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On Quadratic Scalarization of Vector Optimization Problems in Banach Spaces

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We study vector optimization problems in partially ordered Banach spaces and suppose that the objective mapping possesses a weakened property of lower semicontinuity and make no assumptions on the interior of the ordering cone. We discuss the so-called adaptive scalarization of such problems. We show that the corresponding scalar nonlinear optimization problems can be by-turn approximated by quadratic minimization problems. Such regularization is especially attractive from a numerical point of view because it gives the possibility to apply rather simple computational methods for the approximation of the entire set of efficient solutions.

Keywords: vector-valued optimization; lower semicontinuity; partial ordered spaces; efficient solutions; scalarization approach

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1. Introduction

The main goal of this paper is to discuss one class of vector optimization problems in Banach spaces in the case when the objective vector-valued mapping possesses a weakened property of lower semicontinuity and the ordering cone is non-solid in the objective space. We consider vector optimization problems in a special setting, which involves topological properties of the objective space, and discuss the problem of their scalarization. We deal with the case when the objective mapping takes values in a real Banach space Y partially ordered by a pointed cone Λ with possibly empty topological interior. In contrast to the classical setting of the vector optimization problems (see [1, 3]): "Minimize $f(x)$ with respect to the cone Λ subject to constraints $x \in X_{ad} \subset X$, $f : X \to Y$, " we study the problem in the following formulation

$$\text{Find } \operatorname{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x) \tag{1}$$

and associate this problem with the quaternary $\langle X_{ad}, f, \Lambda, \tau\rangle$, where the essential counterpart is the choice of the topology τ on the objective space Y.

We also extend the concept of lower semicontinuity for vector-valued mappings, which is compatible with optimization problems in the form (1), and discuss the

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existence of the so-called (Λ,τ)-efficient solutions to the problem (1). In particular, we show that the topological properties of the spaces (X,σ) and (Y,τ), where this problem is considered, play an essential role. In view of this, our main aim is to discuss a nonlinear scalarization of vector optimization problems (1) with the so-called epi-lower semicontinuous mappings. Thus, in spite of the fact that the scalarization of vector optimization problems (1) in the form of weighted sum takes a rather simple form, this method has some disadvantages. In fact, this approach allows to determine all efficient points by an appropriate parameter choice only for convex problems (see, for instance [1, 4–6]). Moreover, for the objective vector-valued mapping with weakened properties of lower semicontinuity the inclusion $\operatorname{Argmin}_{x\in X_{ad}} \langle f(x),\lambda\rangle_{Y;V} \subseteq \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda) \quad \forall\, \lambda\in K\setminus 0_V$ is not generally valid. Hence, the scalar problems

$$\text{Minimize } f_\lambda(x) = \langle f(x),\lambda\rangle_{Y;V} \text{ subject to } x\in X_{ad} \tag{2}$$

may produce the appearance of the so-called pseudo-solutions or extra solutions to the original one. Besides, one of the most important question is that an "even" choice of elements $\lambda\in\Lambda^*\setminus 0_V$ does not guarantee an "even" distribution of the solutions $x^*\in X_{ad}$ to the corresponding vector optimization problem (1) on the set $\mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$. So, we cannot ensure that each solution $x^*\in \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$ can be attained by solutions to the scalar problem (2). To our best knowledge, no answers to these questions have been found in the case when $f: X_{ad}\to Y$ possesses a weakened property of lower semicontinuity and Λ is a non-solid cone in Y.

Our main goal in this paper is to consider other type of scalarization for the vector optimization problem (1). We develop an adaptive approach to the scalarization in the spirit of Pascoletti–Serafini nonlinear scalarization method. An advantage of this approach is that it allows to approximate the whole set of (Λ,τ)-efficient solutions. Moreover, we show that scalar optimization problems which come from Pascoletti–Serafini approach can be approximated by quadratic minimization problems. Such regularization is especially attractive from a numerical point of view because it gives the possibility to apply rather simple computational methods for numerical calculations.

2. Notation and Preliminaries

Let X and Y be two real Banach spaces. We suppose that these spaces, as topological spaces, are endowed with some topologies $\sigma=\sigma(X)$ and $\tau=\tau(Y)$, respectively. For a subset $A\subset Y$ we denote by $\mathrm{int}_\tau\, A$ and $\mathrm{cl}_\tau\, A$ its interior and closure with respect to the τ-topology, respectively. We will omit this index if it does not lead to a misunderstanding. Let Λ be a τ-closed convex pointed cone in Y. No assumption is imposed on the topological interior of Λ. Throughout this paper, we suppose that Y is partially ordered with the ordering cone Λ. The cone Λ defines a partial order on Y denoted by $\leq_\Lambda$, that is, for any elements $y,z\in Y$, we write $y\leq_\Lambda z$ whenever $z\in y+\Lambda$ and $y<_\Lambda z$ for $y,z\in Y$, if $z-y\in\Lambda\setminus\{0_Y\}$. We say that a sequence $\{y_k\}_{k=1}^\infty\subset Y$ is decreasing and we use the notation $y_k\searrow$ whenever, for all $k\in\mathbb{N}$, we have $y_{k+1}\leq_\Lambda y_k$. We also say that a sequence $\{y_k\}_{k=1}^\infty\subset Y$ is bounded below if there exists an element $y^*\in Y$ such that $y^*\leq_\Lambda y_k$ for all $k\in\mathbb{N}$.

Now, we recall some basic notions concerning the semi-ordered spaces and set-valued mappings. We say that an element $y^*\in S\subset Y$ is Λ-minimal for the set S (see [1]) if there is no $y\in S$ such that $y<_\Lambda y^*$, that is $S\cap(y^*-\Lambda)=\{y^*\}$. Let $\mathrm{Min}_\Lambda(S)$ denote the family of all Λ-minimal elements of S.

Let us introduce two singular elements $-\infty_\Lambda$ and $+\infty_\Lambda$ in Y. We assume that these elements satisfy the following conditions: (1) $-\infty_\Lambda \preceq y \preceq +\infty_\Lambda$ $\forall y \in Y$ and (2) $+\infty_\Lambda + (-\infty_\Lambda) = 0_Y$. Let $Y^\bullet$ denote a semi-extended Banach space: $Y^\bullet = Y \cup \{+\infty_\Lambda\}$ assuming that $\|+\infty_\Lambda\|_Y = +\infty$ and $y + \lambda(+\infty_\Lambda) = +\infty$ $\forall\, y \in Y$ and $\forall\, \lambda > 0$. The following concept is crucial in this paper.

Definition 2.1: We say that a set E is the efficient infimum of the set $S \subset Y$ with respect to the τ topology of Y (or shortly (Λ,τ)-infimum) if E is the collection of all minimal elements of $\mathrm{cl}_\tau\, S$ in the case when this set is non-empty, and E is equal to $\{-\infty_\Lambda\}$ otherwise.

Hereinafter we denote the (Λ,τ)-infimum for S by $\mathrm{Inf}^{\Lambda,\tau}\, S$. Thus, in view of the given above definition, we have

$$\mathrm{Inf}^{\Lambda,\tau}\, S := \begin{cases} \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S), & \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S) \neq \emptyset, \\ -\infty_\Lambda, & \mathrm{Min}_\Lambda(\mathrm{cl}_\tau\, S) = \emptyset. \end{cases}$$

As was shown in [7] (see also [8]), in vector-value case a plausible situation is the following: $\mathrm{Inf}^{\Lambda,\tau}(S) \neq \emptyset$, $\mathrm{Min}_\Lambda(S) \neq \emptyset$, and $\mathrm{Inf}^{\Lambda,\tau}(S) \cap \mathrm{Min}_\Lambda(S) = \emptyset$, in contrast to the scalar case where the inclusion $\mathrm{Min}_\Lambda(S) \subseteq \mathrm{Inf}^{\Lambda,\tau}\, S$ is always true.

