ON THE PROPERTY OF HOMOTHETIC MEAN VALUE ON PERIODICALLY PERFORATED DOMAINS

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Abstract. In this paper we study the limit properties of one class of periodic functions, as ε tends to zero, defined on ε-periodically perforated domain. The holes have a critical size with respect to ε-sized mesh of periodicity. For this type of functions we derive the so-called homothetic mean-value property.

Key words. perforated domain, measure approach, convergence in variable spaces.

AMS subject classifications. 35B27, 49J20, 49J27

1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open domain, and let ε be a small positive parameter. We define a perforated domain Ω_ε as follows: let $Y = [-1/2, +1/2)^n$; Q and K be compact subsets of Y such that $0 \in \mathrm{int}K \cap \partial Q$,

$$\Theta_\varepsilon = \{\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n \ : \ (\varepsilon Y + \varepsilon\,\mathbf{k}) \cap \Omega \neq \emptyset\}\,; \tag{1.1}$$

$$Y_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{\varepsilon(Y + \mathbf{k})\}\,; \qquad T_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left\{\varepsilon^{n/(n-1)} Q + \varepsilon\,\mathbf{k}\right\}; \tag{1.2}$$

$$S_\varepsilon = \begin{cases} \varepsilon^{n/(n-2)} K \cap \partial(\varepsilon^{n/(n-1)} Q), & n \geq 3, \\ \exp\left(-1/\varepsilon^2\right) K \cap \partial(\varepsilon^{n/(n-1)} Q), & n = 2, \end{cases} \tag{1.3}$$

$$\Gamma_\varepsilon^D = \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{S_\varepsilon + \varepsilon\,\mathbf{k}\}\right] \cap \overline{\Omega}, \qquad \Gamma_\varepsilon^N = \left[\partial T_\varepsilon \setminus \Gamma_\varepsilon^D\right] \cap \overline{\Omega}. \tag{1.4}$$

Then we set $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$. The principal feature of this perforated domain is the fact that the size of the holes $Q_\varepsilon + \varepsilon\,\mathbf{k}$ and their boundaries Γ_ε^D, Γ_ε^N are not proportional to the size of the periodicity cell εY.

The aim of this paper is to study the limit properties of the following classes of periodic functions

$$\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right) d\mu_\varepsilon^{\lambda,h}, \quad \int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\,h(\varepsilon)}\right) d\nu_\varepsilon^{\lambda,h}$$

as ε tends to zero and to obtain an explicit form of these limits. Here $0 < \lambda << h < 1$, $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$, $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 0$, $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ are some singular periodic Borel measures concentrated on the manifolds

$$\Gamma^{\lambda,h} = \lambda K \cap \partial(h\,Q) \quad \text{and} \quad \Lambda^{\lambda,h} = \partial(h\,Q) \setminus \Gamma^{\lambda,h},$$

respectively.

Throughout the present paper we suppose that: Ω is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; the compact set Q has Lipschitz boundary ∂Q, $\mathrm{int}\,Q$ is a strongly connected set, $Q \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_1 \geq 0\}$, and its boundary ∂Q contains the

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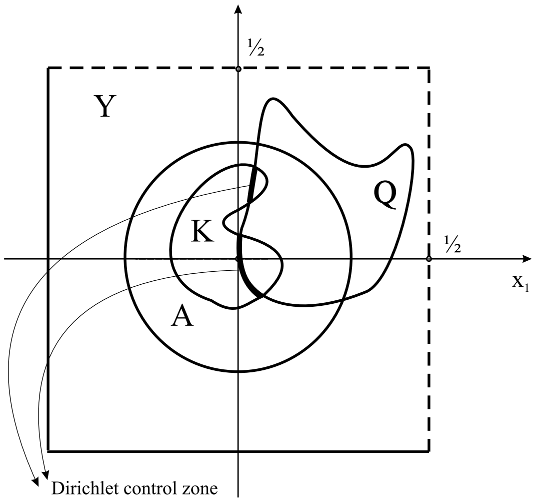

FIG. 1.1. *Example of perforation scheme*

origin; $A = B(\mathbf{0}, r_0)$ is an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \subset\subset Y$ and $K \subset\subset A$ (see Fig. 1.1 for 2-d example). For any subset $E \subset \Omega$ we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}^n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of manifold ∂E on $\mathbb{R}^n$. We always suppose that the sets $K \cap \partial Q^\varsigma$ and $\partial Q \setminus (K \cap \partial Q^\varsigma)$ have non-zero capacity for any $\varsigma > 1$, where $Q^\varsigma = \{\varsigma x, \ \forall\, x = (x_1, \dots, x_n) \in Q\}$ is the homothetic stretching of Q by a factor of ς. Hence $|K \cap \partial Q^\varsigma|_H \neq 0$ for all $\varsigma > 1$.

