Theoretical Aspects of Optimal Control Theory and Partial Differential Equations

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### ASYMPTOTIC ANALYSIS OF AN OPTIMAL BOUNDARY CONTROL PROBLEM FOR LINEAR ELLIPTIC EQUATION IN DOMAIN WITH A ROUGH BOUNDARY

P. I. Kogut<sup>\*</sup>, Yu. Kovalenko<sup>\*\*</sup>

\* Department of Differential Equations, Oles Honchar Dnipro National University, Gagarin av., 72, Dnipro, 49010, e-mail: p.kogut@i.ua
\*\* Department of Differential Equations, Oles Honchar Dnipro National University, Kazakova, 18, build. 14, Dnipro, 49050, e-mail: kovalenkoyu.h@gmail.com

### Abstract

We study the asymptotic behaviour of an optimal boundary control problem for az linear elliptic equation in two-dimensional domain  $\Omega_{\varepsilon}$  with mixed types of boundary conditions. We assume that the boundary of domain  $\Omega_{\varepsilon}$  contains a highly oscillating part with respect to  $\varepsilon$ , and we suppose that the control influence is realized via the Neumann boundary condition posed on the highly oscillating part of boundary. We present some ideas and results concerning the asymptotic analysis of such problems as  $\varepsilon \to 0$  and derive conditions under which the homogenized problem can be recovered in an explicit form. We show that the mathematical description of the homogenized optimal boundary control problem is different from the original one. These differences appear not only in the limit cost functional, geometry of a limit domain, and Neumann boundary conditions, but also in the control constraints.

Keywords: homogenization, asymptotic behaviour, optimal control, rough boundary, singular measure

### 1. Introduction

In this paper we are concerned with the following optimal control problem for linear elliptic equation in two-dimensional domain with mixed (Neumann and Dirichlet) boundary conditions

$$I_{\varepsilon}(u,y) = \frac{1}{2} \int_{\Gamma_{\varepsilon}^{N}} |u|^{2} d\mathcal{H}^{1} + \frac{\alpha}{2} \int_{\Omega_{0}} |y - y_{ad}|^{2} dx \longrightarrow \inf$$
(1.1)

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subjected to the constraints

$$-\Delta y = f \quad \text{in } \Omega_{\varepsilon}, y = 0 \quad \text{on } \Gamma_{\varepsilon}^{D}, \quad \frac{\partial y}{\partial n} = u \quad \text{on } \Gamma_{\varepsilon}^{N},$$

$$(1.2)$$

$$u \in U_{\partial}^{\varepsilon} = \left\{ v \in L^2\left(\Gamma_{\varepsilon}^N\right) \colon \|v\|_{L^2(\Gamma_{\varepsilon}^N)} \le \beta^* \right\}.$$
(1.3)

Here,  $\alpha > 0$ ,  $\beta^* > 0$ ,  $f \in L^2(D)$  and  $y_{ad} \in L^2(\Omega_0)$  are given functions,

$$\Omega_{\varepsilon} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} 0 < x_1 < \mathcal{L} \\ -\varepsilon \widetilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 < \Phi(x_1) \end{array} \right\}$$
$$D = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} 0 < x_1 < \mathcal{L} \\ -1 < x_2 < \Phi(x_1) \end{array} \right\},$$

where  $L \in \mathbb{R}$  is a positive value,  $\Phi \in C^1([0, L])$  and  $F \in C_0^1([0, 1])$  are given functions such that

$$\Phi(x) > 0, \quad \forall \ x \in (0, L), \quad F(y) \in [0, 1], \quad \forall \ y \in [0, 1],$$

 $y \mapsto \widetilde{F}(y)$  is 1-periodic extension of the function  $F : [0,1] \to [0,1]$ , and  $\varepsilon$  is a small positive parameter.

We study the asymptotic behaviour of the optimal control problem (1.1)-(1.3) as the parameter  $\varepsilon$  tends to zero. The characteristic feature of this problem is the fact that the boundary  $\partial\Omega$  of domain  $\Omega_{\varepsilon}$ , where the boundary problem is posed, contains the very highly oscillating part with respect to  $\varepsilon$ , as  $\varepsilon \to 0$ . We consider the optimal control problem assuming that the control influence is realized via the Neumann boundary condition posed on the highly oscillating part of boundary.

Boundary value problems in domains with highly oscillating boundary are prototypes of widely used engineering constructions as well as many other physical and biological systems with very distinct characteristic scales. The computational calculation of the solutions of these problems is very complicated due to the geometry of such domains. Indeed, increase in the size of computational domains naturally leads to longer computing time and makes it very difficult to keep an acceptable level of accuracy. Therefore, asymptotic analysis is one of the main approach to study of boundary value problems in such domains.

We would like to emphasize that in contrast to the approach of Kesavan & Saint Jean Paulin [9] and [10], Saint Jean Paulin & Zoubairi [12] and Conca, Osses & Saint Jean Paulin [4] we do not just look for a limit of optimal control functions and for a limit of minimal values of the cost functionals. Rather, we stay with the optimal control problem in the original sense and look for a homogenized problem as some variational limit of the original one. This limiting problem should be unique (as a result of some passage to the limit), and should preserve the well known variational properties such as the convergence of both optimal solutions and minimal values of a cost function and, of course, should finally have a clearly defined structure including the limit form of a state equation, control and state constraints, a limit cost functional, and should be defined in a "simpler" domain. Our approach, the so-called "direct approach" of calculus of variations, is based on ideas of the theory of  $\Gamma$ -convergence and the concept of variational convergence of constrained minimization problems [11] and its variational properties. The analysis is very much in the spirit of Attuoch [1] and Buttazzo & Dal Maso [2,3].

Using the ideas of the  $\Gamma$ -convergence theory and the concept of the variational convergence of constrained minimization problems, we show that the homogenized problem for the original one can be recovered in the following analytical form:

$$I(u,y) = \frac{|\Delta_{\partial F}|}{2} \int_0^L u^2(s) \, ds + \frac{\alpha}{2} \int_{\Omega_0} (y - y_{ad})^2 \, dx \Longrightarrow \inf$$
(1.4)

subjected to the constraints

$$-\Delta y = f \quad \text{in} \quad \Omega_0, y = 0 \quad \text{on} \quad \Gamma_D, \frac{\partial y}{\partial n} = u |\Delta_{\partial F}| \quad \text{on} \quad \Gamma^N, u \in U_{\partial} = \left\{ v \in L^2(\Gamma^N) : \|v\|_{L^2(\Gamma^N_{\varepsilon})} \le |\Delta_{\partial F}|^{-1/2} \beta^* \right\},$$

$$(1.5)$$

where  $\Gamma^N = \{(x_1, 0) | 0 < x_1 < L\}$ ,  $\Gamma^D = \partial \Omega_0 \setminus |\Gamma^N$ , and  $|\Delta_{\partial F}|$  is the 1-dimensional Hausdorff measure of the arc segment  $\{x_2 = -F(x_1) : 0 \le x_1 < 1\}$ , i.e.

$$|\Delta_{\partial F}| = \int_{1}^{0} \sqrt{1 + [F'(x_1)]^2} \, dx_1.$$

### 2. Statement of the Problem and Some Preliminaries

We define a bounded open subdomain  $\Omega_0$  of  $\mathbb{R}^2$  as follows

$$\Omega_0 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} 0 < x_1 < \mathbf{L} \\ 0 < x_2 < \varPhi(x_1) \end{array} \right\}.$$

In order to describe the domain  $\Omega_{\varepsilon}$  with a rough boundary, we set (see Fig.1)

$$\Omega_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} 0 < x_1 < \mathcal{L} \\ -1 < x_2 \le 0 \end{array} \right\}, \quad D = \Omega_0 \cup \Omega_1.$$



Fig. 1. Example of D-domain

We say that  $\Omega_{\varepsilon}$  is a domain with rough boundary if

$$\Omega_{\varepsilon} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} 0 < x_1 < \mathcal{L} \\ -\varepsilon \widetilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 < \Phi(x_1) \right\},\$$

where it is assumed that  $y \mapsto \widetilde{F}(y)$  is 1-periodic extension of the function  $F : [0,1] \to [0,1]$ , and  $\varepsilon$  is a small parameter (see Fig.2).

*Remark* 2.1. Here in after we suppose that  $\varepsilon$  varies in a strictly decreasing sequence of positive numbers which converges to zero and such that  $N_{\varepsilon} = L/\varepsilon$  are integers. So, when we write  $\varepsilon > 0$ , we consider the elements of this sequence only.