Let X_{ad} be a non-empty subset of the Banach space X, and let $f : X_{ad} \to Y$ be some mapping. Note that the mapping $f : X_{ad} \to Y$ can be associated with its natural extension $\widehat{f} : X \to Y^\bullet$ to the whole space X, where $\widehat{f}(x) = f(x)$, if $x \in X_{ad}$, and $\widehat{f}(x) = +\infty_\Lambda$ otherwise. A mapping $f : X_{ad} \to Y^\bullet$ is said to be bounded below if there exists an element $z \in Y$ such that $z \leq_\Lambda f(x)$ for all $x \in X_{ad}$.

Definition 2.2: We say that a subset A of Y is the efficient infimum of a mapping $f :\ X_{ad} \to Y$ with respect to the τ-topology of Y and denote it by $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}}\, f(x)$, if A is the (Λ,τ)-infimum of the image $f(X_{ad})$ of X_{ad} in Y, i.e.,

$$\mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}}\, f(x) = \mathrm{Inf}^{\Lambda,\tau}\left\{f(x)\ :\ \forall\, x \in X_{ad}\right\}.$$

Let $\{y_k\}_{k=1}^\infty$ be a sequence in Y. Let $\mathrm{L}^\tau\{y_k\}$ be the set of all its cluster points with respect to the τ-topology of Y, that is, $y \in \mathrm{L}^\tau\{y_k\}$ if there is a subsequence $\{y_{k_i}\}_{i=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $y_{k_i} \xrightarrow{\ \tau\ } y$ in Y as $i \to \infty$. If this set is lower unbounded, i.e., $\mathrm{Inf}^{\Lambda,\tau}\, \mathrm{L}^\tau\{y_k\} = -\infty_\Lambda$, we assume that $\{-\infty_\Lambda\} \in \mathrm{L}^\tau\{y_k\}$. If $\mathrm{Sup}^{\Lambda,\tau}\, \mathrm{L}^\tau\{y_k\} = +\infty_\Lambda$, we assume that $\{+\infty_\Lambda\} \in \mathrm{L}^\tau\{y_k\}$.

Following [2], we introduce the following sets

$$\mathcal{N}_\infty := \{N \subseteq \mathbb{N} \mid \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}_\infty^\sharp := \{N \subseteq \mathbb{N} \mid N \text{ infinite}\}.$$

Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence in Y. We write $y_k \xrightarrow{\tau} y_0$, if y_0 is the limit of the sequence $\{y_k\}_{k\in\mathbb{N}}$ with respect to the τ-topology of Y. Moreover, we write $y_k \xrightarrow{\tau,N} y_0$ in the case of convergence of a subsequence designated by an index set $N \in \mathcal{N}_\infty^\sharp$ or $N \in \mathcal{N}_\infty$. It is clear that every subsequence of $\{y_k\}_{k\in\mathbb{N}}$ can be expressed by $\{y_k\}_{k\in N}$, where N belongs to $\mathcal{N}_\infty^\sharp$. In the case of $N \in \mathcal{N}_\infty$, $\{y_k\}_{k\in N}$ denotes a subsequence of $\{y_k\}_{k\in\mathbb{N}}$ that arises by omitting finitely many members. For instance, a subsequence of a subsequence $\{y_k\}_{k\in N}$ $(N \in \mathcal{N}_\infty^\sharp)$ can be expressed by some $\overline{N} \in \mathcal{N}_\infty^\sharp$ with $\overline{N} \subseteq N$ as $\{y_k\}_{k\in\overline{N}}$.

We say that a sequence of pairs $\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset X \times Y$ μ-converges to (x_0, y_0), if $x_k \xrightarrow{\sigma} x_0$ and $y_k \xrightarrow{\tau} y_0$ as $k \to \infty$. Let $x_0 \in X_{ad}$ be a fixed element. In what follows for an arbitrary mapping $f:\ X_{ad}\to Y$ we make use of the following set

$$\mathrm{L}^{\mu}(f,x_0):=\bigcup_{\{x_k\}_{k=1}^{\infty}\in\mathfrak{M}_{\sigma}(x_0)}\mathrm{L}^{\tau}\{f(x_k)\},\tag{3}$$

where $\mathfrak{M}_{\sigma}(x_0)$ is the set of all sequences $\{x_k\}_{k=1}^{\infty}\subset X$ such that $x_k\to x_0$ with respect to the σ-topology of X.

3. Epi-lower semicontinuous mappings in Banach spaces

In this section we make no additional assumptions on the ordering cone Λ and its interior. As before, we suppose that Λ is a τ-closed convex pointed cone in Y. Our aim is to study the lower semicontinuity properties for the so-called locally Λ-bounded below mappings $f:X\to Y^{\bullet}$, where Y is the dual to some separable real Banach space V ($Y=V^{*}$) and Y is endowed with the weak-* topology τ. We also note that the following result is well-know in real analysis: a real-valued function $f:X\to\mathbb{R}$ is lower semicontinuous if, and only if, this mapping has a closed epigraph, i.e. $\mathrm{epi}\,f=\mathrm{cl}_{\mu}\,\mathrm{epi}\,f$. However, this assertion is not true for the vector-valued case (see [3]). In view of this, we introduce the following concept (see [11, 12]).

Definition 3.1: We say that a mapping $f:X\to Y^{\bullet}$ is epi-lower semicontinuous (epi-l.s.c.) at $x_0\in X$ with respect to the μ-topology of $X\times Y$, if

$$f\left(x_0\right)=\inf{}^{\Lambda}L^{\mu}\left(f,x_0\right).\tag{4}$$

Here, $\inf^{\Lambda}A$ denotes the Λ-infimum of a subset A and it is defined as an element of Y such that: for every $y\in Y$, $y\le_{\Lambda}\inf^{\Lambda}A$ if and only if $y\le_{\Lambda}z$ for every $z\in A$. We say that f is epi-l.s.c. on X if f is epi-l.s.c. at each point $x_0\in X$.

Definition 3.2: We say that a mapping $f:X\to Y^{\bullet}$ is locally bounded below with respect to the cone Λ if for every $x_0\in\mathrm{dom}\,f$ there exist an element $b\in Y$ and a neighborhood $\mathcal{U}(x_0)$ of x_0 in X such that $b\le_{\Lambda}f(x)\ \forall\ x\in\mathcal{U}(x_0)$.

Remark 1: As immediately follows from this definition if $Y=\mathbb{R}$ and $\Lambda=\mathbb{R}_{+}$ then (4) implies the relation $\inf^{\Lambda}L^{\mu}\left(f,x_0\right)=\liminf\limits_{x\to x_0}f\left(x\right)$. Hence, in the scalar case, Definition 3.1 is equivalent to the definition of sequential lower semicontinuity of $f:X\to\overline{\mathbb{R}}$ in the classical sense.

Remark 2: Note also that the property of epi-lower semicontinuity was previously used in the literature for a sequence of functions $\{f_w\}_{w\in W}\subset\overline{\mathbb{R}}^{X}$ (see for example [13, 14]), where X and W are topological spaces. In this case they say that the family of functions $\{f_w\}_{w\in W}\subset\overline{\mathbb{R}}^{X}$ is epi-lower semicontinuous at $v\in W$ if $\limsup_{w\to v}\mathrm{epi}\,f_w\subseteq\mathrm{epi}\,f_v$, where the $\limsup_{w\to v}$ is taken in the sense of Kuratowski. Basically, this is equivalent to the relation (see [13])

$$\sup_{U\in\mathcal{N}(x)}\ \sup_{V\in\mathcal{N}(v)}\ \inf_{w\in V}\ \inf_{y\in U}f_w(y)\ge f_v(x)\quad\forall\,x\in X.\tag{5}$$

Therefore, if $X=Y=W=\mathbb{R}$, $f:X\to Y^{\bullet}=\mathbb{R}\cup\{+\infty\}$, and $f_w=f,\forall\,w\in\mathbb{R}$, inequality (5) is reduced to $f(x)\le\sup_{U\in\mathcal{N}(x)}\inf_{y\in U}=:\mathrm{sc}^{-}f(x)f(y)\ \forall\,x\in X$, where $\mathrm{sc}^{-}f:\mathbb{R}\to\mathbb{R}$ represents the so-called lower semicontinuous regularization of the original function $f:\mathbb{R}\to\mathbb{R}$ (see [15]). Since $\mathrm{sc}^{-}f(x)\le f(x)\ \forall\,x\in X$ by default, it follows that in this case the epi-lower semicontinuity in the sense of [14] degenerates into the classical definition of lower semicontinuity of real-valued functions. Hence, in view of Remark **1**, Definition 3.1 and the epi-lower semicontinuity in the sense of [14] coincide in the above case.

