Thermistor Problem: Multi-Dimensional Modelling, Optimization, and Approximation

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ABSTRACT

We consider a problem of an optimal control in coefficients for the system of two coupled elliptic equations also known as thermistor problem which provides a simultaneous description of the electric field $u = u(x)$ and temperature $\theta(x)$. The coefficients of operator div $(A(x)\nabla\theta(x))$ are used as the controls in $L^\infty(\Omega)$. The optimal control problem is to minimize the discrepancy between a given distribution $\theta_d \in L^r(\Omega)$ and the temperature of thermistor $\theta \in W_0^{1,\gamma}(\Omega)$ by choosing an appropriate anisotropic heat conductivity matrix B. Basing on the perturbation theory of extremal problems and the concept of fictitious controls, we propose an "approximation approach" and discuss the existence of the so-called quasi-optimal and optimal solutions to the given problem.

Introduction

Thermistor is a generic name for a device made from materials whose electrical conductivity is highly dependent on temperature. The advantages of thermistors as temperature measurement devices are low cost, high resolution, and flexibility in size and shape. The applications of thermistors can be summarized as follows:

- temperature sensing and control: thermistors provide inexpensive and reliable temperature sensing for a wide temperature range;
- thermal relay and switch: voltage regulation, surge protection;
- indirect measurement of other parameters: when a thermistor is heated its rate of change of temperature depends on its surroundings. This property can be used to monitor other quantities such as liquid level and fluid flow.

In a bounded open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we consider the following steady-state thermistor problem

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div} g \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0, \tag{1}$$

$$-\operatorname{div}(B\nabla\theta) = |\nabla u|^p \quad \text{in} \quad \Omega, \quad \theta|_{\partial\Omega} = 0, \tag{2}$$

$$p(\cdot) = \sigma(\theta(\cdot)) \quad \text{a.e. in} \quad \Omega, \tag{3}$$

where $B \in BV(\Omega)^{N\times N}$ is a given squared matrix.

System (1)–(3) describes the coupling between the electric field with potential u and the temperature θ in an anisotropic thermistor, where its anisotropic heat conductivity is given by a matrix of positive coefficients $B = [b_{ij}(x)]_{i,j=1,\dots,N}$. This model is based on rational mechanics of electrorheological fluids, that takes into account the complex interactions between the electromagnetic fields and the moving liquid. In particular, the electrorheological fluids have the interesting property that their viscosity depends on the electric field in the fluid. A great deal of attention has been paid by many authors in the study of the thermistor problem during the last two decades. The search of the least assumptions on $\sigma(\theta)$, ensuring the (weak) solvability of the system (1)–(3), has been in the agenda of experts for decades. Earlier, existence theorems were proved only under some smallness conditions, e.g., in the case of a sufficiently small Lipschitz constant for the function $\sigma(\theta)$. However, the most essential progress in the study of existence and qualitative properties of solutions to the boundary value problem (1)–(3) was achieved by Zhikov [Zhikov 2011]. It has been shown that the solvability of these systems can be obtained in the multi-dimensional case without any smallness requirements on the function $\sigma(\theta)$ via a regularization approach and further passing to the limit over the parameter of regularization.

Setting of Optimization Problem

Our main goal is two-fold. The first one is to prove an existence result for the thermistor optimal control problem in coefficients with nonlinear state equations containing the p-Laplacian with variable exponent $p = p(x)$. The second one is to provide the asymptotic analysis of a special class of well-defined parametrized optimal control problems with fictitious controls and show that the original problem can be considered as a variational "limit" of the corresponding constrained minimization problems.

In view of this, in a bounded open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega$, we deal with the following optimization problem:

$$\text{Minimize} \quad \left\{ J(B,u,\theta) = \int_\Omega |\theta(x) - \theta_d(x)|^r\, dx \right\} \tag{4}$$

subject to the constraints (1)–(3) and, in addition, $B \in \mathfrak{B}_{ad}$, where

$$\mathfrak{B}_{ad} = \left\{ \begin{array}{c} B \in BV(\Omega)^{N\times N}, \\ m_1 I \leq B(\cdot) \leq m_2 I, \\ \int_\Omega |Db_{ij}| \leq \mu \ \forall\, i,j = \overline{1,N}, \end{array} \right\} \tag{5}$$

$r \in \left(1, \frac{N}{N-2}\right)$ if $N > 2$ and $r \in (1, +\infty)$ for $N = 2$ is a given value, m_1 and m_2 are constants such that $0 < m_1 \leq m_2 < +\infty$, I is the identity matrix in $\mathbb{R}^{N\times N}$, the inequalities (5) are in the sense of the quadratic forms defined by $(B\xi,\xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$, $\theta_d \in L^r(\Omega)$ and $g \in L^\infty(\Omega)^N$ are given distributions, σ is a continuous function such that $\alpha \leq \sigma(y) \leq \beta$ for all $y \in \mathbb{R}$ and the constants α and β satisfy the condition

$$1 < \alpha \leq \beta < \alpha^* = \left\{ \begin{array}{ll} +\infty, & \text{if } \alpha \geq N, \\ \frac{\alpha N}{N-\alpha}, & \text{if } \alpha < N, \end{array} \right. \tag{6}$$

$\mathfrak{B}_{ad}$ stands for the class of admissible controls, and μ is a given positive value. For motivation of BV-choice for the set of admissible controls, we refer to [D'Apice et al. 2008, D'Apice et al. 2010, D'Apice et al. 2012, D'Apice et al. 2014].