Fig. 2. Example of domain  $\Omega_{\varepsilon}$ 

Let  $\partial \Omega_{\varepsilon}$  be the boundary of  $\Omega_{\varepsilon}$ . It is clear that the following decomposition of  $\partial \Omega_{\varepsilon}$  holds

 $\operatorname{true}$ 

$$\partial\Omega_{\varepsilon} = \Gamma_{\varepsilon}^N \cup \Gamma_{\varepsilon}^D$$

where

$$\Gamma_{\varepsilon}^{N} \cap \Gamma_{\varepsilon}^{D} \neq \emptyset, \qquad \Gamma_{\varepsilon}^{N} = \partial \Omega_{\varepsilon} \cap \Omega_{1},$$
  
$$\Gamma_{\varepsilon}^{D} = \partial \Omega_{\varepsilon} \smallsetminus \Gamma_{\varepsilon}^{N} = \begin{cases} \begin{bmatrix} 0\\x_{2} \end{bmatrix}, & \forall 0 \le x_{2} \le \Phi(0), \\ \begin{bmatrix} x_{1}\\x_{2} \end{bmatrix}, & \forall 0 \le x_{1} \le L, \ x_{2} = \Phi(x_{1}), \\ \begin{bmatrix} 1\\x_{2} \end{bmatrix}, & \forall 0 \le x_{2} \le \Phi(L). \end{cases}$$

It is worth to note that the part of boundary  $\Gamma_{\varepsilon}^{N}$  has a such type of oscillations that the ratio of its amplitude to the period of oscillation remains bounded as the small parameter  $\varepsilon$  tends to zero.

The optimal control problem, we are going to consider in  $\Omega_{\varepsilon}$ , is to minimize the cost functional

$$I_{\varepsilon}(u,y) = \frac{1}{2} \int_{\Gamma_{\varepsilon}^{N}} |u|^{2} d\mathcal{H}^{1} + \frac{\alpha}{2} \int_{\Omega_{0}} |y - y_{ad}|^{2} dx \qquad (2.1)$$

subjected to the constraints

$$\begin{aligned} &-\Delta y = f \quad \text{in } \ \Omega_{\varepsilon}, \\ &y = 0 \quad \text{on } \ \Gamma_{\varepsilon}^{D}, \\ &\frac{\partial y}{\partial n} = u \quad \text{on } \ \Gamma_{\varepsilon}^{N}, \end{aligned}$$

$$(2.2)$$

$$u \in U_{\partial}^{\varepsilon} = \left\{ v \in L^2\left(\Gamma_{\varepsilon}^N\right) \colon \|v\|_{L^2(\Gamma_{\varepsilon}^N)} \le \beta^* \right\}.$$
(2.3)

Here,  $\alpha > 0$ ,  $\beta^* > 0$ ,  $f \in L^2(D)$  and  $y_{ad} \in L^2(\Omega_0)$  are given functions.

Let  $H_0^1(D)$  be the classical Sobolev space defined as the closure of  $C_0^\infty(D)$  with respect to the norm

$$\|\varphi\|_{H^1_0(D)} = \left(\int_D |\nabla\varphi|^2 \, dx\right)^{1/2}$$

Since  $\Omega_{\varepsilon} \subset D$  and  $\Gamma^{D}_{\varepsilon} \subset \partial D$  for all  $\varepsilon > 0$ , we set

$$H_0^1\left(\Omega_{\varepsilon};\Gamma^D\right) = \left\{ y \Big|_{\Omega_{\varepsilon}} : \forall y \in H_0^1(D) \right\},\$$

where the index  $\varepsilon$  has been omitted in  $\Gamma^D$  because this part of boundary does not depend on  $\varepsilon > 0$ .

*Remark* 2.2. Taking the definition of  $H_0^1(\Omega_{\varepsilon}; \Gamma^D)$  into account, we can always associate every function  $y \in H^1_0(\Omega_{\varepsilon}; \Gamma^D)$  with its prototype  $\hat{y} \in H^1_0(D)$  such that

$$\widehat{y}(x) = y(x)$$
 a.e. in  $\Omega_{\varepsilon}$ ,  $\widehat{C} \|y\|_{H^1_0(\Omega_{\varepsilon};\Gamma^D)} \ge \|\widehat{y}\|_{H^1_0(D)}$ ,

where

$$\|y\|_{H^1_0(\Omega_{\varepsilon};\Gamma^D)}^2 = \int_{\Omega_{\varepsilon}} |\nabla y|^2 \, dx, \quad \|\widehat{y}\|_{H^1_0(D)}^2 = \int_D |\nabla \widehat{y}|^2 \, dx,$$

and the constant  $\widehat{C}$  depends on  $\Omega_0$  and  $\|y\|_{H^1_0(\Omega_{\varepsilon};\Gamma^D)}$ .

**Definition 2.1.** We say that a pair  $(u_{\varepsilon}, y_{\varepsilon})$  is feasible to the problem (2.1)–(2.3) if

$$u_{\varepsilon} \in U^{\varepsilon}_{\partial}; \quad y_{\varepsilon} \in H^1_0(\Omega_{\varepsilon}; \Gamma^D)$$

and the integral identity

$$\int_{\Omega_{\varepsilon}} (\nabla y_{\varepsilon}, \nabla \varphi) \, dx = \int_{\Omega_{\varepsilon}} f\varphi \, dx + \int_{\Gamma_{\varepsilon}^{N}} u_{\varepsilon} \varphi \, d\mathcal{H}^{1}$$
(2.4)

holds true for each test function  $\varphi \in C_0^{\infty}(D)$ .

Let  $\Xi_{\varepsilon}$  be the set of all feasible pairs to the problem (2.1)–(2.3). We begin with the following result.

**Theorem 2.1.** For every  $\varepsilon > 0$ ,  $f \in L^2(D)$ , and  $u \in L^2(\Gamma^N_{\varepsilon})$  there exist a unique solution  $y \in H^1_0(\Omega_{\varepsilon}; \Gamma^D)$  to the boundary value problem (2.2) such that

$$\|y\|_{H^{1}_{0}(\Omega_{\varepsilon};\Gamma^{D})} \leq \operatorname{diam} D\|f\|_{L^{2}(D)} + \sqrt{\delta_{0}^{-1}\left(\frac{1}{2} + 3\operatorname{diam}^{2}D\right)\|u\|_{L^{2}(\Gamma_{\varepsilon}^{N})}}, \qquad (2.5)$$

where diam D is the Euclidean diameter of D, i.e.

**D**:

diam 
$$D = \sup_{\eta \in D} |\xi - \eta|_{\mathbb{R}^2}$$
, and  $\delta_0 = \frac{1}{\sqrt{1 + (\sup x_{x \in [0,1]} |F'(x)|)^2}}$ 

*Proof.* Since the set  $C_0^{\infty}(D)$  is dense in  $H_0^1(D)$ , it follows that the integral identity (2.4) can be extended (by continuity) to the test functions  $\varphi \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ . As a result, relation (2.4) leads us to the following variational statement of BVP(2.2):

Find 
$$y \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$$
 such that  

$$\int_{\Omega_{\varepsilon}} (\nabla y, \nabla z)_{\mathbb{R}^2} dx = \int_{\Omega_{\varepsilon}} fz \, dx + \int_{\Gamma_{\varepsilon}^N} u\gamma_0(z) \, d\mathcal{H}^1, \quad \forall z \in H_0^1(\Omega_{\varepsilon}; \Gamma^D), \quad (2.6)$$

where  $\gamma_0(z)$  stands for the trace operator.

Let us show that the right hand side of (2.6) is a linear continuous functional with respect to  $z \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ . Indeed,

$$\int_{\Omega_{\varepsilon}} fz \, dx \leq \|f\|_{L^{2}(\Omega_{\varepsilon})} \|z\|_{L^{2}(\Omega_{\varepsilon})} \leq \|f\|_{L^{2}(D)} \|z\|_{L^{2}(\Omega_{\varepsilon})} \leq \text{(by Friedrix inequality)}$$
$$\leq \|f\|_{L^{2}(D)} C \|\nabla z\|_{L^{2}(\Omega_{\varepsilon})^{2}} = C \|f\|_{L^{2}(D)} \|z\|_{H^{1}_{0}(\Omega_{\varepsilon};\Gamma^{D})}, \tag{2.7}$$

with  $C = \operatorname{diam} \Omega_{\varepsilon} \leq \operatorname{diam} D$ .