The following statements provide useful properties of epi-lower semicontinuity.

Theorem 3.3: *Let $f : X \to Y^\bullet$ be a mapping for which its epigraph* epi f *is a sequentially μ-closed subset of $X \times Y$. Then $f : X \to Y^\bullet$ is* epi-*l.s.c. on X.*

Proof: Assume, on the contrary, that there exists an element $x_0 \in X$ such that $f(x_0) \neq \inf^\Lambda L^\mu(f, x_0)$. It means that there is a sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ with the properties

$$x_k \xrightarrow{\sigma} x_0, \quad f(x_k) \xrightarrow{\tau} f^*, \quad f^* \not\geq_\Lambda f(x_0). \tag{6}$$

Moreover, in this case we have $\operatorname{epi} f \ni (x_k, f(x_k)) \xrightarrow{\mu} (x_0, f^*)$. However, as follows from (6), $(x_0, f^*) \notin \operatorname{epi} f$, which contradicts the μ-closure of the set $\operatorname{epi} f$, and this ends the proof. □

Lemma 3.4: *[1, p.29] Let Y be a real linear space with an ordering cone Λ. Assume the cone Λ has nonempty algebraic interior, i.e.*

$$\operatorname{cor}\Lambda = \left\{\overline{y} \in \Lambda \,|\, \forall\, y \in Y\ \exists \overline{\lambda} > 0\ :\ \overline{y} + \lambda y \in \Lambda\ \forall\, \lambda \in [0, \overline{\lambda}]\right\} \neq \emptyset.$$

If Λ is algebraically closed and pointed, then there is a norm $\|\cdot\|$ on Y with the property that for all $y \in \Lambda$

$$x \in [0_Y, y] = \{z \in Y \,|\, 0_Y \leq_\Lambda z \leq_\Lambda y\} \implies \|x\| \leq \|y\|. \tag{7}$$

Taking this result into account we say that the norm $\|\cdot\|$ in Banach space $Y = V^*$ is Λ-monotone if for every $y \in \Lambda$ the property (7) holds true. Note that the Λ-monotonicity property of the norm in a partially ordered Banach space Y is not a greatly restrictive assumption. For instance, if $Y = L^p(\Omega)$ with $p \in (1, +\infty)$, where Ω is an open bounded domain in $\mathbb{R}^N$, is partially ordered by the natural ordering cone Λ, then the norm in these spaces is Λ-monotone (see [1]) in spite of the fact that the corresponding cone Λ has an empty topological interior.

Theorem 3.5: *Let Λ be a τ-closed convex pointed ordering cone in Y such that the norm $\|\cdot\|$ in Y is Λ-monotone. Let $f : X \to Y^\bullet$ be a locally bounded below and* epi-*l.s.c. mapping on X. Then the epigraph of f is sequentially μ-closed.*

Proof: Assume the inverse. Then

$$\exists\, \{(x_k, y_k)\}_{k\in\mathbb{N}} \in \operatorname{epi} f \quad \text{such that} \quad (x_k, y_k) \xrightarrow{\mu} (x_0, y_0) \notin \operatorname{epi} f. \tag{8}$$

It means that

$$x_k \xrightarrow{\sigma} x_0 \text{ in } X, \quad y_k \xrightarrow{\tau} y_0 \text{ in } Y; \tag{9}$$

$$f(x_k) \leq_\Lambda y_k\ \forall k \in \mathbb{N}; \tag{10}$$

$$f(x_0) \not\leq_\Lambda y_0. \tag{11}$$

By (9), the sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded in Y. Hence there exists a constant

$C > 0$ such that

$$\sup_{k \in \mathbb{N}} \|y_k\|_Y \leq C. \tag{12}$$

Since the mapping f is locally bounded below in the sense of Definition 3.2, the condition (10) implies the existence of an element $b \in Y$ and a set $N \in \mathcal{N}_\infty$ such that

$$b \leq_\Lambda f(x_k) \ \ \forall k \in N. \tag{13}$$

Then, inequalities (10), (12), (13), and the Λ-monotonicity of $\|\cdot\|$ in Y imply that the sequence $\{f(x_k)\}_{k \in N}$ is bounded in Y. Hence, by Banach-Alaoglu Theorem, we can extract a subsequence $\{x_k\}_{k \in N_1}$ with $N_1 \in \mathcal{N}_\infty^\sharp$ such that $f(x_k) \overset{\tau, N_1}{\longrightarrow} f^*$ in Y. As a result, passing to the limit in (10) as $N_1 \ni k \to \infty$, we obtain $f^* \leq_\Lambda y_0$. Combining this with (11), we come to the conclusion $f(x_0) \not\leq_\Lambda f^*$. Since $f^* \in L^\mu(f, x_0)$, it follows that $f(x_0) \neq \inf^\Lambda L^\mu(f, x_0)$, which contradicts with the epi-lower semicontinuity of f at the point x_0. This completes the proof. $\square$

The hypotheses that the mapping $f : X \to Y^\bullet$ must be locally bounded below on X and epi-l.s.c. on X are essential in Theorem 3.5, as the following example shows.

Example 3.6 Let $X = \mathbb{R}$, $Y = \ell_2$, and let Λ be the so-called natural ordering cone in ℓ_2 which is defined by $\Lambda = \{(y_1, y_2, \ldots, y_k, \ldots) \in \ell_2 \mid y_k \geq 0 \ \forall\, k \in \mathbb{N}\}$. Let τ be the weak topology of ℓ_2. We define a vector-valued mapping $f : \mathbb{R} \to \ell_2$ as follows:

$$f(x) = \begin{cases} (1, 0, 0, \ldots), & x \neq \frac{1}{k}, \\ (\underbrace{0, 0, \ldots, -k}_{k}, 0, \ldots), & x = \frac{1}{k}. \end{cases} \tag{14}$$

It easily follows from (14) that the mapping f is not locally bounded below at $x = 0$ with respect to the cone Λ. To begin with, we show that f is epi-l.s.c. on $\mathbb{R}$. Indeed, let x_0 be an arbitrary element of $\mathbb{R}$ such that x_0 does not belong to the sequence $\left\{\frac{1}{k}\right\}_{k \in \mathbb{N}}$ and $x_0 \neq 0$. Then there exist a constant $C > 0$ and a neighborhood $\mathcal{V}(x_0)$ of x_0 such that $f(x) = C$ for all $x \in \mathcal{V}(x_0)$. Hence,

$$\inf\nolimits^\Lambda L^\mu(f, x_0) = \inf\nolimits^\Lambda \{f(x_0)\} = f(x_0),$$

that is, the mapping f is epi-l.s.c. at x_0.

Further, we suppose that a given element x_0 admits a representation $x_0 = k^{-1}$ for some $k \in \mathbb{N}$. Then

$$\inf\nolimits^\Lambda L^\mu(f, x_0) = \inf\nolimits^\Lambda \{(1, 0, \ldots), (0, \ldots, -k, 0, \ldots)\} = (0, \ldots, -k, 0, \ldots) = f(x_0).$$

As a result, f is also epi-l.s.c. at x_0.