As for the optimal control problem (4)–(6), to the best of the authors knowledge, the existence of optimal solutions for the above thermistor problem remains an open question. Only very few articles deal with optimal control for the thermistor problem in two dimensional case (see [Hömberg et al. 2010], [Hrynkiv 2009] and references therein). There are several reasons for this:

- it is unknown whether the set of feasible points to the problem (4)–(6) is weakly closed in the corresponding functional space;
- we have no a priori estimates for the weak solutions to the boundary value problem (1)–(3) under conditions (6);

- the asymptotic behaviour of a minimizing sequence to the cost functional (4) is unclear in general;

- the optimal control problem (4)–(6) is ill-posed and relations (1)–(3) require some relaxation (see, for instance, [Durante et al. 2017, Kupenko and Manzo 2016, Kupenko and Manzo in press]).

To circumvent the problems listed above, we propose an "indirect approach" to the solvability of the optimal control thermistor problem in coefficients. Basing on the perturbation theory of extremal problems and the concept of fictitious controls (see, for instance, [Casas et al. 2016, Kogut and Leugering 2011, Kogut et al. 2016]), we prove the existence of so-called quasi-optimal and optimal solutions to the problem (4)–(6) and show that they can be attained by the optimal solutions of some appropriate approximations for the original optimal control problem. The main idea of our approach is based on the fact that weak solutions to the Dirichlet problem (1)–(3) can be attained through a special regularization of the exponent $p = p(x)$ and an approximation of the operator $\mathcal{A}(u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, using its perturbation by the $\varepsilon\Delta_\beta$-Laplacian, and the right-hand side of (2) by its transformation to $\operatorname{div}\left[\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right) u\right] + (g, \nabla u)_{\mathbb{R}^N}$. Here, by attainability of a weak solution (u, θ), we mean the existence of a sequence $\{(u_\varepsilon, \theta_\varepsilon)\}_{\varepsilon>0}$, where (u_ε, θ) are the solutions of "more regular" boundary value problems, such that $(u_\varepsilon, \theta_\varepsilon) \to (u, \theta)$ in some appropriate topology as ε tends to zero.

Preliminaries

We recall the well-known facts for nonlinear elliptic problems with variable exponent and discuss how each of the equations in system (1)–(3) can be interpreted. Assuming that the temperature $\theta = \theta(x)$ is known for some admissible control $B(x)$, we introduce the Sobolev-Orlicz space

$$W_0^{1,p(\cdot)}(\Omega) :=$$

$$\left\{ u \in W_0^{1,1}(\Omega) \;:\; \int_\Omega |\nabla u(x)|^{p(x)}\, dx < +\infty \right\}$$

and equip it with the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)^N}$, where $p(x) = \sigma(\theta(x))$. Here, $|\cdot|$ denotes the Euclidean norm $|\cdot|_{\mathbb{R}^N}$ in $\mathbb{R}^N$, and $L^{p(\cdot)}(\Omega)^N$ stands for the set of all measurable functions $f : \Omega \to \mathbb{R}^N$ such that $\int_\Omega |f(x)|^{p(x)}\, dx < +\infty$. It is well-known that, unlike in classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega)$. Hence, with variable exponent $p = p(x)$ $(1 < \alpha \le p(\cdot) \le \beta)$ it can be associated another Sobolev space, $H = H_0^{1,p(\cdot)}(\Omega)$ as the closure of the set $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$-norm. Since we can lose the density of the set $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ for some (irregular) variable exponents $p(x)$, it follows that a weak solution to the problem (1) is not unique, in general.

Definition 1. *We say that a function $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of the problem* (1) *if*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u, \nabla\varphi\right)_{\mathbb{R}^N}\, dx = \int_\Omega (g, \nabla\varphi)_{\mathbb{R}^N}\, dx, \tag{7}$$

for all $\varphi \in C_0^\infty(\Omega)$, and we say that u is the H-solution of problem (1)*, if $u \in H_0^{1,p(\cdot)}(\Omega)$ and the integral identity* (7) *holds for any test function $\varphi \in H_0^{1,p(\cdot)}(\Omega)$.*

As for the second equation (2), its right-hand side $|\nabla u|^p$ with $p(\cdot) = \sigma(\theta(\cdot))$, a priori belongs to the space $L^1(\Omega)$. In this case, following the L^1-theory of the Dirichlet problem for the Laplace operator, the solution to the boundary value problem (2) can be defined as solution obtained as a limit of approximations (SOLA).