As for the second term in (2.6), we note that the well-known trace theorem for Sobolev spaces states that, for a Lipschitz continuous domain  $\Omega_{\varepsilon}$ , there exists a unique linear continuous map, called the trace operator,

$$\gamma_0: H^1_0(\Omega_{\varepsilon}) \to H^{1/2}(\Gamma^N_{\varepsilon})$$

such that

- 1. for any  $y \in H^1_0(\Omega_{\varepsilon}) \cap C^0(\overline{\Omega}_{\varepsilon})$  one has  $\gamma_0(y) = y \Big|_{\Gamma^N_{\varepsilon}}$ ;
- 2. the following inequality

$$\delta_{\varepsilon} \|\gamma_0(y)\|_{L^2(\Gamma_{\varepsilon}^N)}^2 \le \|\mu_{\varepsilon}\|_{C^1(\overline{\Omega}_{\varepsilon})} \left(\sigma^{1/2} \|\nabla y\|_{L^2(\Omega_{\varepsilon})^2}^2 + (1+\sigma^{-1/2}) \|y\|_{L^2(\Omega_{\varepsilon})}^2\right)$$
(2.8)

holds true for all  $\sigma \in (0,1)$  and  $y \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ , where  $\mu_{\varepsilon}$  is a vector field on  $\overline{\Omega}_{\varepsilon}$  such that

$$\mu_{\varepsilon} \in C^1(\overline{\Omega}_{\varepsilon}; \mathbb{R}^2), \quad (\mu_{\varepsilon}, n_{\varepsilon})_{\mathbb{R}^2} \ge \delta_{\varepsilon} \quad \text{on } \Gamma_{\varepsilon}^N,$$

 $n_{\varepsilon}$  is the outer unit normal vector.

For the details, we refer to P. Grisvard [8, Th. 1.5.1.10].



Fig. 3.

To begin with, let us show that there exists a positive constant  $\delta_0$  such that

$$(\mu, n_{\varepsilon}(x))_{\mathbb{R}^2} \ge \delta_0 \quad \text{for each } x \in \Gamma^N_{\varepsilon}$$

$$(2.9)$$

and  $\delta_0$  is independent of  $\varepsilon$ , where  $\mu = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ . Indeed, let  $x^*$  be an arbitrary point of the boundary  $\Gamma_{\varepsilon}^N$ . Following the definition of  $\Gamma_{\varepsilon}^N$ , we may suppose that

$$x^* = \varepsilon(k+y),$$

where  $k \in \mathbb{Z}$  is an integer, and  $y \in [0, 1]$ .

Then the tangent vector to  $\Gamma_{\varepsilon}^{N}$  at x can be represented as follows (see Fig.3)

$$\nu(x^*) = \begin{bmatrix} 1\\ \frac{d}{dx} \left( -\varepsilon \widetilde{F} \left( \frac{x}{\varepsilon} \right) \Big|_{x=x^*} \right) \end{bmatrix} = \begin{bmatrix} 1\\ -\widetilde{F}'(\frac{x^*}{\varepsilon}) \end{bmatrix} = \begin{bmatrix} 1\\ -\widetilde{F}'(k+y) \end{bmatrix} = \begin{bmatrix} 1\\ -F'(y) \end{bmatrix}$$

(because of the 1-periodicity of the function  $\widetilde{F}$ ).

Hence,

$$n_{\varepsilon}(x^*) = \frac{1}{\sqrt{1 + \left(F'(y)\right)^2}} \begin{bmatrix} -F'(y) \\ -1 \end{bmatrix}$$

is the outer normal vector to  $\Gamma_{\varepsilon}^{N}$  at the point  $x^{*}$ . Taking this fact into account, we deduce from (2.9)

$$(\mu, n_{\varepsilon}(x^*))_{\mathbb{R}^2} = \frac{1}{\sqrt{1 + (F'(y))^2}} \left( -F'(y) \cdot 0 + (-1) \cdot (-1) \right) = \frac{1}{\sqrt{1 + (F'(y))^2}} > 0.$$
(2.10)

Using the fact that  $F \in C_0^1(0, 1)$ , we have

$$||F'||_{C([0,1])} = \max_{0 \le y \le 1} |F'(y)| \le \beta$$

for some  $\beta > 0$ . As a result, we obtain

$$\left(\mu, n_{\varepsilon}(x^*)\right)_{\mathbb{R}^2} = \frac{1}{\sqrt{1 + (F'(y))^2}} \ge \frac{1}{\sqrt{1 + \rho^2}} = \delta_0$$

for all  $x^* \in \Gamma^N_{\varepsilon}$  and all  $\varepsilon > 0$ .

Thus, the a priori estimate (2.8) with  $\mu_{\varepsilon} = \begin{bmatrix} 0\\ -1 \end{bmatrix}$  and  $\sigma = \frac{1}{2}$ , leads us to the following inequality

$$\delta_0 \|\gamma_0(y)\|_{L^2(\Gamma^N_{\varepsilon})}^2 \le \frac{1}{2} \|\nabla y\|_{L^2(\Omega_{\varepsilon})^2}^2 + 3\|y\|_{L^2(\Omega_{\varepsilon})}^2 \quad \forall y \in H^1_0(\Omega_{\varepsilon}; \Gamma^D).$$
(2.11)

Since  $\|y\|_{L^2(\Omega_{\varepsilon})} \leq \operatorname{diam} \Omega_{\varepsilon} \|\nabla y\|_{L^2(\Omega_{\varepsilon})^2}$  by Friedrix inequality, it follows from (2.11) that

$$\|\gamma_{0}(y)\|_{L^{2}(\Gamma_{\varepsilon}^{N})}^{2} \leq \delta_{0}^{-1} \left[\frac{1}{2} \|\nabla y\|_{L^{2}(\Omega_{\varepsilon})^{2}}^{2} + 3\operatorname{diam}^{2}\Omega_{\varepsilon} \|\nabla y\|_{L^{2}(\Omega_{\varepsilon})^{2}}^{2}\right]$$

$$\leq \sqrt{1+\beta^{2}} \left[\frac{1}{2} + 3\operatorname{diam}^{2}D\right] \|y\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})}^{2}.$$
(2.12)

Hence, the second term in the right hand side of (2.6) can be estimated as follows

$$\int_{\Gamma_{\varepsilon}^{N}} u\gamma_{0}(z) d\mathcal{H}^{1} \leq \|u\|_{L^{2}(\Gamma_{\varepsilon}^{N})} \|\gamma_{0}(z)\|_{L^{2}(\Gamma_{\varepsilon}^{N})}$$
  
by (2.12)  
$$\leq \sqrt{(1+\beta^{2})^{1/2} \left(\frac{1}{2}+3\operatorname{diam}^{2} D\right)} \|u\|_{L^{2}(\Gamma_{\varepsilon})} \|z\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})}.$$

Combining this estimate with (2.7), we see that there exists a linear continuous functional

$$G_{\varepsilon} \in \left(H_0^1(\Omega \varepsilon; \Gamma^D)\right)^*$$

such that

$$\left\langle G_{\varepsilon}, z \right\rangle_{\left(H_0^1(\Omega\varepsilon; \Gamma^D)\right)^*; H_0^1(\Omega\varepsilon; \Gamma^D)} = \int_{\Omega_{\varepsilon}} fz \, dx + \int_{\Gamma_{\varepsilon}} u\gamma_0(z) \, d\mathcal{H}^1, \qquad \forall z \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$$

and

$$\|G_{\varepsilon}\|_{\left(H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})\right)^{*}} \leq \|u\|_{L^{2}(\Gamma_{\varepsilon}^{N})} \sqrt{(1+\beta^{2})^{1/2} \left(\frac{1}{2}+3 \operatorname{diam}^{2} D\right)}.$$
 (2.13)

It remains to note that the bilinear form

$$a_{\varepsilon}(y,z) = \int_{\Omega_{\varepsilon}} (\nabla y, \nabla z)_{\mathbb{R}^2} \, dx, \qquad \forall y, z \in H^1_0(\Omega_{\varepsilon}; \Gamma^D)$$

is continuous and coercive on  $H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ .

Hence, by Lax-Milgram Theorem, variational problem (2.6) admits a unique solution  $y_{\varepsilon} \in H^1_0(\Omega_{\varepsilon}; \Gamma^D)$  for every  $u \in L^2(\Gamma^N_{\varepsilon})$  and  $f \in L^2(D)$ . As for the estimate (2.5), it immediate follows the energy equality

$$\|y^2\|_{H_0^1(\Omega_{\varepsilon};\Gamma^D)} = a_{\varepsilon}(y,y) = \langle G_{\varepsilon},y \rangle_{\left(H_0^1(\Omega_{\varepsilon};\Gamma^D)\right)^*;H_0^1(\Omega_{\varepsilon};\Gamma^D)} \le \|G_{\varepsilon}\|_{\left(H_0^1(\Omega_{\varepsilon};\Gamma^D)\right)^*}\|y\|_{H_0^1(\Omega_{\varepsilon};\Gamma^D)}$$
  
d inequality (2.13).

and inequality (2.13).