2. Measure approach to the description of the sets Ω_ε. In this section we describe of the geometry of perforated domain Ω_ε in the terms of singular periodic Borel measures on $\mathbb{R}^n$ following to Zhikov's approach [3].

Let us denote by K^λ and Q^h the homothetic contractions of the sets K and Q at λ^{-1} and h^{-1} times. In what follows it is always assumed that $0 < \lambda << h < 1$. Let the sets $\Gamma^{\lambda,h}$ and $\Lambda^{\lambda,h}$ be defined as follows

$$\Gamma^{\lambda,h} = K^\lambda \cap \partial Q^h, \quad \Lambda^{\lambda,h} = \partial Q^h \setminus \Gamma^{\lambda,h}. \tag{2.1}$$

Let $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ be the normalized periodic Borel measures on $\mathbb{R}^n$ with the periodicity cell Y such that $\mu^{\lambda,h}$ is concentrated on $\Gamma^{\lambda,h}$, $\nu^{\lambda,h}$ is concentrated on $\Lambda^{\lambda,h}$, and both these measures are proportional to the $(n-1)$-dimensional Hausdorff measure. Since these measures are concentrated and uniformly distributed on the corresponding sets, it follows that $\mu^{\lambda,h}(Y \setminus \Gamma^{\lambda,h}) = 0$.

For any function $\varphi \in C^\infty(\mathbb{R}^n)$ we have

$$\int_{\Gamma^{\lambda,h}} \varphi\, d\mathcal{H}^{n-1} = |\Gamma^{\lambda,h}|_H \int_Y \varphi\, d\mu^{\lambda,h} = \lambda^{n-1} |K \cap \partial Q^{h/\lambda}|_H \int_Y \varphi\, d\mu^{\lambda,h}, \tag{2.2}$$

$$\begin{aligned} \int_{\Lambda^{\lambda,h}} \varphi\, d\mathcal{H}^{n-1} &= |\Lambda^{\lambda,h}|_H \int_Y \varphi\, d\nu^{\lambda,h} = \left(|\partial Q^h|_H - |\Gamma^{\lambda,h}|_H\right) \int_Y \varphi\, d\nu^{\lambda,h} = \\ &= \left(h^{n-1}|\partial Q|_H - \lambda^{n-1}|K \cap \partial Q^{h/\lambda}|_H\right) \int_Y \varphi\, d\nu^{\lambda,h}, \end{aligned} \tag{2.3}$$

where by $|C|_H$ we denote the $(n-1)$-dimensional Hausdorff measure of manifold C.

Let δ be the periodic Dirac measure on $\mathbb{R}^n$ concentrated at nodes of the grid $\mathbb{Z}^n$ and such that $\delta(\mathbf{k}) = 1$ for all $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. It is clear that the measures $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ converge weakly to δ as $\max(\lambda, h) \to 0$ (in symbols, $\mu^{\lambda,h} \rightharpoonup \delta$, $\nu^{\lambda,h} \rightharpoonup \delta$), i.e.

$$\left.\begin{aligned} \lim_{\max(\lambda,h)\to 0} \int_Y \varphi \, d\mu^{\lambda,h} &= \int_Y \varphi \, d\delta = \varphi(0), \\ \lim_{\max(\lambda,h)\to 0} \int_Y \varphi \, d\nu^{\lambda,h} &= \int_Y \varphi \, d\delta = \varphi(0) \end{aligned}\right\} \tag{2.4}$$

for every $\varphi \in C^\infty_{per}(Y)$.