Definition 2. *A function $\theta : \Omega \to \mathbb{R}$ is the SOLA to* (2) *if the following two conditions holds:*

- *$u \in L^1(\Omega)$ is a duality solution of* (2) *in the sense of Stampacchia, i.e.*

  $$\int_\Omega \theta\varphi\, dx = \int_\Omega |\nabla u|^p v\, dx, \quad \forall\, \varphi \in L^\infty(\Omega),$$

  where $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the weak solution of

  $$-\operatorname{div}\left(B^t\nabla v\right) = \varphi \quad \text{in } \Omega \quad v = 0 \text{ on } \partial\Omega;$$

- *For any sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $f_n \to |\nabla u|^p$ strongly in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \le$*

$\||\nabla u|^p\|_{L^1(\Omega)} = \|u\|^p_{W_0^{1,p(\cdot)}(\Omega)}$ *for all* $n \in \mathbb{N}$, *we have*

$$\theta_n \to \theta \;\text{ strongly in } L^1(\Omega), \text{ weakly in } W_0^{1,\gamma}(\Omega)$$

for all $\gamma \in [1, \frac{N}{N-1})$, *where* $\theta_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is the weak solution of*

$$-\operatorname{div}(B\nabla\theta_n) = f_n \quad \text{in } \Omega \quad \theta_n = 0 \text{ on } \partial\Omega.$$

The main result concerning the existence of a SOLA to the problem (2), can be stated as follows: if Ω is a bounded domain with sufficiently smooth boundary and $|\nabla u|^{p(\cdot)} \in L^1(\Omega)$, then the Dirichlet problem (2) has the unique SOLA $\theta \in W_0^{1,\gamma}(\Omega)$ with $\gamma \in [1, \frac{N}{N-1})$; moreover, there exists a constant $C = C(\gamma)$ independent of $f = |\nabla u|^{p(\cdot)}$ such that

$$\|\theta\|_{W_0^{1,\gamma}(\Omega)} \le C(\gamma) \int_\Omega |\nabla u(x)|^{p(x)}|\, dx. \tag{8}$$

In fact, if the datum $f = |\nabla u|^p$ is more regular, say $f \in L^{1+\delta}(\Omega)$ for some $\delta > 0$, we have the following result: if $|\nabla u|^p \in L^{1+\delta}(\Omega)$, $0 < \delta < \frac{N-2}{N+2}$ then the unique SOLA of (2) belongs to $W_0^{1,q}(\Omega)$ with $q = \frac{N(1+\delta)}{N-1-\delta} = 1 + \delta + \frac{(1+\delta)^2}{N-1-\delta}$.

The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $\theta_d \in L^r(\Omega)$ and the temperature of thermistor $\theta \in W_0^{1,\gamma}(\Omega)$ by choosing an appropriate anisotropic heat conductivity matrix $B \in \mathfrak{B}_{ad}$. It is assumed here that $r \in \left(1, \frac{N}{N-2}\right)$ where the choice of such range is motivated by Sobolev Embedding Theorem. Namely, in view of the fact that the embedding $W_0^{1,\gamma}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, \frac{N}{N-2})$, the exponents γ and r can be related as follows $\gamma = \frac{Nr}{N+r}$. As a result, for a given $r \in \left(1, \frac{N}{N-2}\right)$ we have $\gamma \in \left[1, \frac{N}{N-1}\right)$.

Characteristic features of the OCP (4)–(6)

Since for a "typical" measurable or even continuous function $\sigma(\theta)$ with properties (6), the set $C_0^\infty(\Omega)$ is not dense in $W_0^{1,p(\cdot)}(\Omega)$, and, hence, no uniqueness of weak solutions to (1)–(3) can be expected, the mapping $B \mapsto (u,\theta)$, where (u,θ) is a weak solution to the boundary value problem (1)–(3), can be multi-valued in general. In view of this, we introduce the set of feasible solutions to the OCP (4)–(6) as follows: $(B,u,\theta,p) \in \Xi_0$ if and only if

$$\left\{\begin{array}{c} B \in \mathfrak{B}_{ad},\; u \in H_0^{1,p(\cdot)}(\Omega),\; \theta \in W_0^{1,\gamma}(\Omega), \\ p \in L^\infty(\Omega), \quad \gamma = \frac{Nr}{N+r}, \\ p(\cdot) = \sigma(\theta(\cdot)) \;\text{ a.e. in } \Omega, \\ u \text{ is the } H\text{-solution of (1)}, \\ \theta \text{ is the SOLA to (2)}. \end{array}\right\} \tag{9}$$

It is clear that $J(B,u,\theta,p) < +\infty$ for all $(B,u,\theta,p) \in \Xi_0$.

So, the characteristic feature of the OCP (4)–(6) is the fact that a priori it is unknown whether the set Ξ_0 is nonempty. Using the assumption (6) and basing on a special technique of the weak convergence of fluxes to a flux, it was established in [Zhikov 2011] that the thermistor problem (1)–(3) for $B = \xi I$, with $\xi \in [m_1, m_2]$, and for any measurable function $\sigma(\theta)$ admits a weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$. However, in this case the inclusion $u \in H_0^{1,p(\cdot)}(\Omega)$ is by no means obvious even in the case of diagonal constant matrix $B \in \mathfrak{B}_{ad}$. Hence, the OCP (4)–(6) requires some relaxation. With that in mind we propose to consider the function $p(\cdot)$ as a fictitious control with some more regular properties and interpret the fulfilment of equality $p(\cdot) = \sigma(\theta(\cdot))$ with some accuracy.

Relaxation of the original OCP

We consider the following extension of the set of feasible solutions to the original OCP. Let $k_0 > 0$ and $\tau \ge 0$ be given constants.