Our next intension is to study the optimal control problem (2.1)-(2.3) that can be represented as the following constrained minimization problem

$$\left\langle \inf_{(u,y)\in\Xi_{\varepsilon}} I_{\varepsilon}(u,y) \right\rangle.$$
(2.14)

With that in mind, we make use of the following observation. Let  $\{(u_k, y_k) \in \Xi_{\varepsilon}\}_{k=1}^{\infty}$  be an arbitrary sequence of feasible solutions to (2.1)-(2.3) such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in} \quad L^2(\Gamma^N_{\varepsilon}), \\ y_k &\rightharpoonup y \quad \text{weakly in} \quad H^1_0(\Omega_{\varepsilon}; \Gamma^D) \end{aligned}$$

as k tends to infinity. Let us show that the limit pair (u, y) lies in the set  $\Xi_{\varepsilon}$  as well. In other words, we are going to prove that  $\Xi_{\varepsilon}$  is sequentially closed subset of  $L^2(\Gamma_{\varepsilon}^N) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ with respect to the product of the weak topologies of  $L^2(\Gamma_{\varepsilon}^N)$  and  $H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ .

To do so, we notice that, for each  $k \in \mathbb{N}$ , the following identities

$$\int_{\Omega_{\varepsilon}} (\nabla y_k, \nabla z)_{\mathbb{R}^2} \, dx = \int_{\Omega_{\varepsilon}} fz \, dx + \int_{\Gamma_{\varepsilon}^N} u_k \gamma_0(z) \, d\mathcal{H}^1 \tag{2.15}$$

hold true for every  $z \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ .

Hence, passing to the limit in (2.15) as  $k \to \infty$ , we obtain

$$\int_{\Omega_{\varepsilon}} (\nabla y, \nabla z)_{\mathbb{R}^2} \, dx = \int_{\Omega_{\varepsilon}} fz \, dx + \int_{\Gamma_{\varepsilon}^N} u\gamma_0(z) \, d\mathcal{H}^1, \quad \forall z \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$$

by definition of the weak convergence in  $L^2(\Gamma_{\varepsilon}^N) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ . Since the boundary value problem (2.2) has a unique weak solution for each  $f \in L^2(D)$  and  $u \in L^2(\Gamma_{\varepsilon}^N)$  (see, for instance, Theorem 2.1), it follows that the pair (u, y) satisfies the relations (2.2) in weak sense. It remains to notice that the norm  $\|\cdot\|_{L^2(\Gamma_{\varepsilon}^N)}$  is sequentially lower semi-continuous with respect to the weak convergence in  $L^2(\Gamma_{\varepsilon}^N)$ , i.e.

$$\liminf_{k \to \infty} \|u_k\|_{L^2(\Gamma_{\varepsilon}^N)} \ge \|u\|_{L^2(\Gamma_{\varepsilon}^N)}.$$
(2.16)

Since  $(u_k, y_k) \in \Xi_{\varepsilon}$  for each  $k \in \mathbb{N}$ , it follows that  $\{u_k\}_{k=0}^{\infty} \subset U_{\delta}^{\varepsilon}$ . Hence, property (2.16) gives

$$\beta^* \geq \liminf_{k \to \infty} \|u_k\|_{L^2(\Gamma^N_\varepsilon)} \geq \|u\|_{L^2(\Gamma^N_\varepsilon)}, \quad i.e \quad u \in U^\varepsilon_\delta$$

Thus, combining the indicated properties of the limit pair (u, y), we can claim that

$$(u,y)\in \Xi_{\varepsilon}$$

i.e the following inference is valid:

Claim 1. The set  $\Xi_{\varepsilon}$  is sequentially closed with respect to the weak topology of

$$L^2(\Gamma^N_{\varepsilon}) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$$

It is worth to notice that the linearity of boundary value problem (2.2) and the convexity of the set  $U^{\varepsilon}_{\delta}$  imply:

**Claim 2.** The set of feasible pairs  $\Xi_{\varepsilon}$  is convex.

The next observation, we are going to make use of, deals with the boundedness of the set  $\Xi_{\varepsilon}$ . Indeed, let (u, y) be an arbitrary feasible solution, i.e  $(u, y) \in \Xi_{\varepsilon}$ . Then

$$\begin{aligned} \|(u,y)\|_{L^{2}(\Gamma_{\varepsilon}^{N})\times H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})} &= \|u\|_{L^{2}(\Gamma_{\varepsilon}^{N})} + \|y\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})} \\ \stackrel{\text{by (2.3)}}{\leq} \beta^{*} + \|y\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})} \stackrel{\text{by (2.4)}}{\leq} \beta^{*} + \operatorname{diam} D\|f\|_{L^{2}(D)} \\ &+ \sqrt{\delta_{0}^{-1}\left(\frac{1}{2} + 3\operatorname{diam}^{2} D\right)} \|u\|_{L^{2}(\Gamma_{\varepsilon})} \\ &\leq \beta^{*} + \operatorname{diam} D\|f\|_{L^{2}(D)} + \beta^{*}\sqrt{\delta_{0}^{-1}\left(\frac{1}{2} + 3\operatorname{diam} D\right)} < +\infty. \tag{2.17}$$

The crucial point we would like to emphasize here, is the fact that the estimate (2.17) is uniform with respect to the small parameter  $\varepsilon > 0$ . So, the following assertion holds true:

Claim 3. The sets  $\{\Xi_{\varepsilon}\}$  is uniformly bounded in  $L^{2}(\Gamma_{\varepsilon}^{N}) \times H^{1}_{0}(\Omega_{\varepsilon};\Gamma^{D})$ .

It remains to indicate the lower semi-continuity property of the cost functional  $I_{\varepsilon} : \Xi_{\varepsilon} \to \mathbb{R}$ . Indeed, for an arbitrary sequence  $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi_{\varepsilon}$  such that

$$(u_k, y_k) \stackrel{k \to \infty}{\rightharpoonup} (u, y)$$
 weakly in  $L^2(\Gamma^N_{\varepsilon}) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ 

We have (by the Relich-Kondrashow Theorem)

$$y_k \to y$$
 strongly in  $L^2(\Omega_{\varepsilon})$  as  $k \to \infty$ . (2.18)

Hence,

$$\begin{split} \liminf_{k \to \infty} I_{\varepsilon}(u_k, y_k) &= \liminf_{k \to \infty} \frac{1}{2} \|u_k\|_{L^2(\Gamma_{\varepsilon}^N)}^2 + \frac{\alpha}{2} \liminf_{k \to \infty} \|y_k - y_{ad}\|_{L^2(\Omega_0)}^2 \\ &\stackrel{\text{by}}{=} \frac{(2.18)}{2} \frac{1}{2} \liminf_{k \to \infty} \|u_k\|_{L^2(\Gamma_{\varepsilon}^N)}^2 + \frac{\alpha}{2} \|y - y_{ad}\|_{L^2(\Omega_0)}^2 \\ &\stackrel{\text{by}}{\geq} \frac{(2.16)}{2} \frac{1}{2} \|u\|_{L^2(\Gamma_{\varepsilon}^N)}^2 + \frac{\alpha}{2} \|y - y_{ad}\|_{L^2(\Omega_0)}^2 = I_{\varepsilon}(u, y). \end{split}$$

As a result, we arrive at the following obvious property of  $I_{\varepsilon}$ .

Claim 4. The cost function  $I_{\varepsilon} : \Xi_{\varepsilon} \to \mathbb{R}$  is strictly convex and lower semi-continuous with respect to the weak topology of  $L^2(\Gamma_{\varepsilon}^N) \times H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ .

We are now in a position to state the main result of this section. Namely, in view of the claims 1-4, the following result is a direct consequence of the the well-known theorem of Calculus of Variations (see A. Fursikov [7], G. Das Maso [5])

**Theorem 2.2.** For every  $\varepsilon > 0, f \in L^2(D)$ , and  $y_{ad} \in L^2(\Omega_0)$ , there exists a unique pair  $(u_{\varepsilon}^0, y_{\varepsilon}^0) \in \Xi_{\varepsilon}$  such that

$$I_{\varepsilon}(u_{\varepsilon}^{0}, y_{\varepsilon}^{0}) = \inf_{(u, y) \in \Xi_{\varepsilon}} I_{\varepsilon}(u, y).$$

The main question, we are going to discuss further on, is about the limit properties of the sequence of optimal pairs

$$\left\{ (u_{\varepsilon}^0, y_{\varepsilon}^0) \in \Xi_{\varepsilon} \right\}_{\varepsilon > 0}$$

In spite of the fact that this sequence is bounded (see (2.17))

$$\sup_{\varepsilon>0} \left[ \|u_{\varepsilon}^{0}\|_{L^{2}(\Gamma_{\varepsilon}^{N})} + \|y_{\varepsilon}^{0}\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})} \right]$$

$$\leq \operatorname{diam} D \cdot \|f\|_{L^{2}(D)} + \beta^{*} \left[ 1 + \sqrt{\delta_{0}^{-1} \left( \frac{1}{2} + 3\operatorname{diam}^{2} D \right)} \right], \qquad (2.19)$$

the study of the asymptotic properties of  $\{(u_{\varepsilon}^0, y_{\varepsilon}^0) \in \Xi_{\varepsilon}\}$  is not trivial. Indeed, for every fixed  $\varepsilon > 0$ , the optimal pair  $(u_{\varepsilon}^0, y_{\varepsilon}^0)$  belongs to the corresponding functional space

$$L^2(\Gamma^N_{\varepsilon}) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$$

and this space at level  $\varepsilon$  varies with  $\varepsilon$ . So that a preliminary problem is to define the convergence formalism of sequences of pairs which belong to different spaces. We note that the way, this convergence should be defined, must be quite flexible. As follows from the definition of the set  $\Omega_{\varepsilon}$ , we can not rely on the existence of an appropriate space  $L^2(\Gamma) \times H^1_0(\Omega)$  so that it contains all spaces  $\{L^2(\Gamma^N_{\varepsilon}) \times H^1_0(\Omega_{\varepsilon};\Gamma^D)\}_{\varepsilon>0}$  and the 'limit' pairs of the sequences in the indicated scale of variable spaces.