It remains to consider the case when $x_0 = 0$. Since $\{(0, 0, \ldots, -k, 0, \ldots)\}_{k \in \mathbb{N}} \subset \ell_2$ is not a τ-convergent sequence in ℓ_2, it follows that $L^\mu(f, 0)$ is a singleton set. Namely, $L^\mu(f, 0) = \{f(0)\}$. Hence, $\inf^\Lambda L^\mu(f, 0) = f(0)$. This proves that the mapping f is epi-l.s.c. on the entire set $\mathbb{R}$.

We are now able to show that the epigraph of the mapping (14) is not a sequentially μ-closed subset of $\mathbb{R} \times \ell_2$. In order to do it, we consider the following sequence $\left\{\left(k^{-1}, 0_{\ell_2}\right)\right\}_{k \in \mathbb{N}} \subset \mathbb{R} \times \ell_2$. Since $0_{\ell_2} \geq_\Lambda f\left(k^{-1}\right) = (0, 0, \ldots, -k, 0, \ldots)$, it follows that each element of this sequence belongs to the epi f. However, it can be easily checked that $\left(k^{-1}, 0_{\ell_2}\right) \xrightarrow{\mu} (0, 0_{l_2})$ in $\mathbb{R} \times \ell_2$ as $k \to \infty$, where $(0, 0_{\ell_2}) \notin \operatorname{epi} f$. Thus, (14) gives an example of epi-l.s.c. mapping on $\mathbb{R}$ for which its epigraph is not a sequentially μ-closed set.

Remark 3: Finally we note that the epi-l.s.c. property of a mapping $f : X \to Y^{\bullet}$ does not imply its sequential lower semicontinuity, in general. At the same time, in the case when the objective space Y is endowed with the weak-$*$ topology, the converse statement is valid (see [11]).

4. Setting of a Vector Optimization Problem and Existence Theorem

Let X_{ad} be a non-empty σ-closed subset of a Banach space X. Let $f : X_{ad} \to Y$ be a given epi-l.s.c. mapping. The vector optimization problem we are going to consider in this section can be stated as follows

$$\text{Find } \; x^{eff} \in X_{ad} \; \text{ such that } \; f(x^{eff}) \in \operatorname{Inf}^{\Lambda,\tau}_{x \in X_{ad}} f(x), \tag{15}$$

where the operator $\operatorname{Inf}^{\Lambda,\tau}_{x \in X_{ad}}$ is defined in Definition 2.2. Since the choice of the τ-topology on the objective space Y is essential, we will associate the optimization problem (15) with the quaternary $\langle X_{ad}, f, \Lambda, \tau\rangle$.

Further we make use of the following concept [7, 8, 16].

Definition 4.1: An element $x^{eff} \in X_{ad}$ is said to be a (Λ, τ)-efficient solution to the problem (15) if $\Big(\left\{f(x^{eff})\right\} - \Lambda\Big) \cap \operatorname{cl}_\tau\Big(f(X_{ad})\Big) = \left\{f(x^{eff})\right\}$. In other words, $x^{eff} \in X_{ad}$ is (Λ, τ)-efficient solution to the problem (15) if x^{eff} realizes the (Λ, τ)-infimum of the mapping $f : X_{ad} \to Y$.

We denote by $\operatorname{Eff}_\tau(X_{ad}; f; \Lambda)$ the set of all (Λ, τ)-efficient solutions to the vector problem (15). Note that there are other notions of optimality for the problem (15) which are rather customary in vector optimization theory [19]. In what follows, we prescribe some additional properties to the ordering cone Λ.

Definition 4.2: Let (Y, τ) be a real topological linear space with an ordering cone Λ. The cone Λ is called Daniell, if every decreasing lower bounded net (i.e. $i \le j \Longrightarrow y_j \le_\Lambda y_i$) τ-converges to its (Λ, τ)-infimum.

A typical example of Daniell cone with respect to the weak topology of $L^p(\Omega)$ $(1 < p < +\infty)$ is the natural ordering cone in $L^p(\Omega)$. A condition ensuring the Daniall property is given by the next lemma [20].

Lemma 4.3: *Let (Y, τ) be a real topological linear space with an ordering cone Λ. If Y has compact intervals $[-z, z]$ and Λ is τ-closed and pointed, then Λ is Daniell.*

Definition 4.4: We say that a sequence $\{x_k\}_{k=1}^\infty \subset X_{ad}$ is minimizing to the vector optimization problem $\langle X_{ad}, f, \Lambda, \tau\rangle$, if $f(x_k) \xrightarrow{\tau} \xi$ in Y, where ξ is an element of $\operatorname{Inf}^{\Lambda,\tau}_{x \in X_{ad}} f(x)$.

We are now in a position to give the main existence result concerning the vector optimization problem (15) (see [7]).

Theorem 4.5: *Let (X, σ) and (Y, τ) be two real topological linear spaces, and let Y be partially ordered with the τ-closed pointed Daniell cone Λ. Let X_{ad} be a non-empty sequentially σ-compact subset of X and let $f : X_{ad} \to Y$ be a given* epi-*l.s.c. mapping. Then the vector optimization problem* $\langle X_{ad}, f, \Lambda, \tau\rangle$ *has a non-empty set of* (Λ,τ)*-efficient solutions and is well-posed in the Tikhonov sense with respect to the* σ*-topology of* X*, i.e. every minimizing sequence* $\{x_k\}_{k=1}^{\infty} \subset X_{ad}$ *has a subsequence* σ*-converging to some element of* $\mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$.

5. On Pascoletti-Serafini Nonlinear Scalarization of Vector Optimization Problems

We assume that X is reflexive and the objective space Y is dual to some separable Banach space V (that is $Y = V^*$). As usual we suppose that these spaces are endowed with some topologies $\sigma = \sigma(X)$ and $\tau = \tau(Y)$, respectively. By default σ is always associated with the weak topology of X, whereas τ is associated with the weak-$*$ topology of Y. Suppose that the space V is partially ordered with a nontrivial pointed ordering cone $K \subset V$ with non-empty algebraic interior $\mathrm{cor}\,(K)$ for which Λ is the dual cone, that is, $\Lambda := K^* := \Big\{ y \in Y \ : \ \langle y, \lambda\rangle_{Y;V} \ge 0 \ \forall\, \lambda \in K \Big\}$. As usual we suppose that Y is partially ordered with a pointed convex cone $\Lambda = K^*$.

As was shown in [7, 16], many open questions concerning the scalarization of vector optimization problem (15) arise even for the simplest situation. Indeed, in spite of the fact that the scalarization in the form

$$f_\lambda(x) = \langle f(x), \lambda\rangle_{Y;V} \to \inf \qquad \text{subject to } \ x \in X_{ad} \subset X, \tag{16}$$

where λ is an element of the cone K, takes rather simple form, it possesses some disadvantages. Namely, the main property of the problem (16) can be characterized as follows (see [7]).

Theorem 5.1 : *Let* X *be a reflexive Banach space, let* V *be a separable Banach space, and let* $Y = V^*$ *be endowed with the weak-*$*$ *topology* τ *and partially ordered with a pointed Daniell cone* $\Lambda = K^*$*, where* K *is a weakly closed ordering cone in* V*. Let also* X_{ad} *be a non-empty bounded weakly closed subset of* X*, and let* $f : X_{ad} \to Y$ *be a* epi-*l.s.c. mapping. Then*

$$\operatorname*{Argmin}_{x\in X_{ad}} \langle f(x), \lambda\rangle_{Y;V} \cap \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda) \neq \emptyset \quad \forall\, \lambda \in K_f^\sigma \setminus 0_V, \tag{17}$$

where K_f^σ *is the co-called cone of* σ*-semicontinuity for the mapping* f*, i.e.*

$$K_f^\sigma := \{\lambda \in K \ : \ f_\lambda \ \textit{ is proper and lower } \ \sigma\textit{-semicontinuous on } \ X_{ad}\}. \tag{18}$$

Since the inclusion $\operatorname*{Argmin}_{x\in X_{ad}} \langle f(x), \lambda\rangle_{Y;V} \subseteq \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)\ \forall\, \lambda \in K_f^\sigma \setminus 0_V$ is not generally valid, it means that the scalar problem (16) may produce the appearance of the so-called pseudo-solutions or extra solutions to the original one. On the other hand, one of the most important question is that an "even" choice of elements $\lambda \in K_f^\sigma \setminus 0_V$ does not guarantee an "even" distribution of the solutions $x^* \in X_{ad}$ to the corresponding vector optimization problem (15) on the set $\mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$. So, we cannot ensure that each solution $x^* \in \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$ can be attained by solutions to the scalar problem (16). To our best knowledge, no answers to these questions have been found in the case when $f : X_{ad} \to Y$ is a epi-l.s.c. mapping and Λ is a non-solid cone in Y (i.e., $\mathrm{int}_\tau \Lambda = \emptyset$).