We introduce also the scaling measures $\mu^{\lambda,h}_\varepsilon$ and $\nu^{\lambda,h}_\varepsilon$ by setting

$$\mu^{\lambda,h}_\varepsilon(B) = \varepsilon^n \mu^{\lambda,h}(\varepsilon^{-1}B), \quad \nu^{\lambda,h}_\varepsilon(B) = \varepsilon^n \nu^{\lambda,h}(\varepsilon^{-1}B)$$

for every Borel set $B \subset \mathbb{R}^n$, and relate the parameters λ, h, and ε by the rule

$$h(\varepsilon) = \varepsilon^{n/(n-1)}, \quad \lambda(\varepsilon) = \varepsilon^{n/(n-2)} \text{ if } n \geq 3, \text{ and } \lambda(\varepsilon) = \exp(-\varepsilon^{-2}) \text{ if } n = 2. \tag{2.5}$$

Then

$$\int_{\varepsilon Y} d\mu^{\lambda,h}_\varepsilon = \varepsilon^n \int_Y d\mu^{\lambda,h} = \varepsilon^n, \quad \int_{\varepsilon Y} d\nu^{\lambda,h}_\varepsilon = \varepsilon^n \int_Y d\nu^{\lambda,h} = \varepsilon^n.$$

It means that the measures $\mu^{\lambda,h}_\varepsilon$ and $\nu^{\lambda,h}_\varepsilon$ weakly converge to the Lebesgue measure:

$$d\mu^{\lambda,h}_\varepsilon \rightharpoonup dx, \quad d\nu^{\lambda,h}_\varepsilon \rightharpoonup dx,$$

that is for every $\varphi \in C^\infty_0(\mathbb{R}^n)$ we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi \, d\mu^{\lambda,h}_\varepsilon = \int_{\mathbb{R}^n} \varphi \, dx, \ \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi \, d\nu^{\lambda,h}_\varepsilon = \int_{\mathbb{R}^n} \varphi \, dx. \tag{2.6}$$

We see that

$$\Gamma^D_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left[K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)} + \varepsilon\mathbf{k}\right], \ \Gamma^N_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left[\partial Q^{h(\varepsilon)} \setminus (K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)}) + \varepsilon\mathbf{k}\right].$$

Then, using the properties (2.2)–(2.3) and setting

$$\sigma(\varepsilon) = \left(h(\varepsilon)^{n-1}|\partial Q|_H - \lambda(\varepsilon)^{n-1}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H\right), \tag{2.7}$$

any integral term $\int_{\Gamma^N_\varepsilon} p_\varepsilon \, \varphi \, d\mathcal{H}^{n-1}$ can be rewritten in the form

$$\begin{aligned} \int_{\Gamma^N_\varepsilon} p_\varepsilon \, \varphi \, d\mathcal{H}^{n-1} &= \sum_{\mathbf{k}\in\Theta_\varepsilon} \int_{\partial Q^{h(\varepsilon)}\setminus\Gamma^{\lambda(\varepsilon),h(\varepsilon)}+\varepsilon\mathbf{k}} p_\varepsilon \, \varphi \, d\mathcal{H}^{n-1} = \\ &= \sigma(\varepsilon) \sum_{\mathbf{k}\in\Theta_\varepsilon} \int_{\varepsilon(Y+\mathbf{k})} \widehat{p}_\varepsilon \, \varphi \, d\nu^{\lambda(\varepsilon),h(\varepsilon)}(x/\varepsilon) = \\ &= \varepsilon^{-n}\sigma(\varepsilon) \sum_{\mathbf{k}\in\Theta_\varepsilon} \int_{\varepsilon(Y+\mathbf{k})} \widehat{p}_\varepsilon \, \varphi \, d\nu^{\lambda(\varepsilon),h(\varepsilon)}_\varepsilon = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \widehat{p}_\varepsilon \, \varphi \, d\nu^{\lambda(\varepsilon),h(\varepsilon)}_\varepsilon \end{aligned} \tag{2.8}$$

for every function $\varphi \in C^\infty_0(\mathbb{R}^n; \Sigma_\varepsilon \cup \Gamma^D_\varepsilon) = \{\psi \in C^\infty_0(\mathbb{R}^n) \ : \ \psi = 0 \text{ on } \Sigma_\varepsilon \cup \Gamma^D_\varepsilon\}$. Here $\widehat{p}_\varepsilon$ is a function of $L^2(\Omega, d\nu^{\lambda,h}_\varepsilon)$ taking the same values with $p_\varepsilon \in L^2(\Gamma^N_\varepsilon)$ on Γ^N_ε. We note that the integral $\int_\Omega \widehat{p}_\varepsilon \varphi \, d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)}$ is well defined. Indeed, since the set Ω is bounded, and $\widehat{p}_\varepsilon \, d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)}$ is a Radon measure, it follows that $\int_\Omega \widehat{p}_\varepsilon \varphi \, d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)}$ is a linear continuous functional on $C_0^\infty(\mathbb{R}^n; \Sigma_\varepsilon \cup \Gamma_\varepsilon^D)$.