Definition 3. *We say that a tuple* (B,u,θ,p) *is quasi-feasible to the OCP* (4)–(6) *if* $(B,u,\theta,p) \in \widehat{\Xi}_0(\tau)$, *where*

$$\left\{\begin{array}{c} B \in \mathfrak{B}_{ad},\; u \in H_0^{1,p(\cdot)}(\Omega), \\ \theta \in W_0^{1,\gamma}(\Omega), p \in \mathfrak{S}_{ad}, \\ \|p - \sigma(\theta)\|_{L^2(\Omega)} \le \tau, \quad \gamma = \frac{Nr}{N+r}, \\ u \text{ is the } H\text{-solution of } (1), \\ \theta \text{ is the weak solution to } (2). \end{array}\right\} \tag{10}$$

$$\mathfrak{S}_{ad} = \big\{ q \in C(\overline{\Omega}) \text{ such that } \quad (11)$$
$$|q(x) - q(y)| \le \omega(|x-y|),$$
$$\forall\, x, y, \in \Omega,\ |x-y| \le 1/2,$$
$$\omega(t) = k_0 / \log(|t|^{-1}),\ 1 < \alpha \le q(\cdot) \le \beta \text{ in } \overline{\Omega} \big\}\,. \quad (12)$$

We also say that $(B^0, u^0, \theta^0, p^0) \in BV(\Omega)^{N\times N} \times H_0^{1,p(\cdot)}(\Omega) \times W_0^{1,\gamma}(\Omega) \times C(\overline{\Omega})$ *is a quasi-optimal solution to the problem* (4)–(6) *if*

$$(B^0, u^0, \theta^0, p^0) \in \widehat{\Xi}_0(\tau) \text{ and}$$
$$J(B^0, u^0, \theta^0, p^0) = \inf_{(B,u,\theta,p) \in \widehat{\Xi}_0(\tau)} J(B, u, \theta, p),$$

and this tuple is called to be optimal if $p^0(\cdot) = \sigma(\theta^0(\cdot))$ *a.e. in* Ω. It is clear that $\widehat{\Xi}_0(\tau) \subset \Xi_0$ for $\tau = 0$ and, moreover, as we will see later on, the set $\widehat{\Xi}_0(\tau)$ is nonempty if only $\tau \ge \sqrt{|\Omega|}(\beta - \alpha)$. It is also worth to emphasize that the condition $p \in \mathfrak{S}_{ad}$ implies that $p(\cdot)$ has some additional regularity. Moreover, in view of the obvious relation $\lim_{t\to 0} |t|^\delta \log(|t|) = 0$ with $\delta \in (0,1)$, it is clear that $p \in C^{0,\delta}(\Omega)$ implies $p \in \mathfrak{S}_{ad}$. Because of this $p \in \mathfrak{S}_{ad}$ is often called a locally log-Hölder continuous exponent. Another point about benefit of the choice of the subset $\mathfrak{S}_{ad}$ is related with the following properties: (i) $\mathfrak{S}_{ad}$ is a compact subset in $C(\overline{\Omega})$ and thus provides uniformly convergent subsequences; (ii) Every cluster point p of a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ is a regular exponent (i.e. in this case the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$), which plays a key role in our further study; (iii) Because of the log-Hölder continuity of an exponent $p \in \mathfrak{S}_{ad}$, the corresponding weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ to the variational problem (7) is such that $|\nabla u|^{(1+\delta)p(\cdot)} \in L^1(\Omega)$ for some $\delta > 0$ and satisfies the estimate

$$\int_\Omega |\nabla u(x)|^{(1+\delta)p(x)}\, dx \le C \int_\Omega |g(x)|^{(1+\delta)p'(x)}\, dx + C, \quad (13)$$

where $\delta > 0$ and $C > 0$ depend only on Ω, α, N, k_0, and $\int_\Omega |g|^{p'}\, dx$. The property (13) is crucial for the proof of existence of quasi-optimal solutions to the problem (4)–(6). It is easy to show that if $u \in W^{1,p(\cdot)}(\Omega)$ is a solution to $\operatorname{div}(A(u)\nabla u) = \operatorname{div} g$ in $\mathcal{D}'(\Omega)$, then

$$\big(A(u)\nabla u, \nabla u\big) = \operatorname{div}\big((A(u)\nabla u - g)u\big) + g \cdot \nabla u,$$

also in $\mathcal{D}'(\Omega)$, where

$$A(u) = |\nabla u(x)|^{p(x)-2}$$
$$\text{or} \quad A(u) = |\nabla u|^{p(x)-2} + \varepsilon |\nabla u|^{\beta - 2}.$$

As a result, it allows to deduce the existence of the unique weak solution to the variational problem

$$-\operatorname{div}(B\nabla\theta) = \operatorname{div}\big((A(u)\nabla u - g)u\big) +$$
$$(g, \nabla u) \quad \text{in } \mathcal{D}'(\Omega)$$

which is also the SOLA to the Dirichlet BVP

$$-\operatorname{div}(B\nabla\theta) = |\big(A(u)\nabla u, \nabla u\big)| \quad \text{in } \Omega, \quad \theta|_{\partial\Omega} = 0.$$