# 3. Description of the sets $\Omega_{\varepsilon}$ and $\Gamma_{\varepsilon}^{N}$ in the terms of singular measures

We devote this section to the construction of measure-theoretical tools aimed to the description of the class of feasible solutions to OCP (2.1)-(2.3) in the terms of so-called singular periodic Borel measure on  $\mathbb{R}^2$ . With that in mind, we follow Zhikov's approach (see [13]), and introduce the following sets

$$\Delta_{\partial F} = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid \begin{array}{c} 0 \le x_1 < 1\\ x_2 = -F(x_1) \end{array} \right\}$$
(3.1)

 $\operatorname{and}$ 

$$\Delta_F = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \Big| \begin{array}{c} 0 \le x_1 < 1 \\ -F(x_1) < x_2 \le 0 \end{array} \right\}$$
(3.2)

Let  $\mu$  and  $\nu$  be periodic finite positive Borel measures in  $\mathbb{R}^2$ . Let

$$Y = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \Big| \begin{matrix} 0 \le x_1 < 1 \\ -\infty < x_2 \le +\infty \end{matrix} \right\}$$

be the cell of periodicity for  $\mu$  and  $\nu$ . We assume that the Borel measures  $\mu$  and  $\nu$  are the probability measures in  $\mathbb{R}^2$  such that,  $\mu$  and  $\nu$  are concentrated and uniformly distributed on the sets

$$\Delta_{\partial F}$$
 and  $\Delta_F$ 

respectively. Hence,

$$\int_{Y} d\mu = \int_{Y} d\nu = 1. \tag{3.3}$$

*Remark* 3.1. Note that by definition  $\mu(Y \setminus \Delta_{\partial F}) = 0$ .

Hence,  $\mu$  is singular with respect to the Lebesque measure  $L^2$ . Moreover, any functions, having the same value on the set  $\Delta_{\partial F}$  coincide as elements of  $L^2(Y, d\mu)$ . Here, the Lebesque space  $L^2(Y, d\mu)$  can be defined in a usual way with the corresponding norm

$$||f||_{L^2(Y,d\mu)} = \left(\int_Y |f(s)|^2 \, d\mu\right)^{1/2}$$

Let  $|\Delta_{\partial F}|$  be the 1-dimensional Hausdorff measure of the arc  $\{x_2 = -F(x_1) : 0 \le x_1 < 1\}$ , i.e.

$$|\Delta_{\partial F}| = \int_{1}^{0} \sqrt{1 + [F'(x_1)]^2} \, dx_1.$$

Then, by definition of  $\mu$ , we have

$$\int_{\substack{x_2=-F(x_1)\\0\le x_1<1}} \varphi \, d\mathcal{H}^1 = |\Delta_{\partial F}| \cdot \int_Y \varphi \, d\mu \tag{3.4}$$

for each smooth function  $C_0^{\infty}(\mathbb{R})^2$ .

The similar reasoning can be applied to Y-periodic probability measure  $\nu$  in  $\mathbb{R}^2$ . In particular, for any  $\varphi \in C_0^{\infty}(\mathbb{R})^2$ , we have

$$\int_{\left\{\begin{array}{c} -F(x_1) < x_2 \le 0\\ 0 \le x_1 < 1\end{array}\right\} = \Delta_F} \varphi \, dx = \alpha^2(\Delta_F) \int_Y \varphi \, d\nu = \left[\int_0^1 \int_{-F(x_1)}^0 dx_2 dx_1\right] \cdot \int_Y \varphi \, d\nu. \tag{3.5}$$

*Remark* 3.2. Since  $\alpha^2(\Delta_F) \neq 0$  and

$$\int_{\Delta_F} \varphi \, dx = \int_{\Omega_1} \varphi(x) \mathcal{X}_{\Delta_F}(x) \, dx = \alpha^2(\Delta_F) \int_{\Omega_1} \varphi \, d\nu$$

for any  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ , it follows that the measure  $\nu$  is absolutely continuous with respect to the two-dimensional Lebesque measure on  $\mathbb{R}^2$ . Here,  $\mathcal{X}_{\Delta_F}$  stands for the characteristic function of the set  $\Delta_F$ , i.e.

$$\mathcal{X}_{\Delta_F}(x) = \begin{cases} 1, & x = (x_1, x_2) \in \Delta_F, \\ 0, & \text{otherwise.} \end{cases}$$

Let S be any Borel set in  $\mathbb{R}^2$ . We introduce two scaling measures  $\mu_{\varepsilon}$  and  $\nu_{\varepsilon}$  by the rules

$$\mu_{\varepsilon}(\mathbb{S}) = \varepsilon \mu(\varepsilon^{-1} \mathbb{S}) \tag{3.6}$$

and

$$\nu_{\varepsilon}(\mathbb{S}) = \varepsilon^2 \nu(\varepsilon^{-1} \mathbb{S}), \qquad (3.7)$$

respectively. It is clear that the measures  $\mu_{\varepsilon}$  and  $\nu_{\varepsilon}$  have the same period  $\varepsilon Y$ . Moreover, the direct calculations show that

$$\mu_{\varepsilon}(\varepsilon Y) := \int_{\varepsilon Y} d\mu_{\varepsilon} \stackrel{\text{by}}{=} \frac{\varepsilon}{\varepsilon} \int_{\varepsilon Y} d\mu \left(\frac{\cdot}{\varepsilon}\right) \stackrel{\text{by}}{=} \frac{\varepsilon}{\varepsilon} \cdot \frac{1}{|\Delta_{\partial F}|} \int_{\varepsilon \Delta_{\partial F}} d\mathcal{H}^{1}\left(\frac{\cdot}{\varepsilon}\right) = \\ = \frac{\varepsilon}{|\Delta_{\partial F}|} \cdot \int_{\left\{(x,y)^{T} \in \mathbb{R}^{2} \left| \left[ x = \varepsilon r, \ y = -\varepsilon F(r) \right] \right\}} d\mathcal{H} \left(\frac{\cdot}{\varepsilon}\right) \\ = \frac{\varepsilon}{|\Delta_{\partial F}|} \int_{0}^{1} \sqrt{(x'(r))^{2} + (y'(r))^{2}} d\left(\frac{r}{\varepsilon}\right) \\ = \frac{\varepsilon}{|\Delta_{\partial F}|} \int_{0}^{1} \sqrt{\varepsilon^{2} + \varepsilon^{2} (F'(r))^{2}} \cdot \frac{1}{\varepsilon} dr = \frac{\varepsilon}{|\Delta_{\partial F}|} \int_{0}^{1} \sqrt{1 + (F'(r))^{2}} dr \\ = \frac{\varepsilon}{|\Delta_{\partial F}|} \cdot |\Delta_{\partial F}| = \varepsilon \cdot 1 = \varepsilon \int_{Y} d\mu = \varepsilon \mu(Y)$$

$$(3.8)$$

and

$$\nu_{\varepsilon}(\varepsilon Y) := \int_{\varepsilon Y} d\nu_{\varepsilon} \stackrel{\text{by}}{=} \stackrel{(3.7)}{=} \varepsilon^{2} \int_{\varepsilon Y} d\nu \left(\frac{\cdot}{\varepsilon}\right)$$

$$\stackrel{\text{by}}{=} \stackrel{(3.5)}{=} \varepsilon^{2} \cdot \frac{1}{\alpha^{2}(\Delta_{F})} \int_{0}^{\varepsilon} \int_{-\varepsilon F(\varepsilon x_{1})}^{0} d\left(\frac{x_{2}}{\varepsilon}\right) d\left(\frac{x_{1}}{\varepsilon}\right) = \begin{cases} \frac{x_{1}}{\varepsilon} = y_{1}\\ \frac{x_{2}}{\varepsilon} = y_{2} \end{cases}$$

$$= \varepsilon^{2} \cdot \frac{1}{\alpha^{2}(\Delta_{F})} \int_{0}^{1} \int_{-F(y_{1})}^{0} dy_{2} dy_{1} = \varepsilon^{2} \cdot \frac{1}{\alpha^{2}(\Delta_{F})} \alpha^{2}(\Delta F)$$

$$= \varepsilon^{2} \cdot 1 = \varepsilon^{2} \int_{Y} d\nu = \varepsilon^{2} \nu(Y). \qquad (3.9)$$

Let us show that the properties (3.8) and (3.9) imply the weak-\* compactness of the sequences  $\{\mu_{\varepsilon}\}_{\varepsilon>0}$  and  $\{\nu_{\varepsilon}\}_{\varepsilon>0}$  in the space of Radon measures  $\mathcal{M}(\mathbb{R}^2)$ . It is worth to remind that if  $\sigma$  is a Borel measure in  $\mathbb{R}^2$  and  $\sigma(K) < +\infty$  for every compact subset  $K \subset \mathbb{R}^2$ , then  $\sigma$  is a Radon measure (i.e.  $\sigma \in \mathcal{M}(\mathbb{R}^2)$ ).

