^\sharp_{ij}(x), & \text{if } \left|a^\sharp_{ij}(x)\right| \leq \left|a^{*,k}_{ij}(x)\right| \text{ and } x\in\mathfrak{W}(A^{\sharp,skew}),\\ a^{*,k}_{ij}(x), & \text{if } \left|a^\sharp_{ij}(x)\right| > \left|a^{*,k}_{ij}(x)\right| \text{ and } x\in\mathfrak{W}(A^{\sharp,skew}),\\ 0, & \text{otherwise,}\end{cases} \tag{73}$$

for all $i,j\in\{1,\dots,N\}$ and $k\in\mathbb{N}$.

Since the strong convergence $A^*_k\to A^*$ in $L^2(\Omega;\mathbb{S}^N_{skew})$ implies (up to a subsequence) the pointwise convergence of this sequence a.e. in Ω, and $A^{\sharp,skew}\preceq A^*$, it follows that conditions (72)–(73) lead us to the following conclusion:

$$A_k := A_k^{sym}+A_k^{skew}\to A^{\sharp,sym}+A^{\sharp,skew} =: A^\sharp \ \text{ in }\ L^2(\Omega;\mathbb{M}^N), \tag{74}$$
$$A_k^{sym}\to A^{\sharp,sym}\quad \text{in}\ \ L^p(\Omega;\mathbb{S}^N_{sym}),\ \forall\, p\in[1,+\infty), \tag{75}$$
$$A_k^{skew}\to A^{\sharp,skew}\quad \text{in}\ \ L^2(\Omega;\mathbb{S}^N_{skew}). \tag{76}$$

Let $\{y_k=y(A_k,f)\}_{k\in\mathbb{N}}$ be the corresponding solutions to the regularized boundary value problems (46). Then by applying the arguments of the proof of Theorem 3.2, it can be shown that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $H^1_0(\Omega)$ and there exists an element $\widehat{y}\in D(A^\sharp)$ such that $(A^\sharp,\widehat{y})\in\Xi$ and, within a subsequence, $y_k\rightharpoonup\widehat{y}$ in $H^1_0(\Omega)$. Our aim is to show that $\widehat{y}=y^\sharp$ and that the following identity

$$I(A^\sharp,y^\sharp) = \limsup_{k\to\infty} I_k(A_k,y_k) \tag{77}$$

holds true.

Indeed, since $(A^\sharp,y^\sharp)\in\Xi$ and $(A^\sharp,\widehat{y})\in\Xi$, it follows that $y=y^\sharp-\widehat{y}$ is a solution of the homogeneous problem

$$\begin{aligned} -\operatorname{div}\left(A\nabla y\right) &= 0 \quad \text{in}\ \ \Omega,\\ y &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{78}$$

Following our initial assumptions, we have $[y,y]_A=0\ \forall\, y\in D(A)$ and for each matrix $A\in\mathfrak{A}_{ad}\subset L^2\big(\Omega;\mathbb{M}^N\big)$. Hence, the problem (78) has only trivial solution, since for this solution we have

$$\int_\Omega \left(\nabla y, A^{\sharp,sym}\nabla y\right)_{\mathbb{R}^N}dx = -[y,y]_{A^\sharp} = 0.$$

Thus, $y^\sharp = \widehat{y}$. To prove the equality (77), we use of the idea of D.Cioranescu and F.Murat (see [2]). Taking into account the property (74), compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, and the energy identities (53) and (37), we get

$$\begin{aligned}\lim_{k\to\infty} I_k(A_k, y_k) &= \lim_{k\to\infty}\left[\|y_k - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y_k, A_k^{sym}\nabla y_k)_{\mathbb{R}^N}\, dx\right]\\ &= \left\|y^\sharp - y_d\right\|^2_{L^2(\Omega)} + \lim_{k\to\infty}\int_\Omega (\nabla y_k, A_k^{sym}\nabla y_k)_{\mathbb{R}^N}\, dx\\ &\overset{\text{by (53) and (74)}}{=} \left\|y^\sharp - y_d\right\|^2_{L^2(\Omega)} + \lim_{k\to\infty}\left[\langle f, \widehat{y}_k\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}\right]\\ &\overset{\text{by (71)}}{=} \left\|y^\sharp - y_d\right\|^2_{L^2(\Omega)} + \left\langle f, y^\sharp\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} - [y^\sharp, y^\sharp]_{A^\sharp}\\ &\overset{\text{by (37)}}{=} \left\|y^\sharp - y_d\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y^\sharp, A^{\sharp,sym}\nabla y^\sharp\right)_{\mathbb{R}^N}\, dx = I(A^\sharp, y^\sharp).\end{aligned}$$

