ON VECTOR-VALUED APPROXIMATION OF STATE CONSTRAINED OPTIMAL CONTROL PROBLEMS FOR NONLINEAR HYPERBOLIC CONSERVATION LAWS

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Abstract. We study one class of nonlinear fluid dynamic models with controls in the initial condition and the source term. The model is described by a nonlinear inhomogeneous hyperbolic conservation law with state and control constraints. We consider the case when the greatest lower bound of the cost functional can be unattainable on the set Ξ of admissible pairs or the set Ξ is possibly empty. Using the methods of vector-valued optimization theory, we show that this optimal control problem admits the existence of the so-called weakened approximate solution which can be interpreted as generalized solution to some vector optimization problem of special form.

1. Introduction

In recent years, the interest of the scientific community for supply chains modeling (see [5, 11–13]) and control has become greater and greater, in order to optimize the production processes. Several questions can be faced in the design of optimal supply chain: the control of the maximum processing rates, or the processing velocities, or the input flow in such way to minimize the value the queues attain and to achieve an expected outflow, or in the case of a supply network the optimal routing of parts such that inventory costs are minimized. The aim of this article is to analyze an optimal control problem described by a nonlinear conservation law with state and control constraints. It is well known that the conservation laws, taking the form of hyperbolic partial differential equations, appear in a variety of applications that offer control or identification of parameters, including the control of traffic and water flows, the modeling of supply chains, gas pipelines, blood flows, etc. The analysis of conservation laws is a very active research area. The main difficulty in dealing with them is the fact that the solution of such systems may develop discontinuities (after a finite time), that propagate in

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time even for smooth initial and boundary conditions (see [4, 20, 21]). To the best knowledge of authors, the existence of optimal controls and their approximation to the problems of conservation laws with state constraints is an open problem even for the simplest situation.

In [8] considering the case when the objective space is the Banach space $L^p_{loc}(R^N)$ partially ordered by the natural ordering cone of positive elements, sufficient conditions for the existence of efficient controls have been derived. When the original control problem is not regular and may fail to have the entropy solutions, the regularization approach and the existence of the so-called efficient regularizators to the original vector valued optimization problem have been discussed. Influx-rates in the equation taking the form of impulse functions has been treated in [10]. Using the vanishing viscosity method and the so-called principle of fictitious controls, it has been shown that entropy solutions to the original Cauchy problem can be approximated by optimal solutions of special optimization problems.

In this paper we carry out our analysis for the following class of optimal control problems (OCPs) with state and control constraints

$$\text{Minimize} \quad J(u,y) \ \text{ subject to } \ u \in \mathcal{U}_{ad},\ y \in \mathcal{Y}_{ad}, \tag{1}$$

where $y = y(u)$ is the entropy solution of the nonlinear inhomogeneous conservation law

$$y_t + \sum_{i=1}^{n} (f_i(y))_{x_i} = g(t,x,y,u_1), \quad (t,x) \in (0,T) \times \mathbb{R}^n \stackrel{\text{def}}{=} \Omega_T, \tag{2}$$

$$y(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \tag{3}$$

Here, J is an objective functional, $u = (u_0, u_1)$ is the control in the initial condition and the source term, $\mathcal{U}_{ad}$ is the set of admissible controls, and $\mathcal{Y}_{ad}$ is the set of admissible states. Hereinafter we specify these sets as follows

$$\begin{aligned} \mathcal{U}_{ad} = \Big\{ & u = (u_0,u_1) \in L^\infty(\mathbb{R}^n) \times L^\infty(\Omega_T)^m \ : \\ & \max\left\{ \|u_0\|_{BV(\mathcal{O})}, \|u_1\|_{BV((0,T)\times\mathcal{O})^m} \right\} \le \gamma \quad \forall\, \mathcal{O} \subset \mathbb{R}^n \Big\}; \end{aligned} \tag{4}$$

$$\mathcal{Y}_{ad} = \Big\{ y \in C([0,T]; L^1_{loc}(\mathbb{R}^n)) \ : \ l(y(t,x)) \le 0 \quad \text{a.e. in } \ D_T \Big\}, \tag{5}$$

where $l : \mathbb{R} \to \mathbb{R}$ is a continuous operator. In mostly applications this operator takes the form $l(y) = y - \alpha$, with some positive constant α.

Remark 1. We note that the assumption of boundedness of the total variation in (4) is rather strict. It means that the original optimal control problem (1)–(5) admits at least one solution provided admissible controls are measurable functions only. Since our main aim in this paper is to study the asymptotic behaviour of approximative solutions to (1)–(5), which we take in a special form (see (4.7)), and to show that any cluster point of such solutions can be considered as a minimizer for the lower semicontinuous regularization of the original problem, we use the property that any sequence of admissible controls is compact with respect to the strong convergence in L^1 norm. This is the reason we assume that, for simplicity, the admissible controls are described by conditions (4). In fact the boundedness of the total variation gives the strong compactness in L^1, which is the crucial hypothesis. In spite of the fact that the introduced assumption (4) is rather restrictive, the problem (1)–(4) admits interesting applications such as optimization of traffic flows on networks, of supply chains, etc. (see, for instance, [5, 9, 12–14]).

We also assume that the flux-function

$$f = (f_1, f_2, \dots, f_n) : \mathbb{R} \to \mathbb{R}^n \quad \text{ is locally Lipschitz.} \tag{6}$$

Furthermore, $g \in L^\infty(\Omega_T; C^{0,1}_{loc}(\mathbb{R} \times \mathbb{R}^n))$ and for all $M_u > 0$ there are constants $C_1, C_2 > 0$ such that

$$g(t,x,y,u_1)\,\mathrm{sgn}\,(y) \le C_1 + C_2|y| \quad \forall\,(t,x,y,u_1) \in \Omega_T \times \mathbb{R} \times [-M_u, M_u]^m\,, \tag{7}$$

where sgn is the sign function.

Our prime interest is to discuss the so-called vector-valued approximation approach to the construction of weakened approximate solutions for the above problem.

We admit that the original OCP (1)–(3) may fail to have an exact solution (u^{opt}, y^{opt}) — namely, the greatest lower bound of the cost functional can be unattainable on the set Ξ of admissible pairs or the set Ξ is possibly empty. To construct solutions close to the set of admissible solutions and guaranteeing the proximity of the cost functional to its greatest lower bound, we apply the so-called vector-valued approximation of the original OCP. To do so, we consider a special vector optimization problem and show that this problem possesses some characteristic properties. In particular, it enables us to study the regularity of the original OCP, leading to the construction of the so-called weakened approximate solutions.

2. Notation and Preliminaries

Let $n \ge 1$ and $m \ge 1$ be two fixed positive integers. Let D be a bounded open domain in $\mathbb{R}^n$. For a given $T > 0$, we set $\Omega_T = (0,T) \times \mathbb{R}^n$ and $D_T = (0,T) \times D$. Let $L^p_{loc}(\Omega_T)$, with $1 \le p \le \infty$, be the locally convex space of all measurable functions $q : \Omega_T \to \mathbb{R}$ such that $q|_{(0,T)\times K} \in L^p((0,T) \times K)$ for all compact sets $K \subset \mathbb{R}^n$.

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$ and let $v : \mathcal{O} \to \mathbb{R}$ be an element of $L^1(\mathcal{O})$. Define

$$\int_{\mathcal{O}} |Dv| = \sup\Big\{ \int_{\mathcal{O}} v\,\mathrm{div}\,\vec{\varphi}\,dx \ : \ \vec{\varphi} \in C^1_0(\mathcal{O})^n,\ \|\varphi(x)\|_{C(\mathcal{O})^n} \le 1 \text{ for } x \in \mathcal{O} \Big\}.$$

According to the Radon-Nikodym theorem, if $\int_{\mathcal{O}} |Dv| < +\infty$ then the distribution Dv is a measure and there exist a function $v' \in L^1(\mathcal{O})^n$ and a measure $D_s v$, singular with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n \lfloor \mathcal{O}$ restricted to $\mathcal{O}$, such that

$$Dv = v' \mathcal{L}^n \lfloor \mathcal{O} + D_s v.$$

Definition 2. A function $v \in L^1(\mathcal{O})$ is said to have a bounded variation in $\mathcal{O}$ if the derivative Dv exists in the sense of distributions and belongs to the class of Radon measures with bounded total variation, i.e. $\int_{\mathcal{O}} |Dv| < +\infty$. By $BV(\mathcal{O})$ we denote the space of all functions in $L^1(\mathcal{O})$ with bounded variation.

Under the norm

$$\|v\|_{BV(\mathcal{O})} = \|v\|_{L^1(\mathcal{O})} + \int_{\mathcal{O}} |Dv|,$$

$BV(\mathcal{O})$ is a Banach space. It is well-known the following compactness result for BV-functions.

Proposition 3. *The uniformly bounded sets in BV-norm are relatively compact in $L^1(\mathcal{O})$, that is, if $\{v_k\}_{k=1}^{\infty} \subset BV(\mathcal{O})$ and $\sup_{k\in\mathbb{N}} \|v_k\|_{BV(\mathcal{O})} < +\infty$, then there exists a subsequence of $\{v_k\}_{k=1}^{\infty}$ strongly converging in $L^1(\mathcal{O})$ to some $v \in BV(\mathcal{O})$.*

Definition 4. A sequence $\{v_k\}_{k=1}^{\infty} \subset BV(\mathcal{O})$ weakly converges to some $v \in BV(\mathcal{O})$, and we write $v_k \rightharpoonup v$ iff the two following conditions hold: $v_k \to v$ strongly in $L^1(\mathcal{O})$, and $Dv_k \rightharpoonup Dv$ weakly* in $\mathcal{M}(\mathcal{O})$, where $\mathcal{M}(\mathbb{R}^n)$ stands for the set of all Radon measures on $\mathbb{R}^n$.