Our main goal in this paper is to present the "approximation approach", based on the perturbation theory of extremal problems and the concept of fictitious controls. With that in mind, we make use of the following family of approximated problems: Minimize $J_{\varepsilon,\tau}(B, u, \theta, p)$, where

$$J_{\varepsilon,\tau}(B, u, \theta, p) = \int_\Omega |\theta - \theta_d|^r\, dx$$
$$+ \frac{1}{\varepsilon} \mu_\tau \left(\int_\Omega |p - \sigma(\theta)|^2\, dx \right) \quad (14)$$

subject to the constraints

$$\operatorname{div}\Big(|\nabla u|^{p(x)-2} \nabla u + \varepsilon |\nabla u|^{\beta-2} \nabla u \Big) = \operatorname{div} g \quad \text{in } \Omega, \quad (15)$$

$$u|_{\partial\Omega} = 0, \quad (16)$$
$$-\operatorname{div}(B\nabla\theta) =$$
$$= \operatorname{div}\Big[\Big(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u - g \Big) u \Big] \quad (17)$$
$$+ (g, \nabla u) \quad \text{in } \Omega, \quad \theta|_{\partial\Omega} = 0, \quad (18)$$
$$B \in \mathfrak{B}_{ad}, \quad p \in \mathfrak{S}_{ad}. \quad (19)$$

Here, the function $\mu_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as follows

$$\mu_\tau(s) = 0 \text{ if } 0 \le s \le \tau^2$$

and

$$\mu_\tau(s) = s - \tau^2 \text{ if } s > \tau^2.$$

There are several principle points in the statement of approximated problem (14)–(19) that should be emphasized. The first one is related with $\varepsilon\Delta_\beta$-regularization of $p(\cdot)$-Laplacian. Though this is a standard trick in order to establish the existence of H-solution to the Dirichlet problem (15) with a given exponent $p(\cdot)$, however, this approach does not allow to arrive at the existence of a weak solution $(u,\theta) \in H_0^{1,p(\cdot)}(\Omega) \times W_0^{1,\gamma}(\Omega)$ to the thermistor problem (1)–(3). This can be done if only the exponent $p(\cdot) = \sigma(\theta(\cdot))$ is regular, i.e. if the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, and the energy density $|\nabla u(\cdot)|^{p(\cdot)}$ belongs to the space $L^{1+\delta}(\Omega)$ for some $\delta > 0$ so that the equation (18) holds in the sense of the distributions. With that in mind we consider the condition $p \in \mathfrak{S}_{ad}$ as an additional option for the regularization of the original OCP. The another point that should be indicated, is related with some relaxation of the equation (2). Namely, it is easy to see that after the formal transformations, the equation (2) can be transformed to the following one

$$-\operatorname{div}(B\nabla\theta) = \operatorname{div}\left[\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right)u\right] + (g,\nabla u) \tag{20}$$

in $\mathcal{D}'(\Omega)$. The benefit of such representation and condition $p \in \mathfrak{S}_{ad}$ is the fact that, due to the estimate (13), the expression $\left(|\nabla u|^{\sigma(\theta)-2}\nabla u - g\right)u$ under the divergence sign in (20) is integrable with degree greater than 1. As follows from our further analysis, this property plays an important role in the study of OCP (4)–(6) and we consider the representation (20) as some relaxation of the relation (2).

Some Auxiliary Results

Orlicz spaces

Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \le p(x) \le \beta < \infty$ a.e. in Ω, where α and β are given constants. Let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ be the corresponding conjugate exponent. It is clear that $\beta' \le p'(\cdot) \le \alpha'$ a.e. in Ω, where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p(\cdot)}(\Omega)^N$ the set of all measurable functions $f(x)$ on Ω such that $\int_\Omega |f(x)|^{p(x)}\,dx < \infty$. Then $L^{p(\cdot)}(\Omega)^N$ is a reflexive separable Banach space with respect to the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1}f) \le 1\right\}, \tag{21}$$

where

$$\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx.$$

The dual of $L^{p(\cdot)}(\Omega)^N$ with respect to $L^2(\Omega)$-inner product will be denoted by $L^{p'(\cdot)}(\Omega)^N$. The following estimates are well-known: if $f \in L^{p(\cdot)}(\Omega)^N$ then

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N}, \tag{22}$$

$$\text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} > 1,$$

$$\|f\|^\beta_{L^{p(\cdot)}(\Omega)^N} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N}, \tag{23}$$

$$\text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} < 1,$$

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} = \int_\Omega |f(x)|^{p(x)}\,dx, \tag{24}$$

$$\text{if } \|f\|_{L^{p(\cdot)}(\Omega)^N} = 1,$$

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)^N} - 1 \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)^N} + 1, \tag{25}$$

$$\|f\|_{L^\alpha(\Omega)^N} \le (1+|\Omega|)^{1/\alpha}\,\|f\|_{L^{p(\cdot)}(\Omega)^N}. \tag{26}$$

Moreover, due to the duality method, it can be shown that

$$\|f\|_{L^{p(\cdot)}(\Omega)^N} \le (1+|\Omega|)^{1/\beta'}\,\|f\|_{L^\beta(\Omega)^N}, \tag{27}$$

$$\beta' = \frac{\beta}{\beta-1}, \quad \forall\, f \in L^\beta(\Omega)^N.$$

We make use of the following results.