This concludes the proof. □

Our next observation shows that variational solutions do not exhaust the entire set of all possible solutions to the original OCP (23)–(24). With that in mind, we adopt the following concept.

Definition 3.5. We say that a pair $(A_0, y_0) \in \Xi$ is a non-variational solution to OCP (23)–(24) if

$$I(A_0, y_0) = \inf_{(A,y)\in\Xi} I(A, y), \quad (A_0, y_0) \in \Xi, \quad \text{and} \tag{79}$$

$$\int_\Omega (\nabla y_0, A_0^{sym}\nabla y_0)_{\mathbb{R}^N}\, dx \neq \langle f, y_0\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}\,. \tag{80}$$

Lemma 3.6. *Assume that there exists a matrix $A_0 \in \mathfrak{A}_{ad}$ and an element $v \in D(A_0)$ with property $[v, v]_{A_0} < 0$. Then there are distributions $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$ such that the optimal control problem*

$$\textit{Minimize } I(A, y) = \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y - \nabla y_d, A^{sym}(\nabla y - \nabla y_d))_{\mathbb{R}^N}\, dx \tag{81}$$

$$\textit{subject to the constraints (21)–(22) and } A \in \mathfrak{A}_{ad} \subset L^2(\Omega; \mathbb{M}^N) \tag{82}$$

has a non-variational solution in the sense of Definition 3.5.

Proof. We consider the OCP (81)–(82) with

$$y_d = v \quad \text{and} \quad f = -\operatorname{div} A_0\nabla v.$$

Since $v \in D(A_0)$, it follows that $v \in H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$. It is easy to see that y_d is a solution to the boundary value problem (21)–(22) under $A = A_0$. Moreover, since $I(A_0, y_d) = 0$, it follows that (A_0, y_d) is the optimal pair to the above OCP.

By contradiction, we assume that (A_0, y_d) can be attained by optimal solutions to regularized problems as in (65). As follows from Theorem 3.2 (see also Remark 2), each attainable solution $(A^\sharp, y^\sharp)$ to this OCP satisfies the inequality $[y^\sharp, y^\sharp]_{A^\sharp} \geq 0$. Since $[y_d, y_d]_{A_0} := [v, v]_{A_0} < 0$, it means that the pair (A_0, y_d) is not attainable and we come into conflict with Proposition 6. The proof is complete. □

Remark 4. As follows from Theorem 3.2, Proposition 6, and Lemma 3.6 none of non-variational solutions can be attainable through the limit of optimal solutions to the regularized problem (44)–(46).

4. **Example of a non-variational optimal solution.** Let Ω be the unit ball in $\mathbb{R}^3$, $\Omega = \left\{x \in \mathbb{R}^3 \ : \ \|x\|_{\mathbb{R}^3} < 1\right\}$. Let us consider the following OCP:

$$\text{Minimize } I(A,y) = \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega \left(\nabla y - \nabla y_d, A^{sym}(\nabla y - \nabla y_d)\right)_{\mathbb{R}^N}\, dx \quad (83)$$

$$\text{subject to the constraints (21)–(22) and } A \in \mathfrak{A}_{ad} \subset L^2(\Omega;\mathbb{M}^N), \quad (84)$$

where the distributions $A^* \in L^2(\Omega, \mathbb{S}^3_{skew})$, $f \in H^{-1}(\Omega)$, and $y_d \in H^1_0(\Omega)$ will be defined later on. Our intention is to show that in this case the above problem admits a non-variational solution, i.e. there exists an admissible pair $(A^0, y^0) \in \mathfrak{A}_{sd} \times H^1_0(\Omega)$ such that

$$I(A^0, y^0) = 0 = \inf_{(A,y)\in\Xi} I(A,y) \quad \text{and} \quad [y^0, y^0]_A = -\zeta < 0, \quad (85)$$

where ζ is a given positive value.