In view of this, our main goal in this section is to consider another type of scalarization for vector optimization problems (15) with epi-l.s.c. mappings f :

$X_{ad} \to Y$ and possibly empty topological interior of the ordering cone Λ in Y, which inherits the ideas of the Pascoletti-Serafini approach [17]. We recall that the epi-lower semicontinuity property of the objective mapping f should be considered as the weakened property of lower semicontinuity for vector-valued mappings in Banach spaces [11]. Hereinafter, we assume that $f : X_{ad} \to Y$ is locally bounded below with respect to the cone Λ.

Let $\{\eta_k\}_{k=1}^{\infty}$ be a Λ-decreasing sequence in Y, i.e. $\eta_{k+1} \leq_\Lambda \eta_k$ for all $k \in \mathbb{N}$. We associate the original vector optimization problem (15) with the following collection of sets: $\Theta_{\eta_k} = \{x \in X_{ad} \ : \ f(x) \leq_\Lambda \eta_k\} \quad \forall \ k \in \mathbb{N}$. It is clear that the sequence $\{\Theta_{\eta_k}\}_{k=1}^{\infty}$ is monotone: $\Theta_{\eta_{k+1}} \subseteq \Theta_{\eta_k}$ $\forall k \in N$. Since the epigraph of $f : X_{ad} \to Y$ is sequentially μ-closed (see [11, 12]), the sets Θ_{η_k} are σ-closed for every $k \in \mathbb{N}$.

Our next observation deals with the limiting properties of the sequence of sets $\{\Theta_{\eta_k}\}_{k=1}^{\infty}$. To do so, we recall the sequential version of the set convergence in the sense of Kuratowski.

Definition 5.2: The sequential K-lower and K-upper limits of the sequence $\{\Theta_{\eta_k}\}_{k=1}^{\infty}$ are defined as

$$K_s\!-\liminf_{k\to\infty} \Theta_{\eta_k} = \left\{x \in X \ : \ \exists\, x_k \xrightarrow{\sigma} x, \ \exists\, k_0 \in \mathbb{N}, \ \forall\, k \geq k_0 \ : \ x_k \in \Theta_{\eta_k}\right\},$$
$$K_s\!-\limsup_{k\to\infty} \Theta_{\eta_k} = \left\{x \in X \ : \exists\, n_k \to +\infty, \exists\, x_k \to x, \forall\, k \in \mathbb{N} \ : \ x_k \in \Theta_{\eta_{n_k}}\right\},$$

respectively. Now, we say that the sequence $\{\Theta_{\eta_k}\}_{k=1}^{\infty}$ K_s-converges to Θ if

$$K_s\!-\liminf_{k\to\infty} \Theta_{\eta_k} = K_s\!-\limsup_{k\to\infty} \Theta_{\eta_k} = \Theta.$$

Proposition 5.3: *Let $\{\eta_k\}_{k=1}^{\infty} \subset Y$ be a Λ-decreasing sequence such that $\eta_k \xrightarrow{\tau} \eta^*$ in Y. Assume that K is a reproducing cone in V,*

$$\{x \in X_{ad} \ : \ f(x) \leq_\Lambda \eta^*\} \neq \emptyset, \tag{19}$$

and X_{ad} is a nonempty sequentially σ-closed subset of X. Then Θ_{η^*} is the sequential K-limit of the sequence $\{\Theta_{\eta_k}\}_{k=1}^{\infty}$ as $k \to \infty$.

Proof: In view of condition (19) and monotonicity of the sequence $\{\eta_k\}_{k=1}^{\infty} \subset Y$, we have $\{x \in X_{ad} \ : \ f(x) \leq_\Lambda \eta^*\} \subseteq \{x \in X_{ad} \ : \ f(x) \leq_\Lambda \eta_k\} \subseteq \Theta_{\eta_k}$ $\forall\, k \in \mathbb{N}$. Hence, $\Theta_{\eta_k} \neq \emptyset$ for all $k \in \mathbb{N}$. Let $\{x_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be any sequences such that $x_k \in \Theta_{\eta_{n_k}}$ for all $k \in \mathbb{N}$, $n_k \to \infty$ as $k \to \infty$, and the sequence $\{x_k\}_{k=1}^{\infty}$ σ-converges to some element $x^* \in X$. Then, the condition of epi-lower semicontinuity of f implies that $f(x_k) \leq_\Lambda \eta_{n_k}$ for all $k \in \mathbb{N}$.

Further we note that the cone K is reproducing in V with nonempty quasi-interior $K^\sharp$. Then, following Peressini [18] and Boenwein [20], we have that in the dual space $Y = V^*$ the ordering cone $\Lambda = K^*$ is normal with respect to the norm topology of Y, that is, $0 \leq_\Lambda y <_\Lambda z \Longrightarrow \|y\|_Y < \|z\|_Y$. Since $f(x_k) \leq_\Lambda \eta_{n_k}$ for all $k \in \mathbb{N}$ and the sequence $\{\eta_{n_k}\}_{k=1}^{\infty}$ monotonically τ-converges to η^* as k tends to ∞ and, hence, $\{\eta_{n_k}\}_{k=1}^{\infty}$ is bounded, it follows that $f(x_k) \leq_\Lambda \eta_{n_1}$ $\forall\, k \in \mathbb{N}$. Therefore, if the sequence $\{f(x_k)\}_{k=1}^{\infty}$ is bounded from below, then normality property implies its boundedness in Y. Hence, by Banach-Alaoglu theorem, we may assume that the sequence $\{f(x_k)\}_{k=1}^{\infty}$ is τ-convergent. Otherwise, if $\inf^\Lambda \{f(x_k)\}_{k=1}^{\infty} = -\infty_\Lambda$ then the inequality $f(x^*) \leq_\Lambda \eta^*$ is obvious.

So, it has a sense to consider only the first case. Let $\xi \in Y$ be the τ-limit of the sequence $\{f(x_k)\}_{k=1}^{\infty}$. Since $f(x_k) \leq_\Lambda \eta_{n_k}$ for all $k \in \mathbb{N}$, $\eta_{n_k} \searrow \eta^*$, and $f(x_k) \xrightarrow{\tau} \xi$, it follows that $\xi \le_\Lambda \eta^*$. Besides, taking into account that fact that $\xi \in \mathrm{L}^\mu(f, x^*)$ and $f(x^*) \overset{\text{by (4)}}{\le_\Lambda} z \ \forall z \in \mathrm{L}^\mu(f, x^*)$, we conclude: $f(x^*) \le_\Lambda \eta^*$. Since X_{ad} is a τ-closed subset of X, it follows that $x^* \in X_{ad}$. Combining this fact with inequality $f(x^*) \le_\Lambda \eta^*$, we obtain $x^* \in \Theta_{\eta^*}$. Hence, $K_s\!-\limsup_{k\to\infty} \Theta_{\eta_k} \subseteq \Theta_{\eta^*}$ by Definition 5.2.