In the proposition below we give a compactness result related to this convergence, together with the lower semicontinuity property (see [15]).

Proposition 5. *Let $\{v_k\}_{k=1}^{\infty}$ be a sequence in $BV(\mathcal{O})$ strongly converging to some v in $L^1(\mathcal{O})$ and satisfying $\sup\limits_{k\in\mathbb{N}} \displaystyle\int_{\mathcal{O}} |Dv_k| < +\infty$. Then*

(i) *$v \in BV(\mathcal{O})$ and $\displaystyle\int_{\mathcal{O}} |Dv| \le \liminf_{k\to\infty} \int_{\mathcal{O}} |Dv_k|$;*
(ii) *$v_k \rightharpoonup v$ in $BV(\mathcal{O})$.*

Throughout the paper we will often use the concepts of the weak and strong convergence in $L^1(D_T)$. Let $\{g_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^1(D_T)$. We recall that $\{g_k\}_{k\in\mathbb{N}}$ is called equi-integrable on D_T, if for any $\delta > 0$ there is $\tau = \tau(\delta)$ such that $\int_S |g_k|\, dz < \delta$ for every measurable subset $S \subset D_T = (0,T) \times D$ of Lebesgue measure $|S| < \tau$. Then the following assertions are equivalent for $L^1(D_T)$-bounded sequences:

(i) a sequence $\{g_k\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(D_T)$;
(ii) the sequence $\{g_k\}_{k\in\mathbb{N}}$ is equi-integrable.

3. Statement of the Problem and Main Motivation

We focus on the fluid dynamic model, expressed by the Cauchy problem for nonlinear inhomogeneous conservation law (2)–(3). To begin with, we define the sets $\mathcal{U}_{ad}$ and $\mathcal{Y}_{ad}$ of admissible controls and states as in (4)–(5). As it was mentioned in Introduction, the characteristic feature of the initial value problem (2)–(3) is that even for arbitrary smooth functions u_0, u_1, and g, the solution of (2)–(3) may develop discontinuous after a finite time (see [20, 21]), which makes it necessary to consider weak solutions. At the same time, weak solutions of (2)–(3) are, in general, not unique and, in order to select the "physically" relevant solution, some additional conditions must be imposed. Following [14, 20, 21], we introduce the entropy-admissibility condition, coming from physical considerations, as follows.

Definition 6. For a given $u = (u_0, u_1) \in \mathcal{U}_{ad}$, a function $y \in L^\infty(\Omega_T)$ is called an entropy solution of (2)–(3) if for all $c \in \mathbb{R}$ and

$$\eta(\lambda) := |\lambda - c|, \quad q_i(\lambda) := \operatorname{sgn}(\lambda - c)\left(f_i(\lambda) - f_i(c)\right), \quad \forall\, i = 1, \dots, n \tag{8}$$

the entropy inequality

$$(\eta(y))_t + \sum_{i=1}^{n} (q_i(y))_{x_i} \le \operatorname{sgn}(y - c) g(t, x, y, u_1) \quad \text{in} \quad \mathcal{D}'(\Omega_T) \tag{9}$$

holds and if the initial data u_0 are assumed on the sense

$$\lim_{t \to 0+} \frac{1}{t} \int_0^t \|y(\tau, \cdot) - u_0\|_{L^1(K)}\, d\tau = 0 \quad \text{for all compact} \quad K \subset \mathbb{R}^n. \tag{10}$$

The existence and uniqueness of entropy solutions for (2)–(3) under assumptions given above was shown by Volpert [24] and Kruzhkov [20] (see also [6,25]). The following result collects important properties of the control-to-state mapping $u \to y(u)$

Theorem 7 ([25])**.** *Let $u = (u_0, u_1) \in \mathcal{U}_{ad}$ be given controls and let the conditions* (6)–(7) *hold. Then for every $u \in \mathcal{U}_{ad}$ there is at most one entropy solution*

$$y = y(u) \in C([0, T]; L^1_{loc}(\mathbb{R}^n)) \times L^\infty(\Omega_T)$$

satisfying (9)–(10)*. Moreover, the mapping*

$$\mathcal{U}_{ad} \ni u \to y(u) \in C([0, T]; L^1(D)) \tag{11}$$

is Lipschitz continuous and if g has a compact support $\operatorname{supp}_x(g) \subset\subset \mathbb{R}^n$ *w.r.t x then $y \in L^\infty(0, T; BV(D))$.*

Definition 8. We say that a pair

$$(u, y) \in \left[L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\Omega_T)\right] \times \left[C([0, T]; L^1_{loc}(\mathbb{R}^n)) \times L^\infty(\Omega_T) \times L^\infty(0, T; BV(D))\right]$$

is an admissible solution to the OCP (1)–(3) if $u = (u_0, u_1) \in \mathcal{U}_{ad}$, $y \in \mathcal{Y}_{ad}$, $J(u, y) < +\infty$, and $y = y(u)$ is the corresponding entropy solution to (2)–(3) in the sense of Definition 6.

Let Ξ be the set of all admissible pairs to the problem (1)–(3). We say that a pair (u^0, y^0) is optimal for the problem (1)–(3) if

$$(u^0, y^0) \in \Xi \quad \text{and} \quad J(u^0, y^0) = \inf_{(u,y)\in\Xi} J(u, y).$$

The following assumption is crucial in this section:

(A1): The OCP (1)–(3) is regular in the following sense: there exists at least one pair (u, y) such that $(u, y) \in \Xi$.

For our further analysis we set

$$\mathbb{U} = L^1(D) \times L^1(D_T), \quad \mathbb{Y} = C([0, T]; L^1(D)),$$
$$\mathbb{U}_{loc} = L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\Omega_T), \quad \mathbb{Y}_{loc} = C([0, T]; L^1_{loc}(\mathbb{R}^n)).$$

Then the sufficient conditions for the existence of an optimal solution to the problem (1)–(3) can be stated as follows.

Theorem 9. *Assume the initial assumptions* (6)–(7) *hold true and the cost functional* $J : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ *is sequentially lower semicontinuous with respect to the norm topology of* $\mathbb{U} \times \mathbb{Y}$. *Then the OCP* (1)–(3) *admits an optimal solution* $(u^0, y^0) \in \Xi$ *if and only if this problem is regular.*

Proof. To begin with, we show that the cost functional $J : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ is bounded below on the set Ξ. Let us assume the converse. Then there exists a sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}} \in \Xi$ such that $J(u_k, y_k) < -k$ for all $k \in \mathbb{N}$. By the initial assumptions, we have $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}_{ad}$; hence, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in

$$[BV_{loc}(\mathbb{R}^n) \times BV_{loc}((0, T) \times \mathbb{R}^n)] \cap [L^\infty(\mathbb{R}^n) \times L^\infty((0, T) \times \mathbb{R}^n)] .$$

Thus, we may assume that (see Proposition 3) $u_k \to u$ in $\mathbb{U}_{loc}$ and, therefore, $u \in \mathcal{U}_{ad}$. Since the mapping (11) is Lipschitz continuous, by the Ascoli-Arzelà Theorem it follows that $\{y_k = y(u_k)\}_{k\in\mathbb{N}} \in \mathcal{Y}_{ad}$ is a bounded sequence in $\mathbb{Y}_{loc}$ and there exists an element $y \in \mathbb{Y}_{loc}$ such that, passing to a subsequence if necessary, we obtain $y_k \to y$ in $\mathbb{Y}_{loc}$. Then, having used the sequential lower semicontinuity of J, we come to the contradiction

$$J(u, y) \le \liminf_{k\to\infty} J(u_k, y_k) < -\infty.$$

Thus, the cost functional $J : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ is bounded below on the set Ξ. Let $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a minimizing sequence for the original problem, i.e.

$$\lim_{k\to\infty} J(u_k, y_k) = \inf_{(u,y)\in\Xi} J(u, y) > -\infty.$$

Following the previous arguments, we may assume that there exists a pair (u^0, y^0) such that

$$u_k \to u^0 \text{ in } L^1(D) \times L^1(D_T) \quad \text{and} \quad y_k \to y^0 \text{ in } C([0,T]; L^1(D)). \tag{12}$$

Taking this fact into account, we can pass to the limit in (9)–(10) as $k \to \infty$. As a result, we immediately come to the conclusion that $y = y(u^0)$ is an entropy solution to the Cauchy problem (2)–(3) under $u = u^0$. In order to prove the inclusion $(u^0, y^0) \in \Xi$, we note that the property $(12)_2$ implies the convergence $y_k(t,x) \to y^0(t,x)$ almost everywhere in D_T. Hence, $y^0 \in \mathcal{Y}_{ad}$ by the continuity property of the operator $l : \mathbb{R} \to \mathbb{R}$. To conclude the proof, it remains to apply the lower semicontinuity property of J

$$-\infty < J(u^0, y^0) \le \underline{\lim}_{k\to\infty} J(u_k, y_k) = \inf_{(u,y)\in\Xi} J(u,y).$$

Thus, the pair (u^0, y^0) is optimal for the problem (1)–(3). □

As follows from Theorem 9, the existence of optimal solutions to the problem (1)–(3) can be obtained by using the compactness arguments and the regularity assumption (A1). However, because of the state constraints $y \in \mathcal{Y}_{ad}$ the regularity of the OCP (1)–(3) (see (A1)) is an open question even for the simplest situation. So, the first question to be answered for this problem is about admissibility: does there exist at least one pair $(u, y) \in \mathbb{U}_{loc} \times \mathbb{Y}_{loc}$ such that $u = (u_0, u_1) \in \mathcal{U}_{ad}$, $y \in \mathcal{Y}_{ad}$, $J(u, y) < +\infty$, and $y = y(u)$ is an entropy solution to (2)–(3) in the sense of Definition 6? In fact, one needs the set of admissible pairs to be sufficiently rich in some sense, otherwise the OCP (1)–(3) becomes trivial. However, from a mathematical point of view, to deal directly with all constraints above presented is typically very difficult and, except for some special cases, this question is largely open [17]. Nevertheless, in many applications it is an important task to find an admissible (or at least an approximately admissible, in a sense to be made precise) pair when both entropy and state constraints for the control-state pairs are given. On the other hand, the set Ξ of admissible solutions may be very "thin" and it is possible that the original problem has no solutions. In view of this, it is reasonable to weaken the requirements on admissible solutions to the original OCP. In particular, it would also be reasonably to assume that the optimality property for the solutions $(u, y(u))$ holds not strictly but rather with some (possibly high) accuracy. Moreover, the greatest lower bound of the cost functional is often unattainable on the given set Ξ. Nevertheless, the absence of a minimum of the functional does not mean that the problem does not make any sense (see, e.g., [17]), since its greatest lower bound exists and hence can be approached with some accuracy. Thus, an extremal problem may have an approximate or suboptimal solution even if it is not solvable.