We divide our analysis into several steps. At the first step we define a skew-symmetric matrix A^* as follows

$$A^*(x) = \begin{pmatrix} 0 & a(x) & 0 \\ -a(x) & 0 & -b(x) \\ 0 & b(x) & 0 \end{pmatrix}, \quad (86)$$

where $a(x) = \dfrac{x_1}{2\|x\|^2_{\mathbb{R}^3}}$ and $b(x) = \dfrac{x_3}{2\|x\|^2_{\mathbb{R}^3}}$. Since

$$\begin{aligned} \|a\|^2_{L^2(\Omega)} &= \int_\Omega \left(\frac{x_1}{2\|x\|^2_{\mathbb{R}^3}}\right)^2 dx \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{\rho^2 \cos^2\varphi \sin^2\psi}{\rho^4} \rho^2 \sin\varphi \, d\psi\, d\varphi\, d\rho < +\infty, \end{aligned}$$

it follows that $a \in L^2(\Omega)$. By analogy, it can be shown that $b \in L^2(\Omega)$. Moreover, it is easy to see that the skew-symmetric matrix A^*, define by (86), satisfies the property $A^* \in H(\Omega, \operatorname{div}; \mathbb{S}^3)$, i.e. $A^* \in L^2(\Omega; \mathbb{S}^3_{skew})$ and $\operatorname{div} A^* \in L^1(\Omega;\mathbb{R}^3)$. Indeed, in view of the definition of the divergence $\operatorname{div} A^*$ of a skew-symmetric matrix, we have $\operatorname{div} A^* = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, where $d_i = \operatorname{div} a^*_i = \dfrac{x_i x_2}{\|x\|^4_{\mathbb{R}^3}}$ and a^*_i is i-th column of A^*. As a result, we get

$$\|\operatorname{div} a^*_i\|_{L^1(\Omega)} = \int_0^1 \int_0^{2\pi} \int_0^\pi \left| \frac{\rho^2 f_i(\varphi,\psi) \sin\varphi \sin\psi}{\rho^4} \right| \rho^2 \sin\varphi\, d\psi\, d\varphi\, d\rho < +\infty,$$

for the corresponding $f_i = f_i(\varphi,\psi)$ $(i = 1,2,3)$. Therefore, $\operatorname{div} A^* \in L^1(\Omega;\mathbb{R}^3)$.

Step 2 deals with the choice of the function $y_d \in H^1_0(\Omega)$. We define it by the rule

$$\begin{aligned} y_d = &\sqrt{\frac{52\zeta}{3\pi(1 - \exp(-2\pi))}} \left(1 - \|x\|^5_{\mathbb{R}^3}\right) \\ &\times \frac{x_2^2}{x_1^2 + x_2^2} \exp\left(-\frac{1}{2} \operatorname{atan2}\left(\frac{x_2}{\|x\|_{\mathbb{R}^3}}, \frac{x_1}{\|x\|_{\mathbb{R}^3}}\right)\right) \quad \text{in } \Omega, \end{aligned} \quad (87)$$

where the two-argument function $\operatorname{atan2}(y,x)$ with the range $[0,2\pi]$ is defined as follows

$$\operatorname{atan2}(y,x)=\begin{cases}\arctan\left(\frac{y}{x}\right)+\pi, & x<0,\\ \arctan\left(\frac{y}{x}\right)+2\pi, & y<0, x>0,\\ \arctan\left(\frac{y}{x}\right), & y\ge 0, x>0,\\ \pi/2, & y>0, x=0,\\ 3\pi/2, & y<0, x=0,\\ 0, & y=0, x=0.\end{cases}$$