To conclude the proof, it remains to establish the inclusion $\Theta_{\eta^*} \subseteq K_s\!-\liminf_{k\to\infty} \Theta_{\eta_k}$. To do so, we fix an arbitrary element $x^* \in \Theta_{\eta^*}$. Due to the assumption (19), we have $f(x^*) \le_\Lambda \eta^*$. Since the sequence $\{\eta_k\}_{k=1}^\infty \subset Y$ is Λ-decreasing, it follows that $f(x^*) \le_\Lambda \eta^* \le_\Lambda \eta_{n_k} \ \forall k \in \mathbb{N}$. Thus, $x^* \in \Theta_{\eta_k}$ for all $k \in \mathbb{N}$ and the desired inclusion immediately follows from Definition 5.2. The proof is complete. □

Before proceeding further, we note that in many applications it has a sense to weaken the requirement on efficient solutions to the vector optimization problem $\langle X_{ad}, f, \Lambda, \tau\rangle$. In particular, we may let the objective mapping to attain its efficient infimum on the set X_{ad} with some error. On the other hand, the set of (Λ,τ)-efficient solutions to such problem can possibly be empty, i.e., the efficient infimum of the objective mapping is unattainable on the given set X_{ad}. Nevertheless, the absence of its infimum does not mean that the vector optimization problem makes no sense, since its efficient infimum exists and hence can be approached with some accuracy. In view of this, it is reasonable to weaken the requirements on the solutions to the vector optimization problem $\langle X_{ad}, f, \Lambda, \tau\rangle$ as follows.

Definition 5.4: We say that an element $x^* \in X_{ad}$ is the (σ,τ)-generalized solution to optimization problem (15), if there exist a sequence $\{x_k\}_{k=1}^\infty \subset X_{ad}$ and an element $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$ such that $x_k \overset{\sigma}{\rightharpoonup} x^*$ in X and $f(x_k) \overset{\tau}{\to} \xi$ in Y.

We denote by $\mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda)$ the set of all (σ,τ)-generalized solutions to the problem $\langle X_{ad}, f, \Lambda, \tau\rangle$. It is clear that

$$\mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda) \subseteq \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda). \tag{20}$$

By $\mathrm{sc}^-_\sigma f_\lambda : X_{ad} \to \mathbb{R}$ we denote the lower σ-semicontinuous envelope of the functional $f_\lambda(x) = \langle f(x), \lambda\rangle_{Y;V}$ with some $\lambda \in K$, that is, $\mathrm{sc}^-_\sigma f_\lambda$ is the greatest lower σ-semicontinuous functional majorized by f_λ on X_{ad}.

Theorem 5.5: *[7] Let X be a reflexive Banach space, σ be the weak topology on X, V be a separable Banach space, and the Banach space $Y = V^*$ be endowed with the weak-$*$ topology τ and partially ordered with a pointed cone $\Lambda = K^*$, where K is a convex pointed cone in V with non-empty algebraic interior* cor$\,(K)$*. Let also X_{ad} be a non-empty sequential σ-compact subset of X, and let $f : X_{ad} \to Y$ be a given mapping (not necessary* epi-*l.s.c. on X_{ad}). Then the following inclusion is valid:*

$$\bigcup_{\lambda\in K^\sharp} \operatorname*{Argmin}_{x\in X_{ad}} \mathrm{sc}^-_\sigma f_\lambda(x) \subseteq \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda). \tag{21}$$

Theorem 5.6: *Let X_{ad} be a non-empty sequential σ-compact subset of X, let $f : X_{ad} \to Y$ be a given* epi-*lower semicontinuous mapping, let $\{\eta_k\}_{k=1}^\infty$ be a Λ-monotonically decreasing sequence in Y such that $\eta_k \overset{\tau}{\to} \xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$, and let K be a reproducing cone in V. Then the sequence $\{\Theta_{\eta_k}\}_{k=1}^\infty$ K_s-converges to Θ_ξ and*

$$\Theta_\xi \subseteq \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda). \tag{22}$$

Proof: To begin with, we show that the set $\{x \in X_{ad} \ : \ f(x) \leq_\Lambda \xi\}$ is nonempty due to the epi-lower semicontinuity of f. Indeed, taking into account the initial assumptions ($\eta_k \xrightarrow{\tau} \xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$), we have

$$\{x \in X_{ad} \ : \ f(x) \leq_\Lambda \eta_k\} \neq \emptyset \quad \forall\, k \in \mathbb{N}.$$

Hence, there exists a sequence $\{x_k\}_{k=1}^\infty$ such that $x_k \in \Theta_{\eta_k}$ for all $k \in \mathbb{N}$. Since X_{ad} is the sequential σ-compact set, it follows that there is an element $x^* \in X_{ad}$ such that, within a subsequence, we have: $\Theta_{\eta_k} \ni x_k \xrightarrow{\sigma} x^*$ as $k \to \infty$. On the other hand, we have an obvious inequality: $\xi \leq_\Lambda f(x_k) \leq_\Lambda \eta_k \leq_\Lambda \eta_1 \ \forall\, k \in \mathbb{N}$. Applying arguments similar to those in the proof of Proposition 5.3, we can assume that $f(x_k) \xrightarrow{\tau} \xi$. Since $\xi \in \inf^\Lambda \mathrm{L}^\mu(f, x^*)$, by epi-lower semicontinuity property of $f : X_{ad} \to Y$, we conclude: $x^* \in \{x \in X_{ad} \ : \ f(x) \leq_\Lambda \xi\}$ and

$$x^* \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda) \ \text{ by Definition 5.4,} \tag{23}$$

i.e., the set $\{x \in X_{ad} \ : \ f(x) \leq_\Lambda \xi\}$ is nonempty. Hence, all preconditions of Proposition 5.3 hold true. Thus, the sequence $\{\Theta_{\eta_k}\}_{k=1}^\infty$ K_s-converges to Θ_ξ as $k \to \infty$.

It remains to establish the inclusion (22). To do so, we fix an arbitrary element $z \in \Theta_\xi$. If Θ_ξ is a singleton then the required conclusion immediately follows from (23). So, we assume that $z \neq x^*$. By Definition 5.2, there exists a sequence $\{z_k\}_{k=1}^\infty$ such that $z_k \xrightarrow{\sigma} z$ in X and $z_k \in \Theta_{\eta_k}$ for all $k \in \mathbb{N}$. Since $\eta_k \searrow \xi$ and $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$ it follows that

$$\eta_k - \Lambda \xrightarrow{K_s} \xi - \Lambda \quad \text{and} \quad (\xi - \Lambda) \cap \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x) = \xi \ \text{by Definition 2.2.}$$

Hence, the sequence $\{f(x_k)\}_{k=1}^\infty$ satisfies conditions $\xi \leq_\Lambda f(z_k) \leq_\Lambda \eta_k \ \forall\, k \in \mathbb{N}$. Therefore, by normality property of Λ and Banach-Alaogly theorem, we have $f(z_k) \xrightarrow{\tau} \xi$. Thus, $z \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda)$ by Definition 5.4. The proof is complete. □

Remark 1: In order to construct a Λ-monotone sequence $\{\eta_k\}_{k=1}^\infty$ (see Theorem 5.6), we can apply the following arguments. Let η_1 be any element of Y such that $\eta_1 >_\Lambda \xi$, where ξ is a given element of $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$. Let $\zeta = \eta_1 - \xi$ be a direction in Y. Then it is reasonably to define $\{\eta_k\}_{k=1}^\infty$ following the rule: $\eta_k := \xi + k^{-1}\zeta$ $\forall\, k \in \mathbb{N}$. Note that, in some sense such choice of elements $\{\eta_k\}_{k=1}^\infty$ comes from the Pascoletti-Serafini approach [17] (see also some generalization of that approach in Gerth and Weidner [21]).