Lemma 1. [[Zhikov 2011], Lemma 13.3] *If a sequence $\{f_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p(\cdot)}(\Omega)$ and $f_k \rightharpoonup f$ in $L^\alpha(\Omega)$ as $k\to\infty$, then $f \in L^{p(\cdot)}(\Omega)$ and $f_k \rightharpoonup f$ in $L^{p(\cdot)}(\Omega)$, i.e.*

$$\lim_{k\to\infty}\int_\Omega f_k\varphi\,dx = \int_\Omega f_k\varphi\,dx, \quad \forall\,\varphi \in L^{p'(\cdot)}(\Omega).$$

Lemma 2. *Let $\{p_k\}_{k\in\mathbb{N}} \subset \mathfrak{S}_{ad}$ and $p \in \mathfrak{S}_{ad}$ be such that $p_k(\cdot) \to p(\cdot)$ uniformly in $\overline{\Omega}$ as $k\to\infty$. If a sequence $\left\{\|f_k\|_{L^{p_k(\cdot)}(\Omega)}\right\}_{k\in\mathbb{N}}$ is bounded and $f_k \rightharpoonup f$ in $L^\alpha(\Omega)$ as $k\to\infty$, then $f \in L^{p(\cdot)}(\Omega)$.*

On the weak convergence of fluxes to flux

A typical situation arising in the study of most optimization problems and which is of fundamental importance in many other areas of nonlinear analysis, can be stated as follows: we have the weak convergence $u_k \rightharpoonup u$ in some Sobolev space $W^{1,\alpha}(\Omega)$ with $\alpha > 1$ and we have the weak convergence of fluxes $A_k(\cdot, \nabla u_k) \rightharpoonup z$ in the Lebesgue space $L^\delta(\Omega)$, $\delta > 1$, where by flux we mean the vector under the divergence sign in an elliptic equation (in our case it is $A_k(\cdot, \nabla u_k) = |\nabla u_k|^{p_k(\cdot)-2}\nabla u_k$ or $A_k(\cdot, \nabla\theta_k) = B_k(\cdot)\nabla\theta_k$). Then the problem is to show that $z = A(\cdot, \nabla u)$, although the validity of this equality is by no means obvious at this stage.

Assume that the fluxes $A_k(x, \xi)$ satisfy the following conditions:

$$A_k : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \tag{28}$$

are Carathéodory vector-valued functions,

$$(A_k(x,\xi) - A_k(x,\eta), \xi - \eta)_{\mathbb{R}^N} \geq 0, \tag{29}$$

$$A_k(x, 0) = 0, \text{ for a.e. } x \in \Omega \text{ and } \forall\, \xi, \eta \in \mathbb{R}^N, \tag{30}$$

$$|A_k(x,\xi)|^{\beta'} \leq C_1|\xi|^\beta + C_2, \tag{31}$$

$$\lim_{k\to\infty} A_k(x,\xi) = A(x,\xi)$$

$$\text{for a.e. } x \in \Omega \text{ and } \forall\, \xi \in \mathbb{R}^N. \tag{32}$$

Let $\{v_k\}_{k\in\mathbb{N}}$ and $\{A_k(\cdot, v_k)\}_{k\in\mathbb{N}}$ be weakly convergent sequences in $L^1(\Omega)^N$, and let v and z be their weak L^1-limits, respectively. In order to clarify the conditions under which the equality $z = A(x, v)$ holds and the fluxes $A_k(\cdot, v_k)$ weakly converge to the flux $A(\cdot, v)$, we cite the following result.

Lemma 3.[[Zhikov 2011], Theorem 4.6] *Assume that $\{u_k\}_{k\in\mathbb{N}}$ and $\{A_k(\cdot, \nabla u_k)\}_{k\in\mathbb{N}}$ are the sequences such that conditions (29)–(31) hold true and*

(i) $u_k \rightharpoonup u$ in $W^{1,\alpha}(\Omega)$ and $u_k \in W^{1,\beta}(\Omega)$ for all $k \in \mathbb{N}$;

(ii) $\sup_{k\in\mathbb{N}} \|A_k(\cdot, \nabla u_k)\|_{L^{\beta'}(\Omega)^N} < +\infty$;

(iii) $\sup_{k\in\mathbb{N}} \|\, (A_k(\cdot, \nabla u_k), \nabla u_k)_{\mathbb{R}^N} \,\|_{L^1(\Omega)} < +\infty$;

(iv) the exponents α and β are related by the condition

$$1 < \alpha \leq \beta < \begin{cases} +\infty, & \text{if } \alpha \geq N-1, \\ \frac{\alpha(N-1)}{N-1-\alpha}, & \text{if } \alpha < N-1. \end{cases} \tag{33}$$

Then, up to a subsequence, the fluxes weakly converge to the flux

$$A_k(\cdot, \nabla u_k) \rightharpoonup A(\cdot, \nabla u) \quad \text{in} \quad L^{\beta'}(\Omega)^N.$$

It is worth to note that in the case of equality $\alpha = \beta$, Lemma 3 becomes the well-known result of Tartar and Murat also known as the div-curl Lemma.