4. On Vector-Valued Regularization of OCP (1.1)–(1.3)

Taking into account the motivation given above, in this section we will issue from the supposition that the original OCP (1)–(3) may fail to have an exact solution $(u^{opt}, y^{opt}) \in \Xi$ — namely, the greatest lower bound of the cost functional can be unattainable on the set Ξ of admissible pairs or the set Ξ is possibly empty. To construct solutions close to the set of admissible solutions and guaranteeing the proximity of the cost functional to its greatest lower bound, we apply the so-called vector-valued approximation of the original OCP. To do so, we introduce some optimality notion in partially ordered spaces.

We associate with the set $D_T = (0,T) \times D$ the objective space $L^1(D_T)$. By default we suppose that $L^1(D_T)$, as topological space, is endowed with the weak topology. Let τ be the weak topology of $L^1(D_T)$. Given a subset $S \subset L^1(D_T)$, we denote by $\operatorname{int}_\tau S$ and $\operatorname{cl}_\tau S$ its interior and closure with respect to the τ-topology, respectively. We also assume that $L^1(D_T)$ is partially ordered by the natural ordering cone of positive elements Λ, which is defined as

$$\Lambda = \left\{ f \in L^1(D_T) \ : \ f(x) \geq 0 \text{ almost everywhere on } \ \Omega \right\}. \qquad (13)$$

Definition 10. (see [16]) An element $y^* \in S \subset L^1(D_T)$ is said to be minimal of the set S, if there is no $y \in S$ such that $y \leq_\Lambda y^*,\ y \neq y^*$, that is

$$S \cap (y^* - \Lambda) = \{y^*\}.$$

Let $\operatorname{Min}_\Lambda(S)$ denote the family of all minimal elements of S. Let us introduce two singular elements $-\infty_\Lambda$ and $+\infty_\Lambda$ in $L^1(D_T)$. We assume that these elements satisfy the following conditions:

$$1) -\infty_\Lambda \preceq y \preceq +\infty_\Lambda,\ \forall y \in L^1(D_T); \quad 2) +\infty_\Lambda + (-\infty_\Lambda) = 0.$$

Let $Y^\bullet$ denote the semi-extended Banach space: $Y^\bullet = L^1(D_T) \cup \{+\infty_\Lambda\}$ assuming that $\|+\infty_\Lambda\|_{L^1(D_T)} = +\infty$ and $y - \lambda(+\infty_\Lambda) = -\infty\ \forall\, y \in L^1(D_T)$ and $\forall\, \lambda \in \mathbb{R}_+$.

Definition 11. We say that a set E is the efficient infimum of a set $S \subset L^1(D_T)$ with respect to the τ-topology of $L^1(D_T)$ (or shortly (Λ,τ)-infimum), if E is the collection of all minimal elements of $\operatorname{cl}_\tau S$ in the case when this set is non-empty, and E is equal to $\{-\infty_\Lambda\}$ in the opposite case.

Hereinafter the (Λ,τ)-infimum for S will be denoted by $\operatorname{Inf}^{\Lambda,\tau} S$. Thus, in view of the definition given above, we have

$$\operatorname{Inf}^{\Lambda,\tau} S := \begin{cases} \operatorname{Min}_\Lambda(\operatorname{cl}_\tau S), & \operatorname{Min}_\Lambda(\operatorname{cl}_\tau S) \neq \emptyset, \\ -\infty_\Lambda, & \operatorname{Min}_\Lambda(\operatorname{cl}_\tau S) = \emptyset. \end{cases}$$

Let X_∂ be a nonempty subset of a Banach space X, and $I : X_\partial \to L^1(D_T)$ be some mapping. Note that the mapping $I : X_\partial \to L^1(D_T)$ can be associated with its natural extension $\widehat{I} : X \to Y^\bullet$ to the entire space X, where

$$\widehat{I}(x) = \begin{cases} I(x), & x \in X_\partial, \\ -\infty_\Lambda, & x \notin X_\partial. \end{cases} \tag{14}$$

We say that a mapping $I : X_\partial \to Y^\bullet$ is bounded below if there exists an element $z \in L^1(D_T)$ such that $z \leq_\Lambda I(x)$ for all $x \in X_\partial$.

Definition 12. A subset A of $L^1(D_T)$ is said to be the efficient infimum of a mapping

$$I : \ X_\partial \to L^1(D_T)$$

with respect to the τ-topology of $L^1(D_T)$ and is denoted by $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x)$, if A is the (Λ,τ)-infimum of the image $I(X_\partial)$ of X_∂ in $L^1(D_T)$, that is,

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x) = \mathrm{Inf}^{\Lambda,\tau} \left\{ I(x) \ : \ x \in X_\partial \right\}.$$

Remark 13. It is clear now that if $a \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x)$ then

$$\mathrm{cl}_\tau \ \{I(x) : \ x \in X_\partial\} \cap (a - \Lambda) = \{a\}$$

provided $\mathrm{Min}_\Lambda \left[\mathrm{cl}_\tau \ \{I(x) \ : \ x \in X_\partial\}\right] \neq \emptyset$.

Let $\{y_k\}_{k=1}^\infty$ be a sequence in $L^1(D_T)$. Let $\mathrm{L}^\tau\{y_k\}$ denote the set of all its τ-cluster points in $L^1(D_T)$, that is, $y \in \mathrm{L}^\tau\{y_k\}$ if there is a subsequence $\{y_{k_i}\}_{i=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $y_{k_i} \stackrel{\tau}{\to} y$ in $L^1(D_T)$ as $i \to \infty$. If this set is lower unbounded, i.e., $\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau\{y_k\} = -\infty_\Lambda$, we assume that $\{-\infty_\Lambda\} \in \mathrm{L}^\tau\{y_k\}$. Let $x_0 \in X_\partial$ be a fixed element. In what follows for an arbitrary mapping $I : \ X_\partial \to L^1(D_T)$, we define the following sets:

$$\mathrm{L}^{\sigma\times\tau}(I, x_0) := \bigcup_{\{x_k\}_{k=1}^\infty \in \mathfrak{M}_\sigma(x_0)} \mathrm{L}^\tau\{I(x_k)\}, \tag{15}$$

$$\mathrm{L}^{\sigma\times\tau}_{\min}(I, x_0) := \mathrm{L}^{\sigma\times\tau}(I, x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x), \tag{16}$$

where $\mathfrak{M}_\sigma(x_0)$ is the set of all sequences $\{x_k\}_{k=1}^\infty \subset X$ such that $x_k \to x_0$ with respect to a σ-topology of X.

We are now able to introduce the notion of the lower limit for the vector-valued mappings.

Definition 14. We say that a subset $A \subset L^1(D_T) \cup \{\pm\infty_\Lambda\}$ is the Λ-lower sequential limit of the mapping $I : X_\partial \to L^1(D_T)$ at the point $x_0 \in X_\partial$ with respect to the product topology $\sigma \times \tau$ of $X \times L^1(D_T)$, and we use the notation $A = \liminf^{\Lambda,\tau}_{x \stackrel{\sigma}{\to} x_0} I(x)$, if

$$\liminf^{\Lambda,\tau}_{x \stackrel{\sigma}{\to} x_0} I(x) := \begin{cases} \mathrm{L}^{\sigma\times\tau}_{\min}(I, x_0), & \mathrm{L}^{\sigma\times\tau}_{\min}(I, x_0) \neq \emptyset, \\ \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^{\sigma\times\tau}(I, x_0), & \mathrm{L}^{\sigma\times\tau}_{\min}(I, x_0) = \emptyset. \end{cases} \tag{17}$$

Remark 15. Note that in the scalar case ($I : X_\partial \to \mathbb{R}$) the sets

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x) \quad \text{and} \quad \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^{\sigma\times\tau}(I,x_0)$$

are singletons. Therefore, if $\mathrm{L}^{\sigma\times\tau}_{\min}(I,x_0) \neq \emptyset$ then we have

$$\mathrm{L}^{\sigma\times\tau}_{\min}(I,x_0) = \mathrm{L}^{\sigma\times\tau}(I,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x)$$
$$= \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^{\sigma\times\tau}(I,x_0) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} I(x) = \mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^{\sigma\times\tau}(I,x_0).$$

Hence the choice rules in (17) coincide and we come to the classical definition of the lower limit.

