On Optimal Control Problem for Conservation Law Modelling One Class of Highly Re-Entrant Production Systems

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Abstract. We discuss the optimal control problem stated as the minimization in the L^2-sense of the mismatch between the actual out-flux and a demand forecast for a hyperbolic conservation law that models a highly re-entrant production system. The output of the factory is described as a function of the work in progress and the position of the so-called push-pull point (PPP) where we separate the beginning of the factory employing a push policy from the end of the factory, which uses a pull policy.

Introduction

We analyze an optimal control problem (OCP) for a highly re-entrant production system which is described by a scalar nonlinear conservation law. Typically, the semi-conductor systems are characterized by the very high volume (number of parts manufactured per unit time) and the very large number of consecutive production steps. This fact motivates to consider the scalar nonlinear conservation laws for the simulation of such processes. Partial differential equations, which are related with nonlinear conservation laws, are rather popular due to their superior analytic properties and availability of efficient numerical tools for simulation. For more detailed discussions of these models we refer to [1, 5, 10, 11, 12, 13, 14, 15, 16].

From the optimization point of view, in manufacturing systems the natural control input is the in-flux. For the time been, in mostly cases only the influence of changing the inflow on the output has been studied. However, the output of a factory can be changed via dispatch policies. Specifically, re-entrant production creates the opportunity to set priority rules for the various stages of production competing for capacity at the same machines. This dispatch policy, as it was indicated in [2], typically allows for two models of operations — the so-called push and pull policies. A puch policy, also known as first buffer first step, is typically assigned to the front of the factory. A pull policy gives priority to later or fixed production steps over the earlier production steps. The step where push policy switches to pull policy is called the push-pull point (PPP). Moving the PPP leads not only to a change in dispatch rules, but also it has significant effect on the total output. This fact motivates to consider the PPP as a control variable. As for the natural control objectives, it is reasonable to minimize the error-signal that is the difference between a given demand forecast and the actual out-flux of manufacturing system.

A modern introduction to the study of hyperbolic conservation laws and especially of the control systems governed by such laws can be found in [3]. Fundamental are questions of wellposedness, regularity properties of solutions, controllability, existence, uniqueness and regularity of optimal controls. Existence of solutions, regularity and wellposedness of nonlinear conservation laws have been widely studied under diverse sets of hypotheses, see e.g. [4] and the references therein. Further results can be found in [8, 9, 17, 18, 20].

The characteristic feature of OCP, we deal with in this article, is the fact that we consider an optimal control problem for the nonlinear conservation law with a nonlocal character of the velocity and with three different control actions — the in-flux, the PPP, and the so-called clearing functions V_1 and V_2.

Statement of the Problem

Let $\alpha_2 > \alpha_1 > 0$ and $\alpha_3 > 0$ be given constants. Let A_{ad} be the following subset of $C^1([0,\infty))$

$$A_{ad} = \left\{ V \in C^1([0,\infty)) \;\middle|\; \begin{array}{c} 0 \le \alpha_1 \le V(x) \le \alpha_2 \;\forall\, x \in [0,\infty), \\ \|V'\|_{C^0([0,\infty))} \le \alpha_3. \end{array} \right\} \tag{1}$$

We denote $\rho(t,x)$ the product density at the stage $x \in [0,1]$ and time $t \in [0,T]$. Here, $x=0$ refers to the point of raw material and $x=1$ to the finished product.

Definition 1 *We say that a mapping $F : [0,T] \times [0,1] \mapsto [0,\infty)$ is the clearing function if there exists a point $x^* \in [0,1]$ and functions $V_1, V_2 \in A_{ad}$ such that*

$$F(t,x) := \rho(t,x)\left[H(x^*-x)V_1\left(W^{push}(t,x)\right) + H(x-x^*)V_2\left(W^{pull}(t,x)\right)\right], \tag{2}$$

where $H(x)$ stands for the Heaviside function and

$$W^{pull}(t,x) = \int_x^1 \rho(t, y-x+x^*)\,dy, \quad W^{push}(t,x) = \int_0^x \rho(t,y)\,dy. \tag{3}$$