On Consistency of Approximated Optimal Control Problems in Coefficients

Let ε be a small parameter. Assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers which converges to 0. Let $\tau \geq 0$ be a given constant. We consider the collection of approximated optimal control problems in coefficients for nonlinear elliptic equations (14)–(19). For every $\varepsilon > 0$ we denote by $\widehat{\Xi}_\varepsilon$ the set of all feasible points to the problem (14)–(19).

Definition 4. *We say that (B, u, θ, p) is a feasible point to the problem (14)–(19) if $B \in \mathfrak{B}_{ad}$, $p \in \mathfrak{S}_{ad}$, and $u \in W_0^{1,\beta}(\Omega)$ and $\theta \in W_0^{1,\gamma}(\Omega)$ are the solutions to the following variational problems*

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \tag{34}$$

$$-\operatorname{div}(B\nabla\theta) = (g, \nabla u)$$
$$+\operatorname{div}\left[\left(|\nabla u|^{p(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u - g\right) u\right] +_{\mathbb{R}^N}, \tag{35}$$

where each of these relations we consider as equalities in $\mathcal{D}'(\Omega)$

The following lemma reflecting the consistency of approximated optimal control problem (14)–(19) [Kupenko and Manzo 2015].

Lemma 4. Let $\theta_d \in L^r(\Omega)$ and $g \in L^\infty(\Omega)^N$ be given distributions with $r \in \left(1, \frac{N}{N-2}\right)$, let $\sigma \in C(\mathbb{R})$ be a given function satisfying the conditions (6), and let τ be an arbitrary non-negative value. Then the approximated optimal control problem (14)–(19) is consistent for each $\varepsilon > 0$, i.e. $\widehat{\Xi}_\varepsilon \neq \emptyset$.

As an obvious consequence of the reasoning given in proof of Lemma 4, we can draw the following inference.

Corollary. *For given $\tau \geq 0$ and $\varepsilon > 0$, let $(B, u, \theta, p) \in \widehat{\Xi}_\varepsilon$ be a feasible point to the problem (14)–(19). Then θ the unique SOLA to the Dirichlet problem*

$$-\xi \operatorname{div}(\nabla\theta_{\varepsilon,k}) = |\nabla\widehat{u}|^\beta \;\text{ in }\; \Omega, \quad \widehat{\theta}\Big|_{\partial\Omega} = 0. \tag{36}$$

The next results are crucial for our analysis.

Lemma 5. *The set of fictitious controls $\mathfrak{S}_{ad}$ is convex, bounded and compact with respect to the strong topology of $C(\overline{\Omega})$.*

Lemma 6. *Let $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ be an arbitrary sequence. Then there exist a distribution $u_\varepsilon \in W_0^{1,\beta}(\Omega)$, an exponent $p_\varepsilon \in \mathfrak{S}_{ad}$, and a subsequence of $\{u_{\varepsilon,k}\}_{k\in\mathbb{N}}$, still denoted by the suffix (ε, k), such that*

$$\varepsilon\|u_\varepsilon\|^\beta_{W_0^{1,\beta}(\Omega)} \le 2^{\alpha'+1}\left(\int_\Omega |g|^{\alpha'}\,dx + |\Omega|\right), \quad (37)$$

$$u_{\varepsilon,k} \rightharpoonup u_\varepsilon \ \text{ in } \ W_0^{1,\beta}(\Omega) \ \text{ as } \ k\to\infty, \quad (38)$$

$$u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega). \quad (39)$$

Lemma 7. *Let $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ be an arbitrary sequence, and let $p_\varepsilon \in \mathfrak{S}_{ad}$ and $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ be such that $p_{\varepsilon,k}(\cdot) \to p_\varepsilon(\cdot)$ uniformly in $\overline{\Omega}$ and $u_{\varepsilon,k} \rightharpoonup u_\varepsilon$ in $W_0^{1,\beta}(\Omega)$ as $k\to\infty$. Then, up to a subsequence, we have the weak convergence of fluxes to a flux:*

$$|\nabla u_{\varepsilon,k}|^{p_{\varepsilon,k}-2}\nabla u_{\varepsilon,k} + \varepsilon|\nabla u_{\varepsilon,k}|^{\beta-2}\nabla u_{\varepsilon,k}$$
$$\rightharpoonup |\nabla u_\varepsilon|^{p_\varepsilon-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon \quad \text{in} \quad L^{\beta'}(\Omega)^N. \quad (40)$$

Lemma 8. *Let $p_\varepsilon \in \mathfrak{S}_{ad}$ and $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ be as in Lemma 8. Then u_ε is the unique weak solution to the Dirichlet problem*

$$\operatorname{div}\left(|\nabla u|^{p_\varepsilon(x)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right) = \operatorname{div} g \quad \text{in} \quad \Omega,$$
$$u|_{\partial\Omega} = 0.$$

Lemma 9. *Let $\{(B_{\varepsilon,k}, u_{\varepsilon,k}, \theta_{\varepsilon,k}, p_{\varepsilon,k})\}_{k\in\mathbb{N}} \subset \widehat{\Xi}_\varepsilon$ be a sequence such that $B_{\varepsilon,k} \overset{*}{\rightharpoonup} B_\varepsilon$ in $BV(\Omega)^{N\times N}$ and $\theta_{\varepsilon,k} \rightharpoonup \theta_\varepsilon$ in $W_0^{1,\gamma}(\Omega)$ for some $\gamma \in [1, \frac{N}{N-1})$. Then we have*

$$\lim_{k\to\infty}\int_\Omega (B_{\varepsilon,k}\nabla\theta_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\,dx = \int_\Omega (B_\varepsilon\nabla\theta_\varepsilon, \nabla\varphi)_{\mathbb{R}^N}\,dx, \quad (41)$$

for all $\varphi \in C_0^\infty(\Omega)$.