By analogy with [19] (see also [8, 9, 18]), we use the following concept of lower semicontinuity for vector-valued mappings.

Definition 16. We say that a mapping $f : X_\partial \to Y$ is $(\Lambda, \sigma\times\tau)$-lower semicontinuous ($(\Lambda, \sigma\times\tau)$-lsc) at the point $x_0 \in X_\partial$ if $f(x_0) \in \liminf^{\Lambda,\tau}_{x \xrightarrow{\sigma} x_0} \widehat{f}(x)$. A mapping f is $(\Lambda, \sigma\times\tau)$-lsc if f is $(\Lambda, \sigma\times\tau)$-lsc at each point of X_∂.

Before proceeding further, we note that the concept of $(\Lambda, \sigma\times\tau)$-lower semicontinuity for the vector-valued mappings, given above, is more general than the well known extensions of the "scalar" notion of lower semicontinuity to the vector-valued case (see, for example, [1–3, 22, 23]). The main motivation to introduce this concept is the following observation.

Proposition 17 ([19])**.** *Let X be a Banach space, and let Y be a partially ordered Banach space with an ordering closed pointed cone Λ. Let X_∂ be a non-empty subset of X and let $f : X_\partial \to Y$ be a given mapping. If $x^0 \in X_\partial$ is any (Λ,τ)-efficient solution to the vector optimization problem $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_\partial} f(x)$, then the mapping $f : X_\partial \to Y$ is $(\Lambda, \sigma\times\tau)$-lsc at this point for any Hausdorff topology σ on X.*

Turning back to the OCP (1)–(3), we make use of the following assumption:

(A2): The cost functional $J : \mathbb{U}\times\mathbb{Y} \to \mathbb{R}$ in (1) has the representation

$$J(u,y) = \int_0^T \int_D F(u,y)\, dx\, dt, \tag{18}$$

where $F : \mathbb{U}_{loc}\times\mathbb{Y}_{loc} \to L^1(D_T)$ is $(\Lambda, \sigma\times\tau)$-lsc on $\mathbb{U}_{loc}\times\mathbb{Y}_{loc}$ in the sense of Definition 16 and σ stands for the strong topology of $L^1(D)\times L^1((0,T)\times D)$.

Another motivation to consider the cost functional in the form (18) with properties given by Hypothesis (A2), is presented in Remark 18. Further, we note that the assertions

$$|l(y)| + l(y) = 0 \text{ almost everywhere in } D_T \quad \text{and} \quad y \in \mathcal{Y}_{ad}$$

are evidently equivalent on the set $y \in C([0,T]; L^1_{loc}(\mathbb{R}^n))$. As the same time, in a general case, we have $|l(y)| + l(y) \geq 0$ almost everywhere in $D_T = (0,T) \times D$. It means that

$$|l(y)| + l(y) \in \Lambda = \left\{ f \in L^1(D_T) \ : \ f(x) \geq 0 \text{ almost everywhere on } \Omega \right\}$$

.

Taking this observation into account, we introduce the following family of vector-valued penalized problems

$$\text{Find } \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y) = \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \left[F(u,y) + \varepsilon^{-1} \Big(|l(y)| + l(y) \Big) \right], \quad (19)$$

where the set $\overline{\Xi}$ of admissible solutions is defined as follows: $(u,y) \in \overline{\Xi}$ if and only if

$$u \in \mathcal{U}_{ad}, \ y \in [C([0,T]; L^1_{loc}(\mathbb{R}^n)) \times L^\infty(\Omega_T) \times L^\infty(0,T; BV(D)), \quad (20)$$

$$y = y(u) \ \text{ is an entropy solution to (2)–(3) in the sense of Definition 6.} \quad (21)$$

Remark 18. It is clear that $\Xi \subset \overline{\Xi}$ for every $\varepsilon > 0$ and $\overline{\Xi} \neq \emptyset$ by Theorem 7. Moreover, following arguments of the proof of Theorem 9, we have the following important property of the set $\overline{\Xi}$: this set is sequentially compact with respect the w-convergence. Here, a sequence of pairs $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \overline{\Xi}$ is said to be w-convergent to a pair $(u,y) \in \left[L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\Omega_T)\right] \times C([0,T]; L^1_{loc}(\mathbb{R}^n))$ if $u_{0,k} \to u_0$ in $L^1(Q)$, $u_{1,k} \to u_1$ in $L^1((0,T) \times Q)$, and $y_k \to y$ in $C([0,T]; L^1(Q))$ for every open bounded domain $Q \subset \mathbb{R}^n$.

We introduce now the following concept.

Definition 19. A pair $(u^{eff}_\varepsilon, y^{eff}_\varepsilon) \in \overline{\Xi}$ is said to be a (Λ,τ)-efficient solution to the problem (19) if $(u^{eff}_\varepsilon, y^{eff}_\varepsilon)$ realizes the (Λ,τ)-infimum of the mapping $\mathcal{F}_\varepsilon : \overline{\Xi} \to L^1(D_T)$, that is,

$$\mathcal{F}_\varepsilon(u^{eff}_\varepsilon, y^{eff}_\varepsilon)) \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y) = \operatorname{Inf}^{\Lambda,\tau} \left\{ \mathcal{F}_\varepsilon(u,y) \ : \ \forall\, (u,y) \in \overline{\Xi} \right\}.$$

We denote by

$$\begin{aligned} &\operatorname{Eff}(\overline{\Xi}; \mathcal{F}_\varepsilon; \tau; \Lambda) \\ &\quad = \left\{ (u^{eff}_\varepsilon, y^{eff}_\varepsilon) \in \overline{\Xi} \ : \ \mathcal{F}_\varepsilon(u^{eff}_\varepsilon, y^{eff}_\varepsilon) \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}}, \mathcal{F}_\varepsilon(u,y) \right\} \end{aligned} \quad (22)$$

the set of all (Λ)-efficient solutions to the vectorial problem (19).

In what follows, we associate with the vector optimization problem (19) the following scalar minimization problem

$$\begin{aligned} &\mathcal{F}^\lambda_\varepsilon(u,y) = \langle \lambda, \mathcal{F}_\varepsilon(u,y) \rangle_{L^\infty(D_T); L^1(D_T)} \\ &\qquad\qquad \to \inf \ \text{ subject to } \ (u,y) \in \overline{\Xi} \subset \mathbb{U}_{loc} \times \mathbb{Y}_{loc} \end{aligned} \quad (23)$$

where λ is an element of the dual cone

$$K = \Lambda^* = \Big\{ \lambda \in L^\infty(D_T) \ : \ \langle \lambda, f\rangle_{L^\infty(D_T);L^1(D_T)} = \int_0^T \int_D \lambda f \, dx\, dt \geq 0 \ \text{ for all } \ f \in \Lambda \Big\}. \tag{24}$$

Remark 20. Since the set $\overline{\Xi}$ is sequentially w-compact, it follows that the constrained minimization problem (23) has a non-empty set of solutions, provided $\mathcal{F}_\varepsilon^\lambda(\cdot) : \overline{\Xi} \to \overline{\mathbb{R}}$ is a proper lower w-semicontinuous function. However, the characteristic feature of the vector optimization problem (19) is the fact that with any $(\Lambda, w\times\tau)$-lower semicontinuous mapping $\mathcal{F}_\varepsilon : \overline{\Xi} \to L^1(D_T)$, which is neither lower semicontinuous nor quasi-lower semicontinuous on $\overline{\Xi}$, there can be always associated a scalar minimization problem (23) for which the corresponding cost functional $\mathcal{F}_\varepsilon^\lambda : \overline{\Xi} \to \mathbb{R}$ is not lower w-semicontinuous on $\overline{\Xi}$. Indeed, let (u^0, y^0) be a pair of $\overline{\Xi}$ where the quasi-lower semicontinuity of $\mathcal{F}_\varepsilon$ fails. Then there exists at least one element $a^* \in \mathrm{cl}_\tau\left(\mathcal{F}_\varepsilon(\overline{\Xi})\right)$ such that

$$\begin{gathered} a^* \in \liminf^{\Lambda,\tau}_{(u,y)\xrightarrow{w}(u^0,y^0)} \mathcal{F}_\varepsilon(u,y), \\ \mathcal{F}_\varepsilon(u^0,y^0) \in \liminf^{\Lambda,\tau}_{(u,y)\xrightarrow{w}(u^0,y^0)} \mathcal{F}_\varepsilon(u,y), \\ \text{and } \ a^* \not\geq \mathcal{F}_\varepsilon(u^0,y^0). \end{gathered} \tag{25}$$

Let $\{(u_k, y_k)\}_{k=1}^\infty \subset \overline{\Xi}$ be a sequence such that

$$(u_k, y_k) \xrightarrow{w} (u^0, y^0) \ \text{ in } \ \mathbb{U}_{loc} \times \mathbb{Y}_{loc} \quad \text{and} \quad \mathcal{F}_\varepsilon(u_k, y_k) \xrightarrow{\tau} a^* \ \text{ in } \ L^1(D_T).$$

Since $a^* \not\geq_\Lambda \mathcal{F}_\varepsilon(u^0,y^0)$, it follows that $a^* - \mathcal{F}_\varepsilon(u^0,y^0) \notin \Lambda$ and, hence, there exists a vector $\lambda^* \in K$ such that

$$\left\langle \lambda^*, a^* - \mathcal{F}_\varepsilon(u^0,y^0)\right\rangle_{L^\infty(D_T);L^1(D_T)} < 0.$$