Let $\eta \in Y$ and $\zeta \in Y$ be given elements. Our next intension is to consider the following parametrized scalar optimization problem

$$\inf_{(x,\gamma)\in\Omega_{\eta,\zeta}} \Phi(x,\gamma), \tag{24}$$

where $\Phi(x,\gamma) = \gamma$ and the set $\Omega_{\eta,\zeta} \subset X \times \mathbb{R}$ is defined as follows

$$\Omega_{\eta,\zeta} = \{(x,\gamma) \in X_{ad} \times \mathbb{R} \ | \ f(x) \leq_\Lambda \eta + \gamma\zeta\}. \tag{25}$$

In order to give a sense to this problem, we note that the elements $\eta \in Y$ and $\zeta \in Y$ in (25) should be related by the following conditions:

(A_1) There exists at least one element $x \in X_{ad}$ such that $f(x) \leq_\Lambda \eta + \zeta$.

The following result inherits the main ideas of the Pascoletti-Serafini approach and shows that it can be extended to the case of vector optimization problems in Banach spaces with epi-lower semicontinuous objective mappings and non-solid ordering cone.

Theorem 5.7: *Let X_{ad} be a non-empty sequential σ-compact subset of X, let $f : X_{ad} \to Y$ be a locally bounded from below and* epi-*lower semicontinuous objective mapping, and let K be a reproducing cone in V. Then*

(i) If $x^0 \in \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$ then there exist elements $\eta^0 \in Y$ and $\zeta^0 \in Y$ such that the pair $(x^0, 0)$ is optimal for the scalar problem (24)–(25), moreover,

$$(x^0, 0) \in \Omega_{\eta^0,\zeta^0} \quad \textit{and} \quad \inf_{(x,\gamma)\in\Omega_{\eta^0,\zeta^0}} \Phi(x,\gamma) = 0. \tag{26}$$

(ii) If $(x^0, \gamma^0) \in X \times \mathbb{R}$ is a minimizer to the scalar problem (24)–(25) with some $\eta \in Y$ and $\zeta \in Y$ such that $\eta + \gamma^0\zeta \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$, then

$$x^0 \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda). \tag{27}$$

(iii) If $(x^0, \gamma^0) \in X \times \mathbb{R}$ is a unique minimizer to the scalar problem (24)–(25) with some $\eta \in Y$ and $\zeta \in Y$, then

$$x^0 \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda). \tag{28}$$

(iv) If elements $\eta \in Y$ and $\zeta \in Y$ are such that $\Omega_{\eta,\zeta} \neq \emptyset$ then the scalar problem (24)–(25) has a nonempty set of minimizers.

Proof: Part (i). Let $x^0 \in X_{ad}$ be a (Λ, τ)-efficient solution to the vector optimization problem (15). Then $f(x^0) \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$. We set

$$\eta^0 = f(x^0) \quad \text{and} \quad \zeta^0 \ \text{ is an arbitrary nontrivial element of } \ \Lambda. \tag{29}$$

Since $f(x^0) = \eta^0 + 0\,\zeta^0$, it follows that the pair $(x^0, 0)$ is admissible to the scalar problem (24)–(25). In order to prove the second part of (26), we assume, by contradiction, that $(x^0, 0) \notin \operatorname*{Argmin}_{(x,\gamma)\in\Omega_{\eta^0,\zeta^0}} \Phi(x,\gamma)$. Then there is a pair $(\widehat{x}, \widehat{\gamma}) \in \Omega_{\eta^0,\zeta^0}$ with properties: $\widehat{\gamma} = \Phi(\widehat{x}, \widehat{\gamma}) < \Phi(x^0, 0) = 0$. Using this relation and the fact that $(\widehat{x}, \widehat{\gamma}) \in \Omega_{\eta^0,\zeta^0}$, we obtain $f(\widehat{x}) \leq_\Lambda \eta^0 + \widehat{\gamma}\,\zeta^0 < \eta^0 \overset{\text{by (29)}}{=} f(x^0)$, and we come into conflict with the condition $x^0 \in \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$. This proves Part (i).

Part (ii) Let $(x^0, \gamma^0) \in X \times \mathbb{R}$ be a minimizer for the scalar problem (24)–(25). Let $\{\gamma_k\}_{k=1}^\infty \subset \mathbb{R}$ be any monotonically decreasing sequence of numbers such that $\gamma_k \to \gamma^0$ as k tends to ∞. We set $\eta_k = \eta + \gamma_k\zeta$ for every $k \in \mathbb{N}$. It is clear that $\{\eta_k\}_{k=1}^\infty$ is the Λ-decreasing sequence in Y for which $\eta^* = \eta + \gamma^0\zeta$ is its τ-limit. Since $(x^0, \gamma^0) \in \Omega_{\eta,\zeta}$ it follows that

$$x^0 \overset{\text{by (25)}}{\in} \Theta_{\eta^*}, \text{ and, hence, } x^0 \overset{\text{by monotonicity of } \eta_k}{\in} \Theta_{\eta_k}\ \forall\, k \in \mathbb{N}.$$

Therefore, the condition $\eta^* = \eta + \gamma^0\zeta \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$ and Theorem 5.6 imply: $\Theta_{\eta_k} \overset{K_s}{\to} \Theta_{\eta^*}$ and, hence, $x^0 \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda)$.

Part (iii) Let $(x^0, \gamma^0) \in X \times \mathbb{R}$ be a unique minimizer for the scalar problem (24)–(25). In this case we cannot apply Theorem 5.6 because the element

$\eta + \gamma^0\zeta$ does not belong in general to the set $\mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$. At the same time, we have: $f(x^0) = \eta + \gamma^0\,\zeta$. Let $\{(x_k,\gamma_k)\}_{k=1}^\infty$ be a minimizing sequence to the problem (24)–(25), i.e., $x_k \overset{\sigma}{\to} x^0$ and $\gamma_k \to \gamma^0$. We always can make this choice such that the sequence$\{\gamma_k\}_{k=1}^\infty$ is monotonically decreasing. Then, because of the property (A_1), the sequence of images $\{f(x_k)\}_{k=1}^\infty$ is bounded in Y (see arguments in the proof of Theorem 5.6). Hence, by Banach-Alaoglu theorem there exists an element $\xi \in \mathrm{L}^\mu(f,x^0)$ such that $f(x_k) \overset{\tau}{\to} \xi$ as $k \to \infty$. Let us assume by contradiction that $\xi \notin \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$. Then there exists an element $\widehat{x} \in X_{ad}$ ensuring the inequality $f(\widehat{x}) <_\Lambda \xi$. Combining this fact with the property $f(x^0) \le_\Lambda z$ for all $z \in \mathrm{L}^\mu(f,x^0)$, we arrive at the conclusion: $f(\widehat{x}) \le_\Lambda f(x^0)$. At the same time we have $f(x^0) = \eta + \gamma^0\,\zeta$ and $\eta + \gamma^0\zeta \notin \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$. As a result, we get: $\big(f(x^0) - \Lambda\big) \cap f(X_{ad}) = \emptyset$. Comparing this condition with relation $\big(f(x^0) - \Lambda\big) \cap f(X_{ad}) = \emptyset$, we conclude $f(\widehat{x}) = f(x^0)$. This, however, contradicts the uniqueness of the solution (x^0,γ^0) of the scalar problem (24)–(25). Thus, $\xi \in \mathrm{Inf}^{\Lambda,\tau}_{x\in X_{ad}} f(x)$, and hence, $x^0 \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda)$.

For the proof of Part (iv) it is sufficient to observe that the set $\Omega_{\eta,\zeta}$ is a nonempty and sequential compact subset of $X \times \mathbb{R}$ with respect to the product of σ-topology and the topology of pointwise convergence in $\mathbb{R}$ (see property (A_1)). Hence, in view of the linear structure of the objective function $\Phi : \Omega_{\eta,\zeta} \times \mathbb{R} \to \mathbb{R}$, the direct method in the Calculus of Variations immediately implies the required conclusion. The proof is complete. $\square$

6. On Quadratic Regularization of Parametrized Problem (24)–(25)

Our main interest in this section is to reduce the parametrized problem (24)–(25) and show that the quadratic regularization approach can be applied in this case. Let $\eta \in Y$ and $\zeta \in Y$ be given elements related by condition (A_1). We assume that there exist a real Banach space Z and an operator $\vec{P} : Z \to X \times \mathbb{R}^2$ with decomposition $\vec{P}(z) = (P_1(z), P_2(z), P_3(z))$, where $P_1 : Z \to X$, $P_2 : Z \to \mathbb{R}$, and $P_3 : Z \to \mathbb{R}$ are such that

- (B_1) The mappings $P_1 : Z \mapsto X$ and $P_2 : Z \to \mathbb{R}$ are surjective in the following sense: for every $(x,\gamma) \in X \times \mathbb{R}$ and every $c > 0$ there exists an element $z \in Z$ satisfying: $P_1(z) = x$, $P_2(z) = \gamma$, and $\|z\|_Z \ge c$.
- (B_2) $P_3(z) = \|z\|_Z^2$ for all $z \in Z$.