Thus

$$\int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \, d\mathcal{H}^{n-1} = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \widehat{p}_\varepsilon^{\,2} \, d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)}, \tag{2.9}$$

where

$$\varepsilon^{-n}\sigma(\varepsilon) = \left[\frac{h(\varepsilon)^{n-1}}{\varepsilon^n}|\partial Q|_H - \frac{\lambda(\varepsilon)^{n-1}}{\varepsilon^n}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H\right] =$$

$$= \begin{cases} |\partial Q|_H - \varepsilon^{n/(n-2)}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H, & \text{if } n \geq 3, \\ |\partial Q|_H - \dfrac{1}{\varepsilon^2}\exp(-\dfrac{1}{\varepsilon^2})|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H, & \text{if } n = 2. \end{cases} \tag{2.10}$$

By analogy we obtain

$$\int_{\Gamma_\varepsilon^N} y_\varepsilon \, \varphi \, d\mathcal{H}^{n-1} = \varepsilon^{-n}\sigma(\varepsilon) \int_\Omega \widehat{y}_\varepsilon \, \varphi \, d\nu_\varepsilon^{\lambda,h} \ , \tag{2.11}$$

$$\int_{\Gamma_\varepsilon^D} u_\varepsilon \, d\mathcal{H}^{n-1} = \varepsilon^{-n}\lambda(\varepsilon)^{n-1}|K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H \int_\Omega \widehat{u}_\varepsilon \, d\mu_\varepsilon^{\lambda,h} \ . \tag{2.12}$$

for any functions $y_\varepsilon \in L^2(\Gamma_\varepsilon^N)$ and $u_\varepsilon \in L^2(\Gamma_\varepsilon^D)$. Here $\widehat{y}_\varepsilon \in L^2(\Omega; \Sigma_\varepsilon)$ and $\widehat{u}_\varepsilon$ are the functions taking the same values with y_ε and u_ε on and , correspondingly.

At the end of this section we cite some auxiliary results that we feel to be interesting per se. Let

$$\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \left\{ \begin{array}{ll} \exp\left(\dfrac{-n}{n^2 - 3n + 2}\ln\varepsilon\right), & \text{if } \ n \geq 3, \\ \varepsilon^2 \exp\left(\dfrac{1}{\varepsilon^2}\right) & \text{if } \ n = 2. \end{array} \right\} \tag{2.13}$$

Then

$$\varsigma(\varepsilon) \in (1, +\infty) \ \forall \, \varepsilon > 0 \ \text{ and } \ \lim_{\varepsilon \to 0} \varsigma(\varepsilon) = +\infty. \tag{2.14}$$

We are interested the limit behaviour of the sequence $\{|K \cap \partial Q^\varsigma(\varepsilon)|_H\}$ as $\varepsilon \to 0$. We recall that each of the sets $Q^\varsigma(\varepsilon) = \{\varsigma(\varepsilon)x, \ \forall \, x = (x_1, \ldots, x_n) \in Q\}$ is the homothetic stretching of Q at $\varsigma(\varepsilon)$ times.

Proposition 2.1. *There exist an open cone*

$$\Lambda \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 > 0\} \tag{2.15}$$

such that

$$\lim_{\varepsilon \to 0} |K \cap \partial Q^\varsigma(\varepsilon)|_H = |K \cap \partial\Lambda|_H. \tag{2.16}$$

Proof. Indeed, by initial assumptions the origin is a Lipschitz point of the boundary ∂Q and $\operatorname{int} Q$ is a strongly connected set in classical sense. Hence there is a neighbourhood $\mathcal{U}(0)$ such that $\mathcal{U}(0) \cap \operatorname{int}Q$ is a convex set [1],[2]. Hence the set

$$\Lambda = \{x \in \mathbb{R}^n \ : \ x \in t\,[\mathcal{U}(0) \cap \operatorname{int}Q] \ \ \forall \, t \in (0, +\infty) \ \}$$

is a non-empty open cone.