As a result, we have

$$\begin{gathered} \liminf_{k\to\infty} \mathcal{F}_\varepsilon^{\lambda^*}(u_k, y_k) = \lim_{k\to\infty} \langle \lambda^*, \mathcal{F}_\varepsilon(u_k,y_k)\rangle_{L^\infty(D_T);L^1(D_T)} \\ = \langle \lambda^*, a^*\rangle_{L^\infty(D_T);L^1(D_T)} < \left\langle \lambda^*, \mathcal{F}_\varepsilon(u^0,y^0)\right\rangle_{L^\infty(D_T);L^1(D_T)} = \mathcal{F}_\varepsilon^{\lambda^*}(u^0,y^0). \end{gathered}$$

Thus, the lower w-semicontinuity property for $\mathcal{F}_\varepsilon^{\lambda^*}$ fails at (u^0,y^0). Moreover (see [19]), for $(\Lambda, w\times\tau)$-lower semicontinuous mappings $\mathcal{F}_\varepsilon : \overline{\Xi} \to L^1(D_T)$ a situation is possible when none of the scalar functions $\mathcal{F}_\varepsilon^\lambda(u,y) = \langle \lambda, \mathcal{F}_\varepsilon(u,y)\rangle_{L^\infty(D_T);L^1(D_T)}$ is lower w-semicontinuous for any $\lambda \in K^\sharp$. Here $K^\sharp$ is the set of all quasi-interior points of K, i.e. $\lambda \in K^\sharp$ if $\lambda \in K$ and $\langle \lambda, b\rangle_{L^\infty(D_T);L^1(D_T)} > 0$ for all $b \in \Lambda \setminus \{0\}$. Having put $\lambda(t,x) = 1$ a.e. in D_T and $\varepsilon = 0$, this leads us to the conclusion: the cost functional

$J : \mathcal{U}_{ad} \times \mathbb{Y} \to \mathbb{R}$ for the original OCP (1)–(3) can lose the w-lower semicontinuity property even if the mapping $F : \overline{\Xi} \to L^1(D_T)$ is $(\Lambda, w \times \tau)$-lower semicontinuous.

The main property of the scalar minimization problem (23) can be characterized as follows.

Theorem 21. *Assume that the initial assumptions* (6)–(7) *hold true and there are a pair* $(u^0, y^0) \in \overline{\Xi}$ *and an element* $\lambda \in K^\sharp$ *such that*

$$(u^0, y^0) \in \operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \left\langle \lambda, \mathcal{F}_\varepsilon^\lambda(u, y)\right\rangle_{L^\infty(D_T);L^1(D_T)}.$$

Then (u^0, y^0) *is a* (Λ, τ)*-efficient solution to the problem* (19)–(21).

Proof. By the initial assumptions, we have

$$\begin{aligned} &\mathcal{F}_\varepsilon^\lambda(u^0, y^0) - \mathcal{F}_\varepsilon^\lambda(u, y) \\ &= \left\langle \lambda, \mathcal{F}_\varepsilon(u^0, y^0) - \mathcal{F}_\varepsilon(u, y)\right\rangle_{L^\infty(D_T);L^1(D_T)} \le 0, \quad \forall\, (u, y) \in \overline{\Xi}. \end{aligned} \tag{26}$$

Let z be any element of the set $\mathrm{cl}_\tau\, \mathcal{F}_\varepsilon(\overline{\Xi})$. Then there exists a sequence $\{(u_k, y_k)\}_{k=1}^\infty \subset \overline{\Xi}$ such that $\mathcal{F}_\varepsilon(u_k, y_k) \xrightarrow{\tau} z$ in $L^1(D_T)$ as $k \to \infty$. Hence, in view of (26), we get

$$\left\langle \lambda, \mathcal{F}_\varepsilon(u^0, y^0) - \mathcal{F}_\varepsilon(u_k, y_k)\right\rangle_{L^\infty(D_T);L^1(D_T)} \le 0, \quad \forall\, k \in \mathbb{N}. \tag{27}$$

Passing to the limit in (27) as $k \to \infty$, we obtain

$$\left\langle \lambda, \mathcal{F}_\varepsilon(u^0, y^0) - z\right\rangle_{L^\infty(D_T);L^1(D_T)} \le 0, \quad \forall\, z \in \mathrm{cl}\, \mathcal{F}_\varepsilon(\overline{\Xi}). \tag{28}$$

Let us assume that $(u^0, y^0) \notin \mathrm{Eff}(\overline{\Xi}; \mathcal{F}_\varepsilon; \tau; \Lambda)$. Then there exists an element $h \in \mathrm{cl}_\tau\, \mathcal{F}_\varepsilon(\overline{\Xi})$ such that $h <_\Lambda \mathcal{F}_\varepsilon(u^0, y^0)$. So, $\mathcal{F}_\varepsilon(u^0, y^0) - h \in \Lambda\setminus\{0\}$. Hence, by definition of the set $K^\sharp$, $\left\langle \mathcal{F}_\varepsilon(u^0, y^0) - h, \lambda\right\rangle_{L^\infty(D_T);L^1(D_T)} > 0$, and we come to a contradiction with (28). So, $(u^0, y^0) \in \mathrm{Eff}(\overline{\Xi}; \mathcal{F}_\varepsilon; \tau; \Lambda)$ and this concludes the proof. $\square$

In view of Remark 20, it is reasonably now to present some results concerning the solvability of the vector optimization problem (19)–(21). To this end, we can apply Theorem 3.5 from [19]. Since the set $\overline{\Xi} \subset \mathcal{U}_{ad} \times \mathbb{Y}_{loc}$ is sequentially w-compact and, hence, it is bounded (see Theorem 7 and Remark 18), we make the following hypothesis.

(A3): There exists a couple of functions $\varphi, \psi \in L^1(D_T)$ such that the estimate

$$\int_Q |\psi|\, dz \le \|F(u(\cdot,\cdot), y(\cdot,\cdot))\|_{L^1(Q)} \le \int_Q |\varphi|\, dz, \quad \forall\, (u, y) \in \overline{\Xi},\ \forall\, Q \subseteq D \tag{29}$$

is valid.

It leads us to the following assertion.

Theorem 22. *Assume the Hypotheses (A1)–(A3) hold true and* $F : \overline{\Xi} \to L^1(D_T)$ *is a* $(\Lambda, w\times\tau)$*-lower semicontinuous mapping. Then the vector optimization problem* (19)–(21) *has a non-empty set of* (Λ,τ)*-efficient solutions for every* $\varepsilon > 0$.

Proof. To begin with, we note that due to Hypothesis (A3), the sequence $\{F(u_k, y_k)\}_{k\in\mathbb{N}}$ is $L^1(D_T)$-bounded and equi-integrable for any sequence of prototypes $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \overline{\Xi}$. Hence, $\{F(u_k, y_k)\}_{k\in\mathbb{N}}$ is relatively weakly compact in $L^1(D_T)$. In order to apply Theorem 3.5 from [19] to our case, we have to show that the mapping $\mathcal{F}_\varepsilon : \overline{\Xi} \to L^1(D_T)$ possesses the $(\Lambda, w\times\tau)$-lower semicontinuity property. To do so, we note that the mapping $l : \mathbb{R} \to \mathbb{R}_+$ is continuous. Hence, for any w-convergent sequence $(u_k, y_k) \overset{w}{\to} (u, y)$ in $\mathbb{U}_{loc} \times \mathbb{Y}_{loc}$, we have

$$
\begin{aligned}
&\||l(y_k)| + l(y_k) - |l(y)| - l(y)\|_{L^1(D_T)}\\
&\le \Big\||l(y_k)|| - |l(y)||\Big\|_{L^1(D_T)} + \||l(y_k)| - l(y)||\|_{L^1(D_T)}\\
&\le 2\||l(y_k)| - l(y)||\|_{L^1(D_T)} = \int_0^T \int_D |l(y_k)| - l(y)|\,dx\,dt\\
&\le \|l(y_k)| - l(y)\|_{C([0,T];L^1(D))} T \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

Thus, $|l| + l : \mathbb{U}_{loc} \times \mathbb{Y}_{loc} \to L^1(D_T)$ is continuous as a mapping from $\mathbb{U}_{loc} \times \mathbb{Y}_{loc}$ with the topology induced by w-convergence to the Banach space $L^1(D_T)$ endowed with the strong topology. Since

$$
\mathcal{F}_\varepsilon(u, y) = F(u, y) + \varepsilon^{-1}\Big(|l(y)| + l(y)\Big), \quad \forall (u, y) \in \overline{\Xi}
$$

and $F : \overline{\Xi} \to L^1(D_T)$ is the $(\Lambda, w\times\tau)$-lower semicontinuous mapping, we obtain the required property (for more details we refer to [18]). □

Further, we present some results which play an essential role in the study of OCP (1)–(3).

Proposition 23. *If there exist a couple of values* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that* $\varepsilon_1 \ne \varepsilon_2$ *and*

$$
\mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_{\varepsilon_1}(u, y) \cap \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_{\varepsilon_2}(u, y) \ne \emptyset \tag{30}
$$

then, under suppositions of Theorem 22, the original OCP (1)–(3) *is regular, i.e.* $\Xi \ne \emptyset$.