It is easy to see that

$$v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big):=\sqrt{\frac{52\zeta}{3\pi(1-\exp(-2\pi))}}\,\frac{x_2^2}{x_1^2+x_2^2}\,\exp\left(-\frac{1}{2}\operatorname{atan2}\left(\frac{x_2}{\|x\|_{\mathbb{R}^3}},\frac{x_1}{\|x\|_{\mathbb{R}^3}}\right)\right)$$
$$=\sqrt{\frac{52\zeta}{3\pi(1-\exp(-2\pi))}}\,\sin^2\varphi\exp(-\varphi/2),\quad \forall\,\varphi\in[0,2\pi]$$

with respect to the spherical coordinates. Hence, $v_0\in C^1(\partial\Omega)$, and, as immediately follows from (87), it provides that $y_d\in L^2(\Omega)$ and $y_d=0$ on $\partial\Omega$. By direct computations, we get

$$\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)=\frac{1}{\|x\|^3_{\mathbb{R}^3}}\left[\begin{array}{c}\frac{\partial v_0}{\partial z_1}\left(\|x\|^2_{\mathbb{R}^3}-x_1^2\right)-\frac{\partial v_0}{\partial z_2}x_1x_2\\ \frac{\partial v_0}{\partial z_2}\left(\|x\|^2_{\mathbb{R}^3}-x_2^2\right)-\frac{\partial v_0}{\partial z_1}x_1x_2\\ -\frac{\partial v_0}{\partial z_1}x_1x_3-\frac{\partial v_0}{\partial z_2}x_2x_3\end{array}\right],\ \forall\,x\neq 0. \tag{88}$$

Hence, there exists a constant $C^*>0$ such that $\left\|\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right\|_{\mathbb{R}^3}\le\frac{C^*}{\|x\|_{\mathbb{R}^3}}$. Thus,

$$\|\nabla y_d\|_{\mathbb{R}^3}\le\left|v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right|\left\|\nabla\left(1-\|x\|^5_{\mathbb{R}^3}\right)\right\|_{\mathbb{R}^3}$$
$$+\left(1-\|x\|^5_{\mathbb{R}^3}\right)\left\|\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right\|_{\mathbb{R}^3}\le C_1+\frac{C_2}{\|x\|_{\mathbb{R}^3}}.$$

As a result, we infer that $\nabla y_d\in L^2(\Omega;\mathbb{R}^3)$, i.e. we finally have $y_d\in H^1_0(\Omega)$.

Step 3. We show that the function y_d, which was introduced before, belongs to the set $D(A^*)$. To do so, we have to prove the estimate

$$\left|\int_\Omega\left(\nabla\varphi,A^*(x)\nabla y_d\right)_{\mathbb{R}^3}dx\right|\le\widetilde{C}(y_d)\left(\int_\Omega|\nabla\varphi|^2_{\mathbb{R}^3}\right)^{1/2}\quad\forall\,\varphi\in C_0^\infty(\Omega). \tag{89}$$

To this end, we make use of the following transformations

$$\int_\Omega\left(\nabla\varphi,A^*\nabla\psi\right)_{\mathbb{R}^3}dx=-\left\langle\operatorname{div}\left(A^*\nabla\psi\right),\varphi\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}$$
$$=\left\langle\operatorname{div}\left[\begin{array}{c}(a_1^*)^t\nabla\psi\\ (a_2^*)^t\nabla\psi\\ (a_3^*)^t\nabla\psi\end{array}\right],\varphi\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}$$
$$=\sum_{i=1}^3\left\langle\operatorname{div}a_i^*,\varphi\frac{\partial\psi}{\partial x_i}\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}+\underbrace{\int_\Omega\sum_{i=1}^3\sum_{j=1}^3\left(a^*_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)\varphi\,dx}_{\substack{=0\\ \text{since}\quad A^*\in L^2(\Omega;\mathbb{S}^3_{skew})}}$$
$$\text{(due to the fact that }\operatorname{div}A^*\in L^1(\Omega;\mathbb{R}^3))=\int_\Omega\left(\operatorname{div}A^*,\nabla\psi\right)_{\mathbb{R}^3}\varphi\,dx,$$

which are obviously true for all $\psi,\varphi\in C_0^\infty(\Omega)$. Since

$$\left|\int_\Omega (\,\mathrm{div}\,A^*,\nabla\psi)_{\mathbb{R}^3}\,\varphi\,dx\right| = \left|\int_\Omega \left(\nabla\varphi,\,A^*\nabla\psi\right)_{\mathbb{R}^3}dx\right| \le C\|A^*\|_{L^2(\Omega;\mathbb{S}^3_{skew})}\|\psi\|_{H^1_0(\Omega)},$$

it follows that, using the continuation principle, we can extend the previous equality with respect to ψ to the following one

$$\int_\Omega (\nabla\varphi, A^*\nabla y_d)_{\mathbb{R}^3}\;dx = \int_\Omega \varphi\,(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\;dx \quad \forall\,\varphi\in C_0^\infty(\Omega). \tag{90}$$