Assume that the origin is a Lipschitz point of ∂Q, but does not belong to a smooth part of the boundary ∂Q. Then the inclusion $K \cap \Lambda \subset K \cap Q^{\varsigma(\varepsilon)}$ holds true for ε small enough, and it immediately implies the existence of a value $\varepsilon_0 > 0$ such that

$$|K \cap \partial\Lambda|_H = |K \cap \partial Q^{\varsigma}(\varepsilon)|_H \quad \forall\, \varepsilon < \varepsilon_0.$$

If a part of boundary ∂Q containing the origin is smooth it follows that there is a neighbourhood $\mathcal{U}(0)$ of the origin such that $\mathcal{U}(0) \cap \partial Q$ is a graph of a smooth function whose epigraph contains $\mathcal{U}(0) \cap Q$. So that we may always suppose that there is a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq}$ such that $\Psi \in C_0^\infty(\mathbb{R}^{n-1})$ and $x_1 = \Psi(x_2, \dots, x_n)$ for every $x = (x_1, x_2, \dots, x_n) \in \mathcal{U}(0) \cap \partial Q$.

Let $\Lambda = \{x \in \mathbb{R}^n \ : \ x_1 > 0\}$. Then $\partial\Lambda = \{x \in \mathbb{R}^n \ : \ x_1 = 0\}$ and taking into account the homothetic transform we have that $x = (x_1, x_2, \dots, x_n) \in K \cap \varsigma(\varepsilon)\partial Q$ for small enough ε if and only if $x \in \varsigma(\varepsilon)\,(\mathcal{U}(0) \cap \partial Q)$. Hence

$$x_1 = \frac{1}{\varsigma(\varepsilon)}\Psi(\varsigma(\varepsilon)x_2, \dots, \varsigma(\varepsilon)x_n) \quad \text{for every} \quad x \in K \cap \varsigma(\varepsilon)\partial Q.$$

Since $\lim_{\varepsilon\to 0} \varsigma^{-1}(\varepsilon)\Psi(\varsigma(\varepsilon)x_2, \dots, \varsigma(\varepsilon)x_n) = 0$ by definition of Hausdorff measure $\mathcal{H}^{n-1}$ we immediately obtain the required. □

Remark 2.2. *As follows from the proof of this proposition the cone* Λ *can be recovered in an explicit form for the case when the origin belongs to a smooth part of the boundary* ∂Q. *Moreover, as evident consequence of Proposition 2.1 and formula* (2.7) *we have*

$$\lim_{\varepsilon\to 0} \sigma(\varepsilon)/\varepsilon^n = |\partial Q|_H. \tag{2.17}$$

In a similar way it can be proved the following statement.

Proposition 2.3. *Let* $\{\rho'_\varepsilon \in \mathbb{R}\}_{\varepsilon>0}$ *be a numerical sequence such that*

$$\rho'_\varepsilon = |A \setminus Q^{\varsigma(\varepsilon)}|/|A| \quad \forall\, \varepsilon > 0. \tag{2.18}$$

Then $\{\rho'_\varepsilon\}_{\varepsilon>0}$ *is the monotone sequence and there exists a value* $\rho^* \in [1/2, 1)$ *such that* $\lim_{\varepsilon\to 0} \rho'_\varepsilon = \rho^*$.

Remark 2.4. *The example at Fig. 2.1 indicates the case when the origin in not a Lipschitz's point of* ∂Q *and hence* ρ^* *can be equal to* 1.

3. Convergence in the variable space $\mathbb{X}_\varepsilon$. Let us recall the main types of convergence in variable spaces occurring in homogenization theory (see [3],[4],[5]). We cite their with respect to the family of periodic Borel measure $\mu_\varepsilon^{\lambda,h}$ that was introduced in (2) and parameters $\lambda = \lambda(\varepsilon)$ and $h = h(\varepsilon)$ are defined by (2.6). Let $\{v_\varepsilon^{\lambda,h} \in L^2(\Omega, d\mu_\varepsilon^{\lambda,h}\}$ be a bounded sequence, i.e. $\limsup_{\varepsilon\to 0} \int_\Omega (v_\varepsilon^{\lambda,h})^2\, d\mu_\varepsilon^{\lambda,h} < +\infty$.