**Definition 3.1.** Let  $\sigma_{\varepsilon}, \sigma$  be non-negative Radon measures in  $\mathbb{R}^2$ . Then  $\{\sigma_{\varepsilon}\}_{\varepsilon>0}$  is called to be a weakly-\* convergent to  $\sigma$  as  $\varepsilon \to 0$  if

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\sigma_\varepsilon = \int_{\mathbb{R}^2} \varphi \, d\sigma \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^2)$$

We begin with the following limiting properties of the singular measures  $\left\{\mu_{\varepsilon} = \varepsilon \mu\left(\frac{\cdot}{\varepsilon}\right)\right\}_{\varepsilon>0}$ **Proposition 3.1.** For every  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ , we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\mu_\varepsilon = \int_{-\infty}^{+\infty} \varphi(x_1, 0) \, dx_1, \tag{3.10}$$

that is,

$$d\mu_{\varepsilon} \stackrel{*}{\rightharpoonup} \delta_{\{x_2=0\}} dx_1 dx_2 \quad \text{in} \quad M(\mathbb{R}^2), \tag{3.11}$$

where  $\delta_{\{x_2=0\}} dx_1 dx_2$  stands for the product of the 1-D Lebesque measure  $dx_1$  and the Dirac measure  $\delta_{\{x_2=0\}} dx_2$ .

*Proof.* Let  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$  be an arbitrary test function. Let us partition the plane  $\mathbb{R}^2$  onto the strips

$$\varepsilon Y_k = \varepsilon Y = \begin{bmatrix} k \\ 0 \end{bmatrix}, \quad k \in \mathbb{Z}$$

of the width  $\varepsilon$ . Then

$$\int_{\mathbb{R}^{2}} \varphi \, d\mu_{\varepsilon} = \sum_{k \in \mathbb{Z}} \int_{\varepsilon Y_{k}} \varphi \, d\mu_{\varepsilon} = \sum_{k \in \mathbb{Z}} \varphi(x_{k}^{\varepsilon}) \int_{\varepsilon Y_{k}} d\mu_{\varepsilon}$$
  
<sup>by  $\varepsilon$  Y -periodicity of  $\mu_{\varepsilon} \sum_{k \in \mathbb{Z}} \varphi(x_{k}^{\varepsilon}) \int_{\varepsilon Y_{k}} d\mu_{\varepsilon}$   
<sup>by  $(3.8) = \sum_{k \in \mathbb{Z}} \varphi(x_{k}^{\varepsilon}) \varepsilon \mu(Y) = \sum_{k \in \mathbb{Z}} \varepsilon \varphi(x_{k}^{\varepsilon}),$  (3.12)</sup></sup>

where  $x_k^{\varepsilon} \in \varepsilon \left( \Delta_{\partial F} + \begin{bmatrix} k \\ 0 \end{bmatrix} \right)$  are some points at the arc  $\Omega_{\varepsilon} \cap \varepsilon Y_k$ , (the existence of such points immediately follows from the Mean Value Theorem). To be more specific, we set

$$x_k^{\varepsilon} = (x_{1,k}^{\varepsilon}, x_{2,k}^{\varepsilon})^T \in \mathbb{R}^2$$

and

$$x_k^{\varepsilon,0} = (x_{1,k}^{\varepsilon}, 0)^T \in \mathbb{R}^2$$

for all  $k \in \mathbb{Z}$  and  $\varepsilon > 0$ . Here, by definition of the set  $\varepsilon \Delta_{\partial F}$ , we have

$$x_{2,k}^{\varepsilon} = -\varepsilon \cdot F(x_{1,k}^{\varepsilon}). \tag{3.13}$$

Since

$$\varphi(x_k^{\varepsilon}) = \varphi(x_{1,k}^{\varepsilon}, x_{2,k}^{\varepsilon}) = \varphi(x_{1,k}^{\varepsilon}, 0) + \left(\varphi(x_{1,k}^{\varepsilon}, x_{2,k}^{\varepsilon}) - \varphi(x_{1,k}^{\varepsilon}, 0)\right)$$
$$= \varphi(x_k^{0,\varepsilon}) + \left(\varphi(x_k^{\varepsilon}) - \varphi(x_k^{0,\varepsilon})\right)$$
(3.14)

and

$$\begin{aligned} |\varphi(x_k^{\varepsilon}) - \varphi(x_k^{0,\varepsilon})| &\leq M_{\varphi} \|x_k^{\varepsilon} - x_k^{0,\varepsilon}\|_{\mathbb{R}^2} \stackrel{\text{by } (3.13)}{=} M_{\varphi} \cdot \left|\varepsilon F(x_{1,k}^{\varepsilon})\right| \leq \\ &\leq M_{\varphi} \cdot \varepsilon \quad \text{(because } F(y) \in [0,1]\text{)}. \end{aligned}$$
(3.15)

Hence, from (3.12) we conclude

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\mu_{\varepsilon} = \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \varepsilon \varphi(x_k^{\varepsilon}) \stackrel{\text{by } (3.14) - (3.15)}{=} \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \varepsilon \varphi(x_k^{0,\varepsilon}) + \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \varepsilon D(\varepsilon), \quad (3.16)$$

where

$$\lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \varepsilon \varphi(x_k^{0,\varepsilon}) = \int_{\mathbb{R}} \varphi(x_1, 0) \, dx_1 \quad (\text{by construction of the Riemann sum})$$

and

$$\varepsilon |D(\varepsilon)| \le \sum_{k \in \mathbb{Z}} \varepsilon |\varphi(x_k^{\varepsilon}) - \varphi(x_k^{0,\varepsilon})| \le M_{\varphi} \sum_{k \in \mathbb{Z}} \varepsilon^2 = \left\{ k = \frac{1}{\varepsilon} \right\} = M_{\varphi} \cdot \varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$
(3.17)

Indeed, in order to specify the details of the limit passage in (3.16) and (3.17) as  $\varepsilon \to 0$ , we note that  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$  has a compact support. Then, for a given  $\varepsilon > 0$ , there exists a real value R > 0 such that

supp 
$$\varphi \subset B_R(0) = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 : |y| \le R \right\}.$$

So, we can suppose that

$$\sum_{k\in\mathbb{Z}}\varepsilon\varphi(x_k^{0,\varepsilon})=\sum_{k=-[N(\varepsilon)]-1}^{[N(\varepsilon)]+1}\varepsilon\varphi(x_k^{0,\varepsilon}),$$

where  $N(\varepsilon) = \varepsilon^{-1}R$  and [a] stands for the integer part of a (See Fig.4)





It means that

 $\sum_{k=-[N(\varepsilon)]-1}^{[N(\varepsilon)]+1}\varepsilon\varphi(x_k^{0,\varepsilon})$ 

is the Riemann sum of the integral  $\int_{-\infty}^{+\infty} \varphi(x_1, 0) dx_1$ . Hence, we arrive at the desired relation (3.16).

The same reasoning should be applied for substantiation of the limit passage (3.17). Thus, the equality (3.10) holds true for any  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ .

**Proposition 3.2.** For every  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$  the following relation

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\nu_{\varepsilon} = 0 \tag{3.18}$$

holds true, i.e.

$$\nu_{\varepsilon} \stackrel{*}{\rightharpoonup} 0 \quad \text{in} \quad M(\mathbb{R}^2) \quad \text{as} \quad \varepsilon \to 0.$$

*Proof.* Proceeding as we did it in Proposition (3.1), we fix an arbitrary function  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ . Then

$$\begin{split} \int_{\mathbb{R}^2} \varphi \, d\nu_{\varepsilon} &= \sum_{\substack{k \in \mathbb{Z} \\ k = \frac{1}{\varepsilon}}} \int_{\varepsilon Y_k} \varphi \, d\nu_{\varepsilon} = \sum_{k \in \mathbb{Z}} \varphi(x_k^{\varepsilon}) \int_{\varepsilon Y_k} \, d\nu_{\varepsilon} \\ &\stackrel{\text{(by } \varepsilon\text{- periodicity of } \nu_{\varepsilon})}{=} \sum_{k \in \mathbb{Z}} \varphi(x_k^{\varepsilon}) \int_{\varepsilon Y} \, d\nu_{\varepsilon} \stackrel{\text{(by } (3.9))}{=} \varepsilon^2 \sum_{k \in \mathbb{Z}} \varphi(x_k^{\varepsilon}) \nu(Y) \\ &\stackrel{\text{(by } (3.3))}{=} \sum_{k \in \mathbb{Z}} \varepsilon^2 \varphi(x_k^{\varepsilon}), \end{split}$$

where  $x_k^{\varepsilon} \in \varepsilon \left( \Delta_F + \begin{bmatrix} k \\ 0 \end{bmatrix} \right)$  is some point of the set  $\Omega_{\varepsilon} \cap \varepsilon Y_k$ . Since  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ , it follows that there exists a constant  $M_{\varphi} > 0$  such that

$$|\varphi(x)| \le M_{\varphi}, \quad \forall x \in \mathbb{R}^2$$

As a result, we obtain

$$\sum_{k \in \mathbb{Z}} \varepsilon^2 \varphi(x_k^{\varepsilon}) \le M_{\varphi} \cdot \varepsilon^2 \sum_{k \in \mathbb{Z}} 1 = M_{\varphi} \varepsilon^2 \cdot \frac{1}{\varepsilon} = \varepsilon M_{\varphi} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Thus,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\nu_{\varepsilon} = 0,$$

and this concludes the proof.

## 4. On Reformulation of OCP

The main point we are going to touch in this section is to give a new description of the original optimal control problem (2.1)–(2.3) in the terms of the scaling measures  $\mu_{\varepsilon}$  and  $\nu_{\varepsilon}$ .

To do so, let us consider the last term in the integral identity (2.6). Using notation of the previous section, we can write down

$$\int_{\Gamma_{\varepsilon}^{N}} u\gamma_{0}(z) d\mathcal{H}^{1} = \sum_{j=1}^{N_{\varepsilon}=L/\varepsilon} \int_{\left\{ \frac{(j-1)\varepsilon \leq x_{1} < j\varepsilon}{x_{2}=-\varepsilon F\left(\frac{x_{1}-(j-1)\varepsilon}{\varepsilon}\right)} \right\}} u\gamma_{0}(z) d\mathcal{H}^{1}$$

$$\stackrel{(\text{by } (3.4) \text{ and } (3.6) )}{=} \sum_{j=1}^{N_{\varepsilon}} |\Delta_{\partial F}| \cdot \int_{\left\{ \frac{(j-1)\varepsilon \leq x_{1} < j\varepsilon; x_{2} \in \mathbb{R},}{\varepsilon Y + \left[ \varepsilon(j-1) \right],} \right\}} u\gamma_{0}(z) \cdot \varepsilon d\mu \left(\frac{\cdot}{\varepsilon}\right)$$

$$= |\Delta_{\partial F}| \cdot \int_{\varepsilon Y + \left[ \frac{\varepsilon(j-1)}{0} \right]} u\gamma_{0}(z) d\mu_{\varepsilon} = |\Delta_{\partial F}| \int_{\Omega_{0} \cup \Omega_{1}} u\gamma_{0}(z) d\mu_{\varepsilon}. \quad (4.1)$$

Following the similar manner, we have (for the rest terms of identity (2.6))

$$\int_{\Omega_{\varepsilon}} fz \, dx = \int_{\Omega_0} fz \, dx + \int_{\left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \begin{bmatrix} 0 < x_1 < L \\ -\varepsilon \widetilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 \le 0 \end{bmatrix} \right\}} fz \, dx,$$
$$\int_{\Omega_{\varepsilon}} (\nabla y, \nabla z)_{\mathbb{R}^2} \, dx = \int_{\Omega_0} (\nabla y, \nabla z)_{\mathbb{R}^2} \, dx + \int_{\left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \begin{bmatrix} 0 < x_1 < L \\ -\varepsilon \widetilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 \le 0 \end{bmatrix} \right\}} (\nabla y, \nabla z)_{\mathbb{R}^2} \, dx,$$

where

$$\int_{\left\{ \begin{bmatrix} 0 < x_1 < L \\ -\varepsilon \tilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 \le 0 \end{bmatrix} \right\}} fz \, dx = \sum_{j=1}^{N_{\varepsilon}} \int_{\left\{ \begin{bmatrix} (j-1)\varepsilon < x_1 \le j\varepsilon \\ -\varepsilon \tilde{F}\left(\frac{x_1}{\varepsilon}\right) < x_2 \le 0 \end{bmatrix} \right\}} fz \, dx$$

$$\stackrel{(by (3.5), (3.7))}{=} \alpha^2 (\Delta_F) \sum_{j=1}^{N_{\varepsilon}} \int_{\varepsilon Y + \left[ \varepsilon (j-1) \right]} fz \, d\nu_{\varepsilon}$$

$$\left( \text{Since, } \int_{\left\{ \bigcup_{j=1}^{N_{\varepsilon}} \left( \varepsilon Y + \left[ \varepsilon (j-1) \\ 0 \end{bmatrix} \right] \right\}} fz \, d\nu_{\varepsilon} = \int_{D = \Omega_0 \cup \Omega_1} fz \, d\nu_{\varepsilon} \right)$$

$$= \alpha^2 (\Delta_F) \int_{\Omega_1 \cup \Omega_0} fz \, d\nu_{\varepsilon}.$$
(4.2)

17

*Remark* 4.1. The following relation is the direct consequence of (4.2)

$$\frac{1}{\alpha^2(\Delta_F)} \int_{\Omega_1} f(x) z(x) \mathcal{X}_{\Omega_\varepsilon \setminus \Omega_0}(x) \, dx = \int_{\Omega_1} fz \, d\nu_\varepsilon, \tag{4.3}$$

where  $z \in H_0^1(D)$  is an arbitrary distribution, and  $\mathcal{X}_{\Omega_{\varepsilon} \setminus \Omega_0}$  is the characteristic function of the set

$$\Omega_{\varepsilon} \setminus \Omega_{0} = \left\{ (x_{1}, x_{2}) \in D \Big|_{-\varepsilon \widetilde{F}} \left( \frac{x_{1}}{\varepsilon} \right) < x_{2} \le 0 \right\}$$

Taking these transformations into account, we see that the following descriptions

$$\left\{ v \in L^2(\Gamma^N_{\varepsilon}) : \|v\|_{L^2(\Gamma^N_{\varepsilon})} \le \beta^* \right\}$$

and

$$\left\{ v \in L^2(D, d\mu_{\varepsilon}) : \|v\|_{L^2(D, d\mu_{\varepsilon})} \le |\Delta_{\partial F}|^{-1/2} \beta^* \right\}$$

are equivalent. Here,  $L^2(D, d\nu_{\varepsilon})$  is the Lebesgue space with respect to the measure  $\nu_{\varepsilon}$  which is endowed with the norm

$$\|f\|_{L^2(D,d\mu_{\varepsilon})}^2 = \int_D f^2(x) \, d\mu_{\varepsilon}$$

Indeed, to this inference more evident, it is enough to observe that (see (4.1))

$$\int_{\Gamma_{\varepsilon}^{N}} u^{2} d\mathcal{H}^{1} = \left( \int_{D} u^{2} d\mu_{\varepsilon} \right) |\Delta_{\partial F}|.$$
(4.4)

As a result, we can reformulate the OCP (2.1)–(2.3) as follows: Find a pair  $(\hat{u}_{\varepsilon}^{0}, \hat{y}_{\varepsilon}^{0})$  such that

$$\widehat{u}_{\varepsilon}^{0} \in L^{2}(D, d\mu_{\varepsilon}), \quad \widehat{y}_{\varepsilon}^{0} \in H^{1}_{0}(\Omega_{\varepsilon}; \Gamma^{D}), \quad \widehat{I}_{\varepsilon}(\widehat{u}_{\varepsilon}^{0}, \widehat{y}_{\varepsilon}^{0}) = \inf_{(u,y)\in\widehat{\Sigma}_{\varepsilon}} \widehat{I}_{\varepsilon}(u, y), \tag{4.5}$$

where

$$\widehat{I}_{\varepsilon}(u,y) = \frac{|\Delta_{\partial F}|}{2} \|u\|_{L^{2}(D,d\mu_{\varepsilon})}^{2} + \frac{\alpha}{2} \|y - y_{ad}\|_{L^{2}(\Omega_{0})}^{2}$$

and  $\widehat{\Xi}_{\varepsilon}$  is the subset of  $L^2(D, d\mu_{\varepsilon}) \times H^1_0(D)$  such that  $(u, y) \in \widehat{\Xi}_{\varepsilon}$  if and only if the following conditions hold true