Proof. For simplicity we suppose that $\varepsilon_1 < \varepsilon_2$. As Theorem 22 indicates, for given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ the corresponding sets of (Λ,τ)-efficient solutions $\mathrm{Eff}(\overline{\Xi}; \mathcal{F}_{\varepsilon_1}; \Lambda)$ and $\mathrm{Eff}(\overline{\Xi}; \mathcal{F}_{\varepsilon_2}; \tau; \Lambda)$ are nonempty. For now we assume that

$$
\mathrm{Eff}(\overline{\Xi}; \mathcal{F}_{\varepsilon_1}; \tau; \Lambda) \cap \mathrm{Eff}(\overline{\Xi}; \mathcal{F}_{\varepsilon_2}; \tau; \Lambda) = \emptyset.
$$

At the same time, the condition (30) ensures the existence of pairs

$$(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}) \in \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_1};\, \tau;\, \Lambda) \quad \text{and} \quad (u^0_{\varepsilon_2}, y^0_{\varepsilon_2}) \in \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_2};\, \tau;\, \Lambda)$$

such that

$$\mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}) = \mathcal{F}_{\varepsilon_2}(u^0_{\varepsilon_2}, y^0_{\varepsilon_2}). \tag{31}$$

Since $(u^0_{\varepsilon_2}, y^0_{\varepsilon_2}) \notin \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_1};\, \tau;\, \Lambda)$, it follows that

$$\mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_2}, y^0_{\varepsilon_2}) \nleq_\Lambda \mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}). \tag{32}$$

On the other hand, in view of the structure of objective mapping $\mathcal{F}_\varepsilon : \overline{\Xi} \to L^1(D_T)$ and condition (31), we have

$$\mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_2}, y^0_{\varepsilon_2}) \geq_\Lambda \mathcal{F}_{\varepsilon_2}(u^0_{\varepsilon_2}, y^0_{\varepsilon_2}) = \mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}).$$

As a result, combining the last inequality with (32), we come to the contradiction:

$$\mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}) \nleq_\Lambda \mathcal{F}_{\varepsilon_1}(u^0_{\varepsilon_1}, y^0_{\varepsilon_1}).$$

Hence, there exists at least one pair $(u^*, y^*) \in \overline{\Xi}$ such that

$$(u^*, y^*) \in \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_1};\, \tau;\, \Lambda) \cap \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_2};\, \tau;\, \Lambda).$$

Then, in view of relation (30), we obtain

$$\begin{aligned} F(u^*, y^*) + \varepsilon_1^{-1}\Big(|l(y^*)| + l(y^*)\Big) &= \mathcal{F}_{\varepsilon_1}(u^*, y^*) \\ = \mathcal{F}_{\varepsilon_2}(u^*, y^*) = F(u^*, y^*) + \varepsilon_2^{-1}\Big(|l(y^*)| + l(y^*)\Big) &\quad \text{a.e. in } D_T. \end{aligned}$$

Whence $(\varepsilon_1^{-1} - \varepsilon_2^{-1})\Big(|l(y^*)| + l(y^*)\Big) = 0$. Since $\varepsilon_1 \neq \varepsilon_2$ it follows that

$$|l(y^*(t,x))| + l(y^*(t,x)) = 0 \quad \text{for almost all } (t,x) \in D_T.$$

However, this relation is equivalent to the inequality $l(y^*(t,x)) \leq 0$ a.e. in D_T. Taking into account that $y^* = y(u^*) \in C([0,T]; L^1_{loc}(\mathbb{R}^n))$ as an entropy solution of (2)–(3), we conclude: $y^* \in \mathcal{Y}_{ad}$ and, therefore, $(u^*, y^*) \in \Xi$. The proof is complete. □

Proposition 24. *If the original OCP* (1)–(3) *admits at least one optimal solution* $(u^0, y^0) \in \Xi$*, then, under suppositions of Theorem 22, there exists* $\varepsilon_0 > 0$ *such that*

$$(u^0, y^0) \in \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_\varepsilon;\, \tau;\, \Lambda) \quad \forall\, \varepsilon \leq \varepsilon_0. \tag{33}$$

Proof. Let us assume, by contradiction, that there is a monotonically decreasing sequence $\{\varepsilon_k\}_{k=1}^\infty \subset \mathbb{R}_+$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $(u^0, y^0) \notin \mathrm{Eff}(\overline{\Xi};\, \mathcal{F}_{\varepsilon_k};\, \tau;\, \Lambda)$. Then there exists a sequence of pair $\big\{(u^*_k, y^*_k) \in \overline{\Xi}\big\}_{k=1}^\infty$ such that

$$\mathcal{F}_{\varepsilon_k}(u^*_k, y^*_k) <_\Lambda \mathcal{F}_{\varepsilon_k}(u^0, y^0) \equiv F(u^0, y^0), \quad \forall\, k \in \mathbb{N}. \tag{34}$$

Hence,

$$\varepsilon_k^{-1}\Big(|l(y_k^*)| + l(y_k^*)\Big) <_\Lambda F(u^0, y^0) - F(u_k^*, y_k^*) \quad \text{a.e. in } D_T, \ \forall\, k \in \mathbb{N}. \quad (35)$$

Since the set $\overline{\Xi} \subset \mathcal{U}_{ad} \times \mathbb{Y}_{loc}$ is sequentially w-compact (see Theorem 7 and Remark 18) and the mapping $F : \overline{\Xi} \to L^1(D_T)$ possesses the $(\Lambda, w \times \tau)$-lower semicontinuity property, it follows that the right-hand side in (35) is uniformly bounded in $L^1(D_T)$ with respect to $k \in \mathbb{N}$. Hence, passing to the limit in (34) as $k \to \infty$ we come into conflict with the boundedness of $F(u^0, y^0) - F(u_k^*, y_k^*)$ provided $\Big(|l(y_k^*)| + l(y_k^*)\Big) \neq 0$ on some subset of D_T with positive Lebesgue measure. Thus, this contradiction means that there exists $k_* \in \mathbb{N}$ such that $|l(y_k^*)| + l(y_k^*) = 0$ almost everywhere in D_T for all $k > k_*$. Hence, $y_k \in \mathcal{Y}_{ad}\ \forall\, k > k_*$. Taking this observation into account, the inequality (35) implies relation $F(u_k^*, y_k^*) <_\Lambda F(u^0, y^0)$ for $k > k_*$. As a result, we come to the contradiction with the initial assumptions

$$J(u_k^*, y_k^*) :=$$
$$\int_0^T \int_D F(u_k^*, y_k^*)\, dx\, dt < \int_0^T \int_D F(u^0, y^0)\, dx\, dt =: J(u^0, y^0) \quad \forall\, k > k_*.$$

The proof is complete. □

Taking these results into account, we introduce the following concept.

Definition 25. We say that a sequence $\{(u_k, y_k)\}_{k=1}^\infty$ in $\mathbb{U}_{loc} \times \mathbb{Y}_{loc}$ is a weakened minimizing sequence if $u_k \in \mathcal{U}_{ad}$, $y_k = y(u_k)$ is an entropy solution to (2)–(3) in the sense of Definition 6 for every $k \in \mathbb{N}$,

$$u_k \to \widehat{u} \ \text{ in } \ \mathbb{U}_{loc}, \quad y_k \to \widehat{y} \ \text{ in } \ \mathbb{Y}_{loc}, \quad \varlimsup_{k\to\infty} l(y_k)(t,x) \le 0 \ \text{ a.e. in } \ D_T,$$
$$F(u_k, y_k) \xrightarrow{\tau} \xi \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} F(u,y) \quad \text{in} \quad L^1(D_T).$$

Moreover, if for every $\varepsilon > 0$ and any ε-neighborhood $\mathcal{V}_\tau(\xi)$ of ξ in $\langle L^1(D_T), \tau\rangle$ there exist a neighborhood $\mathcal{O}(\widehat{u}, \widehat{y})$ of $(\widehat{u}, \widehat{y})$ in the strong topology of $\mathbb{U}_{loc} \times \mathbb{Y}_{loc}$, a pair $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)$, and a positive value δ such that the conditions

$$(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \mathcal{O}(\widehat{u}, \widehat{y}) \cap (\mathcal{U}_{ad} \times \mathbb{Y}_{loc}), \quad F(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \mathcal{V}_\tau(\xi),$$
$$\widehat{y}_\varepsilon = y(\widehat{u}_\varepsilon) \ \text{ is an entropy solution to (2)–(3) in the sense of Definition 6},$$
$$l(\widehat{y}_\varepsilon(t,x)) \le C\varepsilon \ \text{ a.e. in } \ D_T, \quad C > 0 \quad (36)$$

hold true, then $(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)$ is said to be a weakened ε-approximate solution to the problem (1)–(3).

Further, we consider the following family of scalar optimization problems

$$\operatorname{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y) = \operatorname{sc}_w^- \langle \lambda, \mathcal{F}_\varepsilon(u,y)\rangle_{L^\infty(D_T);L^1(D_T)} \to \inf \quad (37)$$
$$\text{subject to } \ (u,y) \in \overline{\Xi} \subset \mathbb{U}_{loc} \times \mathbb{Y}_{loc}, \quad \lambda \in K. \quad (38)$$

Here, $\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right]:\overline{\Xi}\to\mathbb{R}$ denote the lower w-semicontinuous envelope of the functional $\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)=\langle\lambda,\mathcal{F}_\varepsilon(u,y)\rangle_{L^\infty(D_T);L^1(D_T)}$ with some $\lambda\in K=\Lambda^*$, that is, $\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right]$ is the greatest lower w-semicontinuous functional majorized by $\mathcal{F}_\varepsilon^\lambda$ on $\overline{\Xi}$. Then, following the direct method in the Calculus of Variations and taking into account that the set $\overline{\Xi}\subset\mathcal{U}_{ad}\times\mathbb{Y}_{loc}$ is sequentially w-compact, we get:

Proposition 26. *Every minimizing sequence for*

$$\inf_{(u,y)\in\overline{\Xi}}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)$$

has a w-cluster point which is a minimum point of $\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right]$ *on* $\overline{\Xi}$*.*

We are now able to prove the main result of this article.