1. The weak convergence $v_\varepsilon^{\lambda,h} \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ means that

$$v \in L^2(\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^{\lambda,h}\varphi\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega v\varphi\, dx \text{ for any } \varphi \in C_0^\infty(\Omega);$$

2. The strong convergence $v_\varepsilon^{\lambda,h} \to v$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ means that

$$v \in L^2(\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^{\lambda,h} z_\varepsilon^{\lambda,h}\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega vz\, dx$$

if $z_\varepsilon^{\lambda,h} \rightharpoonup z$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$;

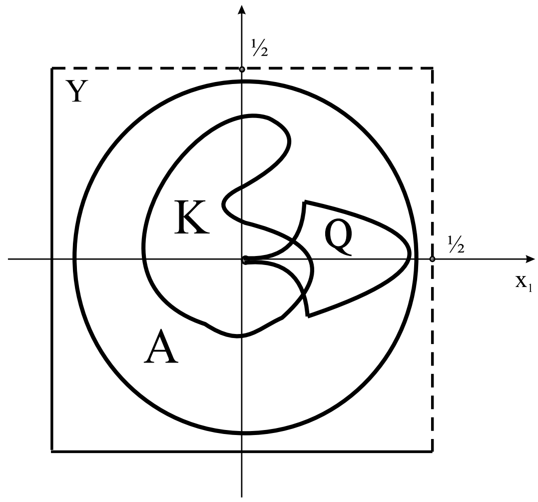

FIG. 2.1. *Example of the set Q with non-Lipschitz boundary at the origin*

3. if a sequence $\{b^{\lambda,h} \in L^2(Y, d\mu^{\lambda,h})\}$ is bounded, then the weak convergence $b^{\lambda,h} \rightharpoonup b$ in $L^2(Y, d\mu^{\lambda,h})$ means that $b \in L^2(Y, d\delta)$ and

$$\lim_{\max(\lambda,h)\to 0} \int_Y b^{\lambda,h}\psi\, d\mu^{\lambda,h} = \int_Y b\psi\, d\delta \quad \text{for any} \quad \psi \in C^\infty_{per}(Y),$$

where $d\delta$ is the Y-periodic Dirac measure concentrated at the origin.

The following properties of convergence in variable spaces hold:

(a) *Compactness criterium:* if a sequence is bounded in $L^2(\Omega, d\mu^{\lambda,h}_\varepsilon)$ then this sequence is compact in the sense of weak convergence;

(b) *Property of lower semicontinuity:* if $v^{\lambda,h}_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu^{\lambda,h}_\varepsilon$ then

$$\liminf_{\varepsilon\to 0} \int_\Omega (v^{\lambda,h}_\varepsilon)^2\, d\mu^{\lambda,h}_\varepsilon \geq \int_\Omega v^2\, dx;$$

(c) *Criterium of strong convergence:* $v^{\lambda,h}_\varepsilon \to v$ if and only if

$$v^{\lambda,h}_\varepsilon \rightharpoonup v \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_\Omega (v^{\lambda,h}_\varepsilon)^2\, d\mu^{\lambda,h}_\varepsilon = \int_\Omega v^2\, dx;$$

(d) Since $\mu^{\lambda,h}_\varepsilon \rightharpoonup dx$ it follows that (see Lemma 1 [5])

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi\, \mu^{\lambda,h}_\varepsilon = \int_\Omega \varphi\, dx \quad \forall \ \in C(\overline{\Omega}) \ \text{ and } \ \limsup_{\varepsilon\to 0} \mu^{\lambda,h}_\varepsilon(F) \leq |F|$$

for any compact set $F \subset \Omega$.

4. The homothetic mean value property. PROPOSITION 4.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Y-periodic such that $g \in L^2(\partial Q, d\mathcal{H}^{n-1})$. Then*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu^{\lambda,h}_\varepsilon = \left(\frac{1}{|\partial Q|_H} \int_{\partial Q} g\, d\mathcal{H}^{n-1}\right) \int_\Omega \varphi\, dx \tag{4.1}$$

for any $\varphi \in C(\overline{\Omega})$*. In particular,*

$$\lim_{\varepsilon\to 0}\int_\Omega g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} = |\Omega| \left(\frac{1}{|\partial Q|_H}\int_{\partial Q} g\, d\mathcal{H}^{n-1}\right).$$

Proof. It is evident that we can restrict our attention to the case when $g \geq 0$. Let us partition the set Ω into cubes εY with edges ε and denote these cubes by the symbols εY^j. Then

$$\begin{aligned}\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} &= \sum \int_{\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} + \\ &+ \sum \int_{\Omega\cap\,\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} = \\ = \sum \varphi(x_j^{\lambda,h}) \int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} &+ \sum \int_{\Omega\cap\,\varepsilon Y^j} \varphi(x)\, g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h}, \end{aligned} \tag{4.2}$$

where $x_j^{\lambda,h}$ is a point of the cube εY^j and the second sum is calculated over the set of 'boundary' cubes. Taking into account the definition of the scaling measure $\nu_\varepsilon^{\lambda,h}$ and Y-periodicity of g we have

$$\begin{aligned}\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} &= \varepsilon^n \int_{\varepsilon Y} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu^{\lambda(\varepsilon),h(\varepsilon)}\left(\frac{x}{\varepsilon}\right) = \\ &= \frac{\varepsilon^n}{\sigma(\varepsilon)} \int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\, d\mathcal{H}^{n-1}, \end{aligned} \tag{4.3}$$

where the set $\Lambda^{\lambda(\varepsilon),h(\varepsilon)}$ is defines by (2.1).