Let us show that $(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\in L^\infty(\Omega)$. In this case, relation (90) would imply the estimate

$$\begin{aligned}\left|\int_\Omega (\nabla\varphi, A^*\nabla y_d)_{\mathbb{R}^3}\;dx\right| &\le \|(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\|_{L^\infty(\Omega)}\int_\Omega |\varphi|\,dx\\ &\le \widetilde{C}(y_d)\left(\int_\Omega |\nabla\varphi|^2_{\mathbb{R}^N}\right)^{1/2}\quad \forall\,\varphi\in C_0^\infty(\Omega),\end{aligned}$$

which means that the element y_d belongs to the set $D(A^*)$.

Indeed, as follows from (88), we have the equality

$$\left(\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big),\frac{x}{\|x\|^3_{\mathbb{R}^3}}\right)_{\mathbb{R}^3} = 0. \tag{91}$$

Thus, the gradient of the function $\nabla v_0(\frac{x}{\|x\|_{\mathbb{R}^3}})$ is orthogonal to the vector field $Q = x/\|x\|^3_{\mathbb{R}^3}$ outside the origin. Therefore,

$$\begin{aligned}(\nabla y_d,\,\mathrm{div}\,A^*)_{\mathbb{R}^3} &:= \left(\nabla\left[\left(1-\|x\|^5_{\mathbb{R}^3}\right)v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right],\frac{x}{\|x\|^3_{\mathbb{R}^3}}\,\frac{x_2}{\|x\|_{\mathbb{R}^3}}\right)_{\mathbb{R}^3}\\ &= \left(\nabla\left(1-\|x\|^5_{\mathbb{R}^3}\right),\frac{x}{\|x\|^3_{\mathbb{R}^3}}\right)_{\mathbb{R}^3} v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\frac{x_2}{\|x\|_{\mathbb{R}^3}}\\ &\quad+\left(1-\|x\|^5_{\mathbb{R}^3}\right)\left(\nabla v_0\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big),\frac{x}{\|x\|^3_{\mathbb{R}^3}}\right)_{\mathbb{R}^3}\frac{x_2}{\|x\|_{\mathbb{R}^3}} = I_1+I_2,\end{aligned} \tag{92}$$

where $I_2=0$ by (91). Since $\nabla\left(1-\|x\|^5_{\mathbb{R}^3}\right) = -5\|x\|^3_{\mathbb{R}^3}x$, $\frac{x_2}{\|x\|_{\mathbb{R}^3}} = \sin\varphi\sin\psi$ with respect to the spherical coordinates, and function v_0 is smooth, it follows that there exists a constant $C_0>0$ such that $|(\nabla y_d,\,\mathrm{div}\,A^*)_{\mathbb{R}^3}|\le C_0$ almost everywhere in Ω. Thus, $(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\in L^\infty(\Omega)$ and we have obtained the required property.

Step 4. Using results of the previous steps, we show that the function y_d satisfies the condition $[y_d,y_d]_{A^*} = -\zeta<0$. Indeed, let $\{\varphi_\varepsilon\}_{\varepsilon\to 0}\subset C_0^\infty(\Omega)$ be a sequence such that $\varphi_\varepsilon\to y_d$ strongly in $H^1_0(\Omega)$. Then by continuity, we have

$$[y_d,y_d]_{A^*} = \lim_{\varepsilon\to 0}\int_\Omega (\nabla\varphi_\varepsilon, A^*\nabla y_d)_{\mathbb{R}^3}\;dx \overset{\text{by (90)}}{=} \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon\,(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\;dx$$

Since $(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\in L^\infty(\Omega)$ and $\varphi_\varepsilon\to y_d$ strongly in $H^1_0(\Omega)$, we can pass to the limit in the right-hand side of this relation. As a result, we get

$$[y_d,y_d]_{A^*} = \int_\Omega y_d\,(\,\mathrm{div}\,A^*,\nabla y_d)_{\mathbb{R}^3}\;dx = \frac{1}{2}\int_\Omega \left(\,\mathrm{div}\,A^*,\nabla y_d^2\right)_{\mathbb{R}^3}\;dx. \tag{93}$$