Theorem 27. *Assume that the initial assumptions* (6)–(7) *hold true and the cost functional* $J:\mathbb{U}\times\mathbb{Y}\to\mathbb{R}$ *in* (1) *has the representation* (18) *with property (A3), where* $F:\mathbb{U}_{loc}\times\mathbb{Y}_{loc}\to L^1(D_T)$ *is a given mapping (not necessary* (Λ,w)*-lsc on* $\overline{\Xi}$*). Then*

$$\bigcup_{\lambda\in K^\sharp}\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)\tag{39}$$

forms a set of weakened ε-approximate solutions to the problem (1)–(3) provided this problem is regular.

Proof. We divide this proof into several steps. Step 1. First we study the property of the sets $\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)$. Let λ be any element of $K^\sharp$ and let $\varepsilon>0$ be a given value. Then, by Proposition 26, there exists at least one pair $(u_\varepsilon^*,y_\varepsilon^*)\in\overline{\Xi}$ such that

$$(u_\varepsilon^*,y_\varepsilon^*)\in\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y).\tag{40}$$

Since $\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)$ is the lower w-semicontinuous envelope of the functional

$$\mathcal{F}_\varepsilon^\lambda(u,y)=\langle\lambda,\mathcal{F}_\varepsilon(u,y)\rangle_{L^\infty(D_T);L^1(D_T)},$$

it follows that there exists a sequence of pairs $\{(u_k^\varepsilon,y_k^\varepsilon)\}_{k=1}^\infty\subset\overline{\Xi}$ such that

$$(u_k^\varepsilon,y_k^\varepsilon)\xrightarrow{w}(u_\varepsilon^*,y_\varepsilon^*)\quad\text{as}\quad k\to\infty\tag{41}$$

and

$$\begin{aligned}&\lim_{k\to\infty}\langle\lambda,\mathcal{F}_\varepsilon(u_k^\varepsilon,y_k^\varepsilon)\rangle_{L^\infty(D_T);L^1(D_T)}\\ &=\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u_\varepsilon^*,y_\varepsilon^*)\overset{\text{by condition (40)}}{\le}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)\\ &\le\langle\lambda,\mathcal{F}_\varepsilon(u,y)\rangle_{L^\infty(D_T);L^1(D_T)}\quad\forall(u,y)\in\overline{\Xi}.\end{aligned}\tag{42}$$

Hypothesis (A3) implies that there exist an integer $\widehat{k} \in \mathbb{N}$ and an element $\widehat{\zeta}_\varepsilon \in L^1(D_T)$ such that

$$\left\langle \lambda, \mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \right\rangle_{L^\infty(D_T);L^1(D_T)} < \left\langle \lambda, \widehat{\zeta}_\varepsilon \right\rangle_{L^\infty(D_T);L^1(D_T)} \quad \forall\, k > \widehat{k}.$$

Since $\lambda \in K^\sharp$, this implies the estimate $\|\mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon)\|_{L^1(D_T)} \le \|\widehat{\zeta}_\varepsilon\|_{L^1(D_T)}$ for all $k > \widehat{k}$. Hence, without loss of generality, we may suppose that the sequence $\{\mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon)\}_{k=1}^\infty$ is bounded in $L^1(D_T)$ and equi-integrable. So, there exist an element $\eta_\varepsilon \in L^1(D_T)$ and a subsequence of $\{\mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon)\}_{k=1}^\infty$ (still denoted by subscript k) such that $\mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \stackrel{\tau}{\to} \eta_\varepsilon$ in $L^1(D_T)$ as $k \to \infty$.

For now we assume that

$$\eta_\varepsilon \notin \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y). \tag{43}$$

Hence, an element $\xi \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y)$ can be found such that $\xi <_\Lambda \eta_\varepsilon$. Therefore, $\eta_\varepsilon - \xi \in \Lambda \setminus \{0\}$, and using the fact that $\lambda \in K^\sharp$, we just come to the inequality

$$\langle \lambda, \eta_\varepsilon \rangle_{L^\infty(D_T);L^1(D_T)} > \langle \lambda, \xi \rangle_{L^\infty(D_T);L^1(D_T)}$$

which is equivalent to

$$\lim_{k\to\infty} \langle \lambda, \mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \rangle_{L^\infty(D_T);L^1(D_T)} > \langle \lambda, \xi \rangle_{L^\infty(D_T);L^1(D_T)}. \tag{44}$$

On the other hand, for the element $\xi \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y)$ there exists a sequence $\{(v_k, z_k)\}_{k=1}^\infty \subset \overline{\Xi}$ such that $\mathcal{F}_\varepsilon(v_k, z_k) \stackrel{\tau}{\to} \xi$ in $L^1(D_T)$. Since the set $\overline{\Xi}$ is sequentially w-compact, we may suppose that $(v_k, z_k) \stackrel{w}{\to} (v^*, z^*) \in \overline{\Xi}$. Then, by inequality (42), we deduce

$$\lim_{k\to\infty} \langle \lambda, \mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \rangle_{L^\infty(D_T);L^1(D_T)} \le \langle \mathcal{F}_\varepsilon(v_i, z_i), \lambda \rangle_{L^\infty(D_T);L^1(D_T)}, \quad \forall\, i \in \mathbb{N}. \tag{45}$$

Passing to the limit in (45) as $i \to \infty$, we get

$$\lim_{k\to\infty} \langle \lambda, \mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \rangle_{L^\infty(D_T);L^1(D_T)} \le \langle \lambda, \xi \rangle_{L^\infty(D_T);L^1(D_T)}.$$

However, this contradicts (44) and, hence, (43). Thus,

$$(u_k^\varepsilon, y_k^\varepsilon) \stackrel{w}{\to} (u_\varepsilon^*, y_\varepsilon^*) \ \text{ and } \ \mathcal{F}_\varepsilon(u_k^\varepsilon, y_k^\varepsilon) \stackrel{\tau}{\to} \eta_\varepsilon \in \operatorname{Inf}^{\Lambda,\tau}_{(u,y)\in\overline{\Xi}} \mathcal{F}_\varepsilon(u,y) \ \text{ in } \ L^1(D_T) \tag{46}$$

as $k \to \infty$.

Step 2. Our next intention is to prove that the sequence $\{(u_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon\to 0} \subset \overline{\Xi}$, which were obtained in the previous step, contains a subsequence, still denoted by the suffix ε, such that $(u_\varepsilon^*, y_\varepsilon^*) \stackrel{w}{\to} (u^*, y^*)$ as $\varepsilon \to 0$ and $(u^*, y^*) \in \Xi$. In the same way as in the proof of Theorem 7, we can conclude that the sequence $\left\{(u_\varepsilon^*, y_\varepsilon^*) \in \overline{\Xi}\right\}_{\varepsilon>0}$ is relatively w-compact and, passing to a subsequence if necessary, we get

$$(u_\varepsilon^*, y_\varepsilon^*) \xrightarrow{w} (u^*, y^*), \quad \text{where} \quad (u^*, y^*) \in \overline{\Xi}. \tag{47}$$

Let us prove that $y^* \in \mathcal{Y}_{ad}$. Let (u, y) be any admissible pair to the original problem, that is, $(u, y) \in \Xi$ (here, $\Xi \neq \emptyset$ by the regularity assumption). Then $\Big(|l(y)| + l(y)\Big) = 0$. Therefore,

$$\mathcal{F}_\varepsilon(u_\varepsilon^*, y_\varepsilon^*) := F(u_\varepsilon^*, y_\varepsilon^*) + \varepsilon^{-1}\Big(|l(y_\varepsilon^*)| + l(y_\varepsilon^*)\Big) \not>_\Lambda \mathcal{F}_\varepsilon(u, y) \equiv F(u, y)$$

for $\varepsilon > 0$ small enough.

By the initial assumptions the set $\{F(u_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon>0}$ is bounded below, say by $z \in L^1(D_T)$. The latter yields $\varepsilon^{-1}\Big(|l(y_\varepsilon^*)| + l(y_\varepsilon^*)\Big) \not>_\Lambda w$, with $w = F(u, y) - z$, i.e. $|l(y_\varepsilon^*)| + l(y_\varepsilon^*) \not>_\Lambda \varepsilon w$. On the other hand, $|l(y_\varepsilon^*)| + l(y_\varepsilon^*) \geq_\Lambda 0_{L^1(D_T)}$ $\forall\, \varepsilon > 0$. Hence, passing to the limit as $\varepsilon \to 0$ in the above relations and using the fact that $\varepsilon w \to 0_{L^1(D_T)}$ and $y_\varepsilon^* \to y^*$ in $C([0, T]; L^1(D))$, we come to the inequality

$$0_{L^1(D_T)} \leq \liminf_{\varepsilon \to 0} \Big(|l(y_\varepsilon^*)| + l(y_\varepsilon^*)\Big) \not>_\Lambda 0_{L^1(D_T)}.$$

Hence, $\liminf_{\varepsilon \to 0} \Big(|l(y_\varepsilon^*)| + l(y_\varepsilon^*)\Big) = 0$ and, in view of the continuity property of l, we obtain

$$\Big(|l(y^*)| + l(y^*)\Big) = \liminf_{\varepsilon \to 0} \Big(|l(y_\varepsilon^*)| + l(y_\varepsilon^*)\Big) = 0.$$

Since this is equivalent to the inequality $l(y^*(t, x)) \leq 0$ almost everywhere in D_T, it follows that the limit pair (u^*, y^*) is an admissible solution to the original OCP (1)–(3). Moreover, using hypothesis (A3) and applying similar arguments as we did it before, it can be shown that the sequence $\{F(u_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon \to 0}$ is relatively τ-compact in $L^1(D_T)$). Hence there exists an element $\eta \in L^1(D_T)$ such that within a subsequence, we have

$$F(u_\varepsilon^*, y_\varepsilon^*) \xrightarrow{\tau} \eta \quad \text{in} \quad L^1(D_T) \quad \text{as} \quad \varepsilon \to 0. \tag{48}$$