Since $\Lambda^{\lambda(\varepsilon),h(\varepsilon)} = \partial Q^{h(\varepsilon)} \setminus \Gamma^{\lambda(\varepsilon),h(\varepsilon)}$ it follows that

$$\int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1} = \int_{\partial Q^{h(\varepsilon)}} g\, d\mathcal{H}^{n-1} - \int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1}.$$

Then due to the definition of the homothetic contraction and formula (2.5) we have

$$\int_{\partial Q^{h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\, d\mathcal{H}^{n-1} = h^{n-1}(\varepsilon)\int_{\partial Q} g\, d\mathcal{H}^{n-1} = \varepsilon^n \int_{\partial Q} g\, d\mathcal{H}^{n-1}, \tag{4.4}$$

$$\int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right)\, d\mathcal{H}^{n-1} = h^{n-1}(\varepsilon)\int_{h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1}. \tag{4.5}$$

Using definition of the set $\Gamma^{\lambda(\varepsilon),h(\varepsilon)}$ (see (2.1)) we obtain

$$h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)} = K^{\lambda(\varepsilon)/h(\varepsilon)} \cap \partial Q = \lambda(\varepsilon)\left(K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}\right).$$

In view of Proposition 2.1 we have

$$\lim_{\varepsilon\to 0}\left|\lambda(\varepsilon)\left(K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)}\right)\right|_H = |K\cap\partial\Lambda|_H \lim_{\varepsilon\to 0}\lambda^{n-1}(\varepsilon) = 0.$$

Thus combining relations (4.3)–(4.5) we conclude

$$\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right)\, d\nu_\varepsilon^{\lambda,h} = \frac{\varepsilon^n}{\sigma(\varepsilon)}\varepsilon^n\left(\int_{\partial Q} g\, d\mathcal{H}^{n-1} + J(\varepsilon)\right), \tag{4.6}$$

$$J(\varepsilon) \leq \|g\|_{L^2(\partial Q, d\mathcal{H}^{n-1})} |K^{\lambda(\varepsilon)/h(\varepsilon)} \cap \partial Q|_H^{1/2} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.7}$$

In result, substituting (4.6) and (4.7) into (4.2) we have

$$\left|\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\, h(\varepsilon)}\right) d\nu_\varepsilon^{\lambda,h} - \frac{\varepsilon^n}{\sigma(\varepsilon)}\left(\int_{\partial Q} g\, d\mathcal{H}^{n-1}\right)\sum \varphi(x_j^{\lambda,h})\,\varepsilon^n\right| \leq$$
$$\leq \frac{\varepsilon^n}{\sigma(\varepsilon)} J(\varepsilon) \sum \varphi(x_j^{\lambda,h})\,\varepsilon^n +$$
$$+ \frac{\varepsilon^n}{\sigma(\varepsilon)}\left(\int_{\partial Q} g\, d\mathcal{H}^{n-1} + J(\varepsilon)\right) \sup_{x\in\Omega} |\varphi|\,\varepsilon^n D(\varepsilon), \quad (4.8)$$

where $D(\varepsilon)$ is a quantity of 'boundary' cubes. and $\varepsilon^n D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$.

Since

$$\lim_{\varepsilon\to 0} \sum \varphi(x_j^{\lambda,h})\,\varepsilon^n = \int_\Omega \varphi\, dx \quad \text{by construction of Riemann sum },$$
$$\lim_{\varepsilon\to 0} \frac{\varepsilon^n}{\sigma(\varepsilon)} = |\partial Q|_H^{-1} \quad \text{by (2.17)},$$
$$\lim_{\varepsilon\to 0} J(\varepsilon) = 0 \quad \text{by (4.7)},$$
$$\lim_{\varepsilon\to 0} \varepsilon^n D(\varepsilon) = 0 \quad \text{by Jordan's measurability property of } \partial\Omega$$

it follows that estimate (4.8) immediately leads us to the required result. □

In a similar way it can be proved the following result.