As follows from this definition $F(t,x)$ can be associated with the flux at the time $t \in [0,T]$ and stage $x \in [0,1]$ in the factory, whereas $x^* \in [0,1]$ is the PPP where the push policy switches to pull policy. Taking into account that in the manufacturing systems the natural control input is the in-flux, we arrive at the following statement of OCP:

$$Minimize\ \left\{ I(u,V_1,V_2,x^*) = \int_0^T |y(t)-y_d(t)|^2, dt + \|V_1''-z_{1,d}\|^2_{L^2(0,a_1)} + \|V_2''-z_{2,d}\|^2_{L^2(0,a_2)} \right\} \tag{4}$$

subject to the constraints

$$\partial_t \rho(t,x) + \partial_x\left(V\rho(t,x)\right) = 0 \quad in \quad Q = (0,T)\times(0,1), \tag{5}$$

$$V = H(x-x^*)V_2\left(\int_x^1 \rho(t,y-x+x^*)\,dy\right) + H(x^*-x)V_1\left(\int_0^x \rho(t,y)\,dy\right), \tag{6}$$

$$\rho(0,x) = \rho_0(x) \quad for \quad x\in[0,1], \quad \rho(t,0)V_1(0) = u, \quad for \quad t \in [0,T], \tag{7}$$

$$y(t) = \rho(t,1)V_2(0), \tag{8}$$

$$V_1, V_2 \in A_{ad}, \quad x^* \in [0,1], \tag{9}$$

$$u \in U_{ad} := \left\{ w \in L^2(0,T) \mid \|w\|_{L^2(0,T)} \le \alpha_4,\ w(x) \ge 0 \quad a.e. on \quad (0,T) \right\}, \tag{10}$$

where

$$a_1 = \sqrt{T}\,\alpha_4 + \|\rho_0\|_{L^\infty(0,1)}, \quad a_2 = \frac{\alpha_2}{\alpha_1}\left(\sqrt{T}\,\alpha_4 \|\rho_0\|_{L^\infty(0,1)}\right), \tag{11}$$

$z_{1,d} \in L^2(0,a_1)$, $z_{2,d} \in L^2(0,a_2)$, $\rho_0 \in L^2(0,1)$, and $y_d \in L^2(0,T)$ are functions, and $y(t)$ is the out-flux corresponding to the in-flux $u \in L^2_+(0,T)$, functions V_1, V_2, and initial data ρ_0. We also suppose $\rho_0 \in L^2(0,1)$ and $y_d \in L^2(0,T)$ are nonnegative almost everywhere, and the constant a in definition of the cost functional is such that $\max\{a_1,a_2\} \le a < +\infty$.

Hereinafter, a tuple

$$(u, V_1, V_2, x^*) \in L^2(0,T) \times C^1([0,a_1]) \times C^1([0,a_2]) \times [0,1] \tag{12}$$

with properties 9–10 we call an admissible control.

Auxiliary Results

Following the recent results in this field [19], we adopt the following definition of a weak solution to the problem 5–7.

Definition 2 *Let $T>0$, $\rho_0 \in L^1(0,1)$, $u \in L^1(0,T)$, $x^* \in [0,1]$, and $V_1, V_2 \in A_{ad}$ be given. We say that a pair $(\rho_1,\rho_2) \in C^0([0,T]; L^1(0,x^*) \times L^1(x^*,1))$ is a weak solution to the Cauchy problem 5–7 if for every $\tau \in [0,T]$ and every test functions $(\varphi_1,\varphi_2) \in C^1([0,T]\times[0,x^*]) \times C^1([0,T]\times[x^*,1])$ such that*

$$\varphi_1(\tau,x)=0, \quad \forall\, x \in [0,x^*], \qquad \varphi_1(t,x^*)=0, \quad \forall\, t \in [0,\tau], \tag{13}$$
$$\varphi_2(\tau,x)=0, \quad \forall\, x \in [x^*,1], \qquad \varphi_2(t,1)=0, \quad \forall\, t \in [0,\tau], \tag{14}$$