Remark 1: In finite dimensional case ($X = \mathbb{R}^N$), we suggest the following structure for the space Z and transformation $\vec{P} : Z \to X \times \mathbb{R}^2$ (see [22])

$$Z = \mathbb{R}^{N+2},$$

$$\vec{P}(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \\ P_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ z_1 & z_2 & \cdots & z_{N+1} & z_{N+2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_{N+1} \\ z_{N+2} \end{bmatrix}.$$

Further, we consider the following family of parametrized scalar optimization problems

$$F(z) = \|z\|_Z^2 \longrightarrow \inf \tag{30}$$

subjected to the constraints $z \in \Xi_{\eta,\zeta}$, where

$$\Xi_{\eta,\zeta} = \left\{ z \in Z \;\middle|\, P_1(z) \in X_{ad}\,, f(P_1(z)) \leq_\Lambda \eta + P_2(z)\zeta, P_2(z) + s \leq \|z\|_Z^2, \right\} \quad (31)$$

and s is a positive constant.

Theorem 6.1 : *Let X_{ad} be a non-empty sequential σ-compact subset of X and let $f : X_{ad} \to Y$ be a locally bounded from below and* epi-*lower semicontinuous on X_{ad} objective mapping, and let K be a reproducing cone in V. Let also Z be a reflexive Banach space endowed with the weak topology and let P_1 and P_2 be continuous mappings in following sense*

$$P_1(z_k) \stackrel{\sigma}{\to} P_1(z) \quad in \quad X \quad and \quad P_2(z_k) \to P_2(z) \quad in \quad \mathbb{R} \quad (32)$$

provided $z_k \to z$ weakly in Z. Assume that the elements $\eta \in Y$ and $\zeta \in Y$ are such that the property (A_1) and Hypotheses (B_1)–(B_2) are valid. Then following assertions hold true:

(a) There exists a constant $s > 0$ such that if $z^0 \in \Xi_{\eta,\zeta}$ is a unique minimizer for $\inf\limits_{z\in\Xi_{\eta,\zeta}} F(z)$ then $x^0 := P_1(z^0) \in \mathrm{GenEff}_{\sigma,\tau}(X_{ad};\, f;\, \Lambda)$.

(b) If $x^0 \in \mathrm{Eff}_\tau(X_{ad};\, f;\, \Lambda)$ then there exist elements $\eta^0 \in Y$, $\zeta^0 \in Y$, $z^0 \in Z$, and $s > 0$ such that $x^0 = P_1(z^0)$, $z^0 \in \Xi_{\eta^0,\zeta^0}$, and z^0 is a minimizer for the regularized problem (30)–(31) under $\eta = \eta^0$ and $\zeta = \zeta^0$.

Proof: To begin with, we note that condition (A_1) ensures the fulfilment of the inequality $\inf_{(x,\gamma)\in\Omega_{\eta,\zeta}} \Phi(x,\gamma) > -\infty$. Since X_{ad} is a σ-compact subset of X and, hence, X_{ad} is bounded, the choice of a positive value s in $(31)_3$ can be realized as

$$s > -\inf_{(x,\gamma)\in\Omega_{\eta,\zeta}} \Phi(x,\gamma) + \sup_{z\in Z} \inf_{P_1(z)\in X_{ad}} \|z\|_Z^2.$$

Let $z^0 \in \Xi_{\eta,\zeta}$ be a minimizer to the regularized problem (30)–(31). We use the following transformation

$$x = P_1(z), \quad \gamma = P_2(z) \quad \kappa = \|z\|_Z^2 := P_3(z) \quad \forall\, z \in \Xi_{\lambda,\eta,\zeta} \quad (33)$$

As a result, the description (31) leads us to the implication: "If $z \in \Xi_{\eta,\zeta}$, then $(x,\gamma) := (P_1(z), P_2(z)) \in \Omega_{\eta,\zeta}$. " So, the pair $(x^0,\gamma^0) := \big(P_1(z^0), P_2(z^0)\big)$ is admissible for the scalar optimization problem (24)–(25). The converse implication

$$\text{If} \quad (x,\gamma) \in \Omega_{\eta,\zeta} \quad \text{then} \quad \exists z \in Z \quad \text{such that } x = P_1(z),\ \gamma = P_2(z),\ z \in \Xi_{\eta,\zeta}$$

holds true due to Hypothesis (B_1). Hence, the images of functions $\Phi : \Omega_{\eta,\zeta} \to \mathbb{R}$ and $P_2 : \Xi_{\eta,\zeta} \to \mathbb{R}$ coincide. Therefore,

$$\inf_{(x,\gamma)\in\Omega_{\eta,\zeta}} \Phi(x,\gamma) \leq P_2(z) \quad \forall\, z \in \Xi_{\lambda,\eta,\zeta}. \quad (34)$$

Moreover, in view of the continuity property (32), we have an obvious deduction: If $z^0 \in \Xi_{\eta,\zeta}$ is a minimizer to the regularized problem (30)–(31), then

$$P_2(z^0) + s \stackrel{\text{by } (31)_3}{=} \|z^0\|_Z^2 \stackrel{\text{by } (34)}{=} \inf_{(x,\gamma)\in\Omega_{\eta,\zeta}} \Phi(x,\gamma) + s.$$

Combining these observations, we arrive at the conclusion:

$$\inf_{(x,\gamma)\in\Omega_{\lambda,\eta,\zeta}} \Phi_{\lambda,\eta,\zeta}(x,\gamma) = \Phi_{\lambda,\eta,\zeta}(x^0,\gamma^0) = \|z^0\|_Z^2 - s,$$

i.e., the pair $(x^0,\gamma^0) := \big(P_1(z^0), P_2(z^0)\big)$ is optimal for the scalar optimization problem (24)–(25). As a result, it remains to apply Theorem 5.7, item (iii). Thus, assertion (a) is valid.

In order to prove assertion (b) of this theorem, we note that if x^0 is a (Λ,τ)-efficient solution to the original vector optimization problem (15), then Theorem 5.7 ensures the existence of elements $\eta^0 \in Y$ and $\zeta^0 \in Y$ such that the pair $(x^0, 0)$ is optimal for the scalar problem (24)–(25) and $\inf_{(x,\gamma)\in\Omega_{\eta^0,\zeta^0}} \Phi(x,\gamma) = 0$. Having put $s > \sup_{z\in Z} \inf_{P_1(z)\in X_{ad}} \|z\|_Z^2$ in (31), we indicate the element $z^0 \in Z$ as follows $P_1(z^0) = x^0$, $P_2(z^0) = 0$, and $s = \|z^0\|_Z^2$. Note that the existence of such elements comes from Hypothesis (B_1). Then direct calculations show that $z^0 \in \Xi_{\eta^0,\zeta^0}$ and

$$0 = \inf_{(x,\gamma)\in\Omega_{\eta^0,\zeta^0}} \Phi(x,\gamma) = \Phi(x^0,0) = P_2(z^0) = \|z^0\|_Z^2 - s.$$

Hence, z^0 is a minimizer for the problem $\inf_{z\in\Xi_{\eta^0,\zeta^0}} F(z)$. This concludes the proof. □

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