Step 3 deals with the limiting properties of the sequence

$$\left\{\mathrm{sc}_w^- \left[\mathcal{F}_\varepsilon^\lambda\right] : \overline{\Xi} \to \mathbb{R}\right\}_{\varepsilon \to 0}.$$

We note that this sequence of functionals is monotonically increasing and for every $\varepsilon > 0$ we can write down

$$\mathrm{sc}_w^- \left[\mathcal{F}_\varepsilon^\lambda\right](u, y) = \mathrm{sc}_w^- \left\langle \lambda, F(u, y)\right\rangle_{L^\infty(D_T); L^1(D_T)} + \varepsilon^{-1} \int_{D_T} \lambda\Big(|l(y)| + l(y)\Big)\, dz \tag{49}$$

(by continuity of $l : \overline{\Xi} \to L^1(D_T)$ as a mapping from $\mathbb{U}_{loc} \times \mathbb{Y}_{loc}$ with the topology induced by w-convergence to the Banach space $L^1(D_T)$ endowed with the strong topology). Hence, this sequence admits the existence of $\Gamma(w)$-limit with the following representation (see [7])

$$\begin{aligned}
&\Gamma(w)-\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)\\
&=\Gamma(w)-\lim_{\varepsilon\to 0}\left[\mathrm{sc}_w^-\left\langle\lambda,F(u,y)\right\rangle_{L^\infty(D_T);L^1(D_T)}\right]\\
&+\lim_{\varepsilon\to 0}\left[\varepsilon^{-1}\int_{D_T}\lambda\Big(|l(y)|+l(y)\Big)\,dz\right]\\
&=\mathrm{sc}_w^-\left\langle\lambda,F(u,y)\right\rangle_{L^\infty(D_T);L^1(D_T)}+\lim_{\varepsilon\to 0}\left[\varepsilon^{-1}\int_{D_T}\lambda\Big(|l(y)|+l(y)\Big)\,dz\right]\\
&=\begin{cases}\mathrm{sc}_w^-\left\langle\lambda,F(u,y)\right\rangle_{L^\infty(D_T);L^1(D_T)}, & (u,y)\in\Xi,\\ +\infty, & (u,y)\in\overline{\Xi}\setminus\Xi.\end{cases}
\end{aligned}\tag{50}$$

Applying the main variational properties of $\Gamma(w)$-limit to the functional

$$\Gamma(w)-\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right]:\overline{\Xi}\to\mathbb{R},$$

we come to the following assertions (see [7]).

1.: Convergence of minimal values:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\min_{(u,y)\in\overline{\Xi}}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)&\overset{\text{by Step 2}}{=}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon^\lambda(u_\varepsilon^*,y_\varepsilon^*)\\
&=\min_{(u,y)\in\overline{\Xi}}\Big(\Gamma(w)-\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)\Big)\\
&\overset{\text{by (50)}}{=}\min_{(u,y)\in\Xi}\Big(\mathrm{sc}_w^-\left\langle\lambda,F(u,y)\right\rangle_{L^\infty(D_T);L^1(D_T)}\Big).
\end{aligned}\tag{51}$$

2.: Convergence of minimizers:

$$\begin{aligned}
(u_\varepsilon^*,y_\varepsilon^*)\overset{w}{\to}(u^*,y^*)&\in\operatorname*{Argmin}_{(u,y)\in\Xi}\Big(\mathrm{sc}_w^-\left\langle\lambda,F(u,y)\right\rangle_{L^\infty(D_T);L^1(D_T)}\Big)\\
&=\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}}\Big(\Gamma(w)-\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u,y)\Big).
\end{aligned}\tag{52}$$

3.: Sequential properties:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right](u_\varepsilon,y_\varepsilon)&\geq\Big(\Gamma(w)-\lim_{\varepsilon\to 0}\mathrm{sc}_w^-\left[\mathcal{F}_\varepsilon^\lambda\right]\Big)(u^*,y^*)\\
&=\mathrm{sc}_w^-\left\langle\lambda,F(u^*,y^*)\right\rangle_{L^\infty(D_T);L^1(D_T)}
\end{aligned}\tag{53}$$

for any sequence $\{(u_\varepsilon,y_\varepsilon)\}_{\varepsilon>0}\subset\overline{\Xi}$ such that $(u_\varepsilon,y_\varepsilon)\overset{w}{\to}(u^*,y^*)$ as $\varepsilon\to 0$.

Step 4. This step concludes the proof. As it was shown before, we have the following property $F(u_\varepsilon^*,y_\varepsilon^*)\overset{\tau}{\to}\eta$ in $L^1(D_T)$ as $\varepsilon\to 0$. Our aim is to prove that $\eta \in \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\Xi} F(u,y)$ in $L^1(D_T)$. To do so, we assume the converse:

$$\eta \notin \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\Xi} F(u,y). \tag{54}$$

Hence, there can be found an element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\Xi} F(u,y)$ such that $\xi <_\Lambda \eta$. Therefore, $\eta - \xi \in \Lambda \setminus \{0\}$, and using the fact that $\lambda \in K^\sharp$, we just come to the inequality

$$\langle \lambda, \eta\rangle_{L^\infty(D_T);L^1(D_T)} > \langle \lambda, \xi\rangle_{L^\infty(D_T);L^1(D_T)}$$

which is equivalent to

$$\lim_{\varepsilon\to 0} \langle \lambda, F(u^*_\varepsilon, y^*_\varepsilon)\rangle_{L^\infty(D_T);L^1(D_T)} > \langle \lambda, \xi\rangle_{L^\infty(D_T);L^1(D_T)}\,. \tag{55}$$

On the other hand, for the element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\Xi} F(u,y)$ there exists a sequence $\{(v_k, z_k)\}_{k=1}^\infty \subset \Xi$ such that $F(v_k, z_k) \overset{\tau}{\to} \xi$ in $L^1(D_T)$. Since the set $\overline{\Xi}$ is sequentially w-compact, we may suppose that $(v_k, z_k) \overset{w}{\to} (v^*, z^*) \in \Xi$.

Further, we note that in view of the convergences (48) and (41) with properties (46), it follows that combining these with (47) and applying the diagonal method, we can extract a sequence $\{k = k(\varepsilon)\}_{\varepsilon\to 0} \subset \mathbb{N}$ such that

$$\overline{\Xi} \ni (u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)}) \overset{w}{\to} (u^*, y^*) \in \Xi \quad \text{as } \varepsilon \to 0,$$
$$F(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)}) \overset{\tau}{\to} \eta \ \text{ in } \ L^1(D_T) \quad \text{as } \varepsilon \to 0.$$

Then, taking into account variational properties (52)–(53), we obtain

$$\mathrm{sc}^-_w \langle \lambda, F(u^*, y^*)\rangle_{L^\infty(D_T);L^1(D_T)} = \min_{(u,y)\in\Xi} \Big(\mathrm{sc}^-_w \langle \lambda, F(u,y)\rangle_{L^\infty(D_T);L^1(D_T)} \Big), \tag{56}$$

$$\begin{aligned} \lim_{\varepsilon\to 0} \Big\langle \lambda, F(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)})\Big\rangle_{L^\infty(D_T);L^1(D_T)} &= \mathrm{sc}^-_w \Big[\langle \lambda, F(u^*, y^*)\rangle_{L^\infty(D_T);L^1(D_T)} \Big] \\ &\overset{\text{by condition (56)}}{\leq} \mathrm{sc}^-_w \Big[\langle \lambda, F(u,y)\rangle_{L^\infty(D_T);L^1(D_T)} \Big] \\ &\leq \langle \lambda, F(u,y)\rangle_{L^\infty(D_T);L^1(D_T)} \quad \forall (u,y) \in \Xi. \end{aligned} \tag{57}$$

As a result, inequality (57) implies

$$\begin{aligned} \lim_{\varepsilon\to 0} \Big\langle \lambda, F(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)})\Big\rangle_{L^\infty(D_T);L^1(D_T)} \\ \leq \langle F(v_i, z_i), \lambda\rangle_{L^\infty(D_T);L^1(D_T)}\,, \quad \forall\, i \in \mathbb{N}. \end{aligned} \tag{58}$$

Passing to the limit in (58) as $i \to \infty$, we get

$$\lim_{k\to\infty} \Big\langle \lambda, F(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)})\Big\rangle_{L^\infty(D_T);L^1(D_T)} \leq \langle \lambda, \xi\rangle_{L^\infty(D_T);L^1(D_T)}\,.$$

However, this contradicts (55) and, hence, (54). Thus,

$$(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)}) \overset{w}{\to} (u^*, y^*) \ \text{ and}$$

$$\mathcal{F}_\varepsilon(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)}) \xrightarrow{\tau} \eta \in \mathrm{Inf}^{\Lambda,\tau}_{(u,y)\in\Xi} F(u,y) \quad \text{in} \quad L^1(D_T)$$

as $k \to \infty$ and, hence, the elements of the sequence $\left\{(u^\varepsilon_{k(\varepsilon)}, y^\varepsilon_{k(\varepsilon)})\right\}_{\varepsilon>0}$ can be considered as weakened ε-approximate solutions to the problem (1)–(3) in the sense of Definition 25. Note that the estimate (36) is ensured by the condition: $l(y^*(t,x)) \le 0$ almost everywhere in D_T. This concludes the proof. □

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