the following integral identities hold true

$$\int_0^\tau \int_0^{x^*} \rho_1(t,x)\left[\partial_t\varphi_1(t,x) + V_1\left(\textstyle\int_0^x \rho_1(t,y)\,dy\right)\partial_x\varphi_1(t,x)\right] dxdt \tag{15}$$
$$+ \textstyle\int_0^\tau u(t)\varphi_1(t,0)\,dt + \int_0^{x^*} \rho_0(x)\varphi_1(0,x)\,dx = 0, \tag{16}$$
$$\int_0^\tau \int_{x^*}^1 \rho_2(t,x)\left[\partial_t\varphi_2(t,x) + V_2\left(\textstyle\int_x^1 \rho_2(t,y-x+x^*)\,dy\right)\partial_x\varphi_2(t,x)\right] dxdt \tag{17}$$
$$+ \textstyle\int_0^\tau \rho_1(t,x^*)V_1\left(\int_0^{x^*}\rho_1(t,y)\,dy\right)\varphi_2(t,x^*)\,dt + \int_{x^*}^1 \rho_0(x)\varphi_2(0,x)\,dx = 0. \tag{18}$$

We have the following result concerning the existence of weak solutions to the Cauchy problem (5)–(7).

Theorem 3 *Let $\rho_0 \in L^\infty(0,1)$, $u \in L^1(0,T)$, $V_1, V_2 \in A_{ad}$, and $x^* \in [0,1]$ be given. Then the Cauchy problem (5)–(7) admits a unique global solution*

$$\rho(t,x) = \begin{cases} \rho_1(t,x), & if \quad t \in [0,T],\ x \in [0,x^*), \\ \rho_2(t,x), & if \quad t \in [0,T],\ x \in (x^*,1] \end{cases} \tag{19}$$

such that

$$(\rho_1,\rho_2) \quad \in C([0,T];L^1(0,x^*)) \times C([0,T];L^1(x^*,1)), \tag{20}$$
$$(\rho_1,\rho_2) \quad \in C([0,x^*];L^1(0,T)) \times C([x^*,1];L^1(0,T)). \tag{21}$$

Existence of Optimal Solutions

In this section we focus on solvability of OCP 4–10. To begin with we note that unknown control functions V_1 and V_2 in 4–9 are supposed to be defined on domains $[0,a_1]$ and $[0,a_2]$, respectively, with constants a_1 and a_2 given by 11.

Definition 4 *We say that a tuple (u,V_1,V_2,x^*,ρ) is a feasible solution to OCP 4–10 if (u,V_1,V_2,x^*) satisfies constraints 9–10, the function $\rho(t,x)$ with properties 19–20 is the corresponding weak solution to the Cauchy problem (5)–(7), and $I(u,V_1,V_2,x^*) < \infty$.*

We denote by Ξ the set of all feasible solutions for the OCP 4–10.

Remark 5 *As follows from 1 and Definition 4, if $(u,V_1,V_2,x^*,\rho) \in \Xi$, then $V_i \in W^{2,2}(0,a_i)$ $(i=1,2)$. Hence, by Sobolev Embedding Theorem, we have $W^{2,2}(0,a_i) \hookrightarrow C^{1,1/2}([0,a_i])$, where $\frac{1}{2}$ stands for the Hölder exponent. Therefore, as a direct consequence of Arzelà-Ascoli Theorem, we have a compact inclusion $C^{1,1/2}([0,a_i]) \hookrightarrow C^1([0,a_i]]$.*

We say that a tuple $(u^0,V_1^0,V_2^0,x^{*,0},\rho^0)$ is an optimal solution to 4–10 if

$$(u^0,V_1^0,V_2^0,x^{*,0},\rho^0) \in \Xi \quad and \quad I(u^0,V_1^0,V_2^0,x^{*,0}) = \inf_{(u,V_1,V_2,x^*,\rho)\in\Xi} I(u,V_1,V_2,x^*).$$

We are now in a position to present our main result.

Theorem 6 *For arbitrary $z_{1,d} \in L^2(0,a_1)$, $z_{2,d} \in L^2(0,a_2)$, $\rho_0 \in L^\infty(0,1)$, $y_d \in L^2(0,T)$, $\alpha_i > 0$, $(i = 1,\ldots,4)$, such that $\alpha_2 > \alpha_1 > 0$, and constants a_i, $(i = 1,2)$ given by 11, the OCP 4–10 admits at least one optimal solution $(u^0, V_1^0, V_2^0, x^{*,0}, \rho^0) \in \Xi$.*

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