Proposition 4.2. *Let Λ be the cone that is defined in Proposition 2.1, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a Y-periodic such that $g \in H^1(K)$. Then*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right) d\mu_\varepsilon^{\lambda,h} = \left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\, d\mathcal{H}^{n-1}\right)\int_\Omega \varphi\, dx \quad (4.9)$$

for any $\varphi \in C(\overline{\Omega})$. In particular,

$$\lim_{\varepsilon\to 0} \int_\Omega g\left(\frac{x}{\varepsilon\,\lambda(\varepsilon)}\right) d\mu_\varepsilon^{\lambda,h} = |\Omega| \left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\, d\mathcal{H}^{n-1}\right)$$

Remark 4.3. *The results of Propositions 4.1–4.2 are the examples of bounded sequences in variable spaces whose weak limits can be recovered in an explicit form.*

Let us consider the following example. Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $Y = [-1/2, +1/2]^2$. We set: K is the cube of the center at the origin and sides $1/2$, Q is the circle of the center $(1/6, 0)$ in radius $1/6$, $h(\varepsilon) = \varepsilon^2$, $\lambda(\varepsilon) = \exp(-\varepsilon^{-2})$. Let $v : Y \to \mathbb{R}$ be a function such that

$$v(x,y) = x^2 + y^2 \ \text{ if } \ (x,y) \in K, \quad v(x,y) = 0 \ \text{ in } \ Y \setminus K.$$

For every positive $\alpha \in (0,1]$, let's introduce the Y-periodic function $V_\alpha \in L^1_{per}(Y)$ as follows

$$V_\alpha(x) = \begin{cases} v(\frac{x}{\alpha}, \frac{x}{\alpha}), & (x,y) \in Y^\alpha, \\ 0, & \text{otherwise}. \end{cases}$$

Then the support of this function satisfies the inclusion $Y \cap \operatorname{supp} V_\alpha \subseteq K^\alpha$. Now we consider the following sequence of Y-periodic functions $\left\{g_\varepsilon \in L^1_{per}(Y) \cap H^1(K)\right\}$, where

$$g_\varepsilon(x,y) = V_{\lambda(\varepsilon)}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \quad \forall\, \varepsilon > 0.$$

The question is: find

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^D}\varphi(x,y)g_\varepsilon(x,y)\,d\mathcal{H}^1,\quad \forall\,\varphi\in C(\overline{\Omega}),\tag{4.10}$$

where

$$\Gamma_\varepsilon^D=\left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon}\{S_\varepsilon+\varepsilon\,\mathbf{k}\}\right]\cap\overline{\Omega},\quad S_\varepsilon=\exp\left(-1/\varepsilon^2\right)K\cap\partial(\varepsilon^2)Q).$$

Since $\Lambda=\big\{(x_1,x_2)\in\mathbb{R}^2\ :\ x_1>0\big\}$, and $K\cap\partial\Lambda=\{(0,x_2)\ :\ x_2\in(-1/4,+1/4)\}$, it follows that (see Proposition (4.2))

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^D}\varphi(x,y)g_\varepsilon(x,y)\,d\mathcal{H}^1&=\lim_{\varepsilon\to 0}\int_{\Omega}\varphi(x,y)V_{\lambda(\varepsilon)}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\,d\mu_\varepsilon^{\lambda,h}=\\&=|\Omega|\left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda}(x^2+y^2)\,d\mathcal{H}^1\right)\int_\Omega\varphi\,dxdy=\\&=\frac{|\Omega|}{2}\int_{-1/4}^{1/4}y^2\,dy\int_\Omega\varphi\,dxdy=\frac{|\Omega|}{192}\int_\Omega\varphi\,dxdy.\end{aligned}$$

REFERENCES

[1] L. Carbone, R. De Arcangelis, *Unbounded functionals in the calculus of variations. Representation, relaxation, and homogenization*, Chapman& Hall/CRC, New York, 2002.

[2] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[3] V.V. Zhikov, *On an extension of the method of two-scale convergence and its applications*, Sbornik *Math.*, **191**(2000),No. 7, 973-1014.

[4] V.V. Zhikov, *Homogenization of elastitic problems on singular structures*, Izvestija: *Math.*, **66**(2002), No. 2, 299-365.

[5] V.V. Zhikov, *On two-scale convergence*, Trudy Seminara imeni I.G. Petrovskogo, Moscow University, **23**(2003), 149–187 (in Russian).