Let $\Omega_\varepsilon = \left\{x \in \mathbb{R}^3 \mid \varepsilon < \|x\|_{\mathbb{R}^3} < 1\right\}$ and let $\Gamma_\varepsilon = \{\|x\|_{\mathbb{R}^3} = \varepsilon\}$ be the sphere of radius ε centered at the origin. Then

$$\begin{aligned}\int_{\Omega_\varepsilon} \left(\operatorname{div} A^*, \nabla y_d^2\right)_{\mathbb{R}^3} dx &\overset{\text{since } y_d \in H_0^1(\Omega)}{=} \int_{\Gamma_\varepsilon} \left(\operatorname{div} A^*, \nu\right)_{\mathbb{R}^3} y_d^2 \, d\mathcal{H}^2 \\ &= \int_{\Gamma_\varepsilon} \left(\operatorname{div} A^*, \nu\right)_{\mathbb{R}^3} \left(1 - \|x\|^5_{\mathbb{R}^3}\right)^2 v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 \\ &= \int_{\Gamma_\varepsilon} \left(\operatorname{div} A^*, \nu\right)_{\mathbb{R}^3} v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) \\ &= \int_{\Gamma_\varepsilon} \left(\frac{x}{\|x\|^3_{\mathbb{R}^3}}, \Big(-\frac{x}{\|x\|_{\mathbb{R}^3}}\Big)\right)_{\mathbb{R}^3} \frac{x_2}{\|x\|_{\mathbb{R}^3}} v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) \\ &= -\varepsilon^{-2} \int_{\Gamma_\varepsilon} \frac{x_2}{\|x\|_{\mathbb{R}^3}} v_0^2\Big(\frac{x}{\|x\|_{\mathbb{R}^3}}\Big) \, d\mathcal{H}^2 + o(1) = -\int_\Gamma b_0(x) v_0^2(x) \, d\mathcal{H}^2 + o(1),\end{aligned}$$

where $b_0 = \sin\varphi \sin\psi$ and and $v_0^2 = \frac{52\zeta}{3\pi(1-\exp(-2\pi))} \sin^4\varphi \exp(-\varphi)$. Since

$$\int_{\partial\Omega} b_0 v_0^2 \, d\mathcal{H}^2 = \frac{52\zeta}{3\pi(1-\exp(-2\pi))} \int_0^{2\pi} \sin^5\varphi \, e^{-\varphi} \, d\varphi \int_0^\pi \sin^2\psi \, d\psi = 2\zeta > 0,$$

it remains to combine this result with (93) and relation

$$\int_\Omega \left(\operatorname{div} A^*, \nabla y_d^2\right)_{\mathbb{R}^3} dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \left(\operatorname{div} A^*, \nabla y_d^2\right)_{\mathbb{R}^3} dx.$$

As a result, we infer $[y_d, y_d]_{A^*} = -\zeta < 0$.

Step 5. This is the last step in our analysis of OCP (83)–(84). Let us define the distribution $f \in H^{-1}(\Omega)$ as follows

$$f = -\operatorname{div}\left(A^\sharp \nabla y_d + A^* \nabla y_d\right), \tag{94}$$

where $A^\sharp$ is an arbitrary symmetric matrix such that $A^\sharp \in \mathfrak{A}_{ad,1}$.

Assume that a compact set Q of $L^2(\Omega; \mathbb{S}^3_{skew})$ contains matrix A^*. Then it is obvious that the matrix $A^0 = A^\sharp + A^*$ is admissible for OCP (83)–(84), i.e. $A^0 \in \mathfrak{A}_{ad}$.

Since $y_d \in D(A^*)$, it follows that $f \in H^{-1}(\Omega)$. Hence, (A^0, y_d) is an admissible pair to the problem (83)–(84). Taking into account that $I(A^0, y_d) = 0$, we finally conclude: the pair $(A^0, y^0) := (A^\sharp + A^*, y_d)$ is a non-variational solution to OCP (83)–(84).

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