To conclude this section, we give the existence result for the approximated OCP (14)–(19).

Theorem 1. Let $\theta_d \in L^r(\Omega)$ and $g \in L^\infty(\Omega)^N$ be given distributions with $r \in \left(1, \frac{N}{N-2}\right)$, let $\sigma \in C(\mathbb{R})$ be a given function satisfying the conditions (6), and let τ be an arbitrary non-negative value. Then the optimal control problem (14)–(19) admits at least one solution for each $\varepsilon > 0$.

Main results

The main result of this paper is the following theorem, where we claim that if the OCP (4)–(6) has a sufficiently regular feasible point, then there exist optimal solutions to the OCP and some of them are the limit as $\varepsilon \searrow 0$ of optimal solutions to (14)–(19).

Theorem 2. *Let Ω be an open bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary. Assume that $\widehat{\Xi}_0(\tau) \ne \emptyset$ for $\tau = 0$, i.e. there exist a matrix $\widehat{B} \in \mathfrak{B}_{ad}$, an exponent $\widehat{p} \in \mathfrak{S}_{ad}$, and a weak solution to the thermistor problem (1)–(3) $(\widehat{u}, \widehat{\theta}) \in W_0^{1,\sigma(\widehat{\theta}(\cdot))}(\Omega) \times W_0^{1,\gamma}(\Omega)$ with $\gamma = \frac{Nr}{N+r}$ and $B(\cdot) = \widehat{B}(\cdot)$ such that $\widehat{p} = \sigma(\widehat{\theta})$ almost everywhere in Ω. Then OCP (4)–(6) has a non-empty set of optimal solutions and some of them can be attained in the following way*

$$B^0_\varepsilon \overset{*}{\rightharpoonup} B^0 \ \text{ in } \ BV(\Omega)^{N\times N}, \quad u^0_\varepsilon \rightharpoonup u^0 \ \text{ in } \ W_0^{1,\alpha}(\Omega), \quad (42)$$

$$\theta^0_\varepsilon \rightharpoonup \theta^0 \ \text{ in } \ W_0^{1,\gamma}(\Omega), \quad p^0_\varepsilon \to p^0 \ \text{ uniformly on } \ \overline{\Omega}, \quad (43)$$

as $\varepsilon \to 0$, where $(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon)$ are the solutions to the approximated problems (14)–(19) with $\tau = \varepsilon$ in (14).

It is clear that the condition $\widehat{p} = \sigma(\widehat{\theta})$ in the statement of Theorem 1, where $\widehat{p}$ has logarithmic modulus of continuity, imposes some additional and rather special constraint on the function $\sigma \in C(\mathbb{R})$. The principle point here is the fact that this relation has to be valid for a particular function $\widehat{\theta}$ and it is not required that the function $\sigma(\theta(\cdot))$ must be at least continuous for every solution $\theta \in W_0^{1,\gamma}(\Omega)$ of (2). It is rather delicate problem to guarantee the fulfilment of the equality $\widehat{p} = \sigma(\widehat{\theta})$ by the direct description of function $\sigma \in C(\mathbb{R})$ even if we make use of the “typical” assumption: σ is a Lipschitz continuous function.

Since it is unknown whether OCP (4)–(6) is solvable or the main assumptions of Theorem 1 are satisfied, it is reasonable to show that this problem admits the quasi-optimal solutions and they can be attained (in some sense) by optimal solutions to special approximated problems. We prove the following result.

Theorem 3. *Let* $\left\{(B^0_\varepsilon, u^0_\varepsilon, \theta^0_\varepsilon, p^0_\varepsilon)\right\}_{\varepsilon>0}$ *be an arbitrary sequence of optimal solutions to the approximated problems (14)–(19). Assume that either there exists a constant* $C^* > 0$ *satisfying condition*

$$\limsup_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p) \le C^* < +\infty$$

or $\tau \ge \sqrt{|\Omega|}(\beta-\alpha)$, *where* $\widehat{\Xi}_\varepsilon$ *stands for the set of feasible solutions to the problem (14)–(19). Then any cluster tuple* $\left(B^0, u^0, \theta^0, p^0\right)$ *in the sense of convergence* (42)–(43) *is a quasi-optimal solution of the OCP* (4)–(6). *Moreover, in this case the following variational property holds*

$$\lim_{\varepsilon\to 0} \inf_{(B,u,\theta,p)\in\widehat{\Xi}_\varepsilon} J_{\varepsilon,\tau}(B,u,\theta,p)$$
$$= J\left(B^0, u^0, \theta^0, p^0\right) = \inf_{(B,u,\theta,p)\in\widehat{\Xi}_0(\tau)} J(B,u,\theta,p).$$

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