(i)  $||u||_{L^2(D,d\mu_{\varepsilon})} \leq |\Delta_{\partial F}|^{-1/2}\beta^*$ 

(ii) the integral identity

$$\int_{\Omega_0} (\nabla \widehat{y}, \nabla \varphi)_{\mathbb{R}^2} dx + \xi \cdot \int_{\Omega_1} (\nabla \widehat{y}, \nabla \varphi)_{\mathbb{R}^2} d\nu_{\varepsilon} = \int_{\Omega_0} f\varphi dx + \xi \cdot \int_{\Omega_1} f\varphi d\nu_{\varepsilon} + |\Delta_{\partial F}| \int_D u\varphi d\mu_{\varepsilon} \quad \text{for all} \quad \varphi \in C_0^\infty(D) \quad (4.6)$$

is valid for any prototype  $\widehat{y} \in H_0^1(D)$  of function  $y \in H_0^1(\Omega_{\varepsilon}, \Gamma^D)$  (see Remark 4.1). Here  $\xi = \alpha^2(\Delta_F)$ . As follows from Theorem 2.2, the OCP (4.5) admits a unique optimal pair  $(\widehat{u}_{\varepsilon}^{0}, \widehat{y}_{\varepsilon}^{0}) \in \widehat{\Xi}_{\varepsilon} \subset L^{2}(D, d\mu_{\varepsilon}) \times H^{1}_{0}(\Omega_{\varepsilon}, P^{\nu})$  for every fixed  $\varepsilon > 0$ ,  $f \in L^{2}(D)$ , and  $y_{ad} \in L^{2}(\Omega_{0})$ . Moreover, in view of the relation (4.4) we see that

$$\begin{aligned} \|u_{\varepsilon}^{0}\|_{L^{2}(\Gamma_{\varepsilon}^{N})}^{2} + \|y_{\varepsilon}^{0}\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})}^{2} &= |\Delta_{\partial F}| \|u_{\varepsilon}^{0}\|_{L^{2}(D,d\mu_{\varepsilon})}^{2} + \|\widehat{y}_{\varepsilon}^{0}\|_{H_{0}^{1}(\Omega_{0};\Gamma^{D})}^{2} \\ &+ \alpha^{2}(\Delta_{F}) \|\nabla\widehat{y}_{\varepsilon}^{0}\|_{L^{2}(\Omega_{1},d\nu_{\varepsilon})^{2}}^{2} \geq C \left[ \|u_{\varepsilon}^{0}\|_{L^{2}(D,d\mu_{\varepsilon})}^{2} + \|\nabla\widehat{y}_{\varepsilon}^{0}\|_{L^{2}(\Omega_{0})^{2}}^{2} + \|\nabla\widehat{y}_{\varepsilon}^{0}\|_{L^{2}(\Omega_{1},d\nu_{\varepsilon})}^{2} \| \right], \end{aligned}$$

where  $\widehat{y}_{\varepsilon}^{0} \in H_{0}^{1}(D)$  is a prototype of  $y_{\varepsilon}^{0}$  in the sense of Remark 4.1, and

$$C = \min\left\{1, |\Delta_{\partial F}|, |\alpha^2(\Delta_F)|\right\}.$$

Hence, due to the estimate (2.19), we have

$$\sup_{\varepsilon > 0} \left[ \|u_{\varepsilon}^{0}\|_{L^{2}(D,d\mu_{\varepsilon})}^{2} + \|\nabla \widehat{y}_{\varepsilon}^{0}\|_{L^{2}(\Omega_{0})^{2}}^{2} + \|\nabla \widehat{y}_{\varepsilon}^{0}\|_{L^{2}(\Omega_{1},d\nu_{\varepsilon})^{2}}^{2} \right] \\
\leq C^{-1} \left( \sup_{\varepsilon > 0} \left[ \|u_{\varepsilon}^{0}\|_{L^{2}(\Gamma_{\varepsilon}^{N})} + \|y_{\varepsilon}^{0}\|_{H_{0}^{1}(\Omega_{\varepsilon},\Gamma^{D})} \right]^{2} \right) \\
\stackrel{(\text{by (2.19))}}{\leq} C^{-1} \cdot 2 \left[ \left( \operatorname{diam} D \|f\|_{L^{2}(D)} \right)^{2} + \left( \beta^{*} \left[ 1 + \sqrt{\delta_{0}^{-1} \left( \frac{1}{2} + 3 \operatorname{diam}^{2} D \right)} \right] \right)^{2} \right]. \quad (4.7)$$

Thus, the sequence  $\{(u_{\varepsilon}^0, \nabla \widehat{y}_{\varepsilon}^0)\}_{\varepsilon \to 0}$  is uniformly bounded in the scale of variable spaces

$$L^2(D, d\mu_{\varepsilon}) \times L^2(\Omega_0)^2 \times L^2(\Omega_1, d\nu_{\varepsilon})^2.$$

Let us recall the definition and the main properties of the weak and strong convergence in the variable  $L^2$ - space [14].

### 5. Convergence in Variable Spaces

By a non-negative Radon measure on  $\Omega \subset \mathbb{R}^2$ , we mean a non-negative Borel measure which is finite on every compact subset of  $\Omega$ . The spaces of all non-negative Radon measures on  $\Omega$  will be denoted by  $\mathcal{M}_+(\Omega)$ . According to the Riesz theory, each Radon measure  $\mu \in \mathcal{M}_+(\Omega)$ can be interpreted as element of the dual of the space  $C_0(\Omega)$  of all continuous functions vanishing of infinity. If  $\mu$  is a non-negative Radon measure on  $\Omega$ , we will use  $L^2(\Omega, d\mu)$  to denote the usual Lebesque space with respect to the measure  $\mu$  with the corresponding norm

$$||f||_{L^2(\Omega, d\mu)} = \left(\int_{\Omega} |f(x)|^2 \, d\mu\right)^{1/2}$$

Let  $\{\mu_{\varepsilon}\}_{\varepsilon>0}$  and  $\mu$  be Radon measures such that  $\mu_{\varepsilon} \stackrel{*}{\rightharpoonup} \mu$  in  $M_{+}(\Omega)$ , i.e.

$$\lim_{\varepsilon \to 0} \int_{\Omega} \varphi \, d\mu_{\varepsilon} = \int_{\Omega} \varphi d\mu \quad \forall \varphi \in C_0(\mathbb{R}^2).$$

If we set  $\Omega = D$  and, for each  $\varepsilon > 0$ , define the measure  $\mu_{\varepsilon}$  as follows:

 $\mu_{\varepsilon} = \nu_{\varepsilon}$ , where  $\nu_{\varepsilon}$  is given by (3.7),

then  $\mu_{\varepsilon} \stackrel{*}{\rightharpoonup} \mu$  in  $\mathcal{M}_{+}(D)$  with  $d\mu = \delta_{\{x_2=0\}} dx_1 dx_2$  (see Proposition 3.1). At the same time, if  $\Omega = \Omega_1$  and  $\mu_{\varepsilon}$  are defined by (3.6), then (see Proposition 3.2)  $\mu_{\varepsilon} \stackrel{*}{\rightharpoonup} 0$  in  $\mathcal{M}_{+}(\Omega_1)$ .

Let us recall the definition and main properties of convergence in the variable  $L^2$ -space.

1. A sequence  $\{v_{\varepsilon} \in L^2(\Omega, d\mu_{\varepsilon})\}_{\varepsilon > 0}$  is called bounded if

$$\limsup_{\varepsilon \to 0} \int_{\Omega} |v_{\varepsilon}|^2 \, d\mu_{\varepsilon} < +\infty.$$

2. A bounded sequence  $\{v_{\varepsilon} \in L^2(\Omega, d\mu_{\varepsilon})\}_{\varepsilon>0}$  converges weakly to  $v \in L^2(\Omega, d\mu)$  if

$$\lim_{\varepsilon \to 0} \int_{\Omega} v_{\varepsilon} \varphi \, d\mu_{\varepsilon} = \int_{\Omega} v \varphi \, d\mu \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

and it is written as  $v_{\varepsilon} \rightharpoonup v$  in  $L^2(\Omega, d\mu_{\varepsilon})$ .

3. The strong convergence  $v_{\varepsilon} \to v$  in  $L^2(\Omega, d\mu_{\varepsilon})$  means that  $v \in L^2(\Omega, d\mu)$  and

$$\lim_{\varepsilon \to 0} \int_{\Omega} v_{\varepsilon} w_{\varepsilon} \, d\mu_{\varepsilon} = \int_{\Omega} v w \, d\mu \quad \text{as} \quad w_{\varepsilon} \rightharpoonup w \quad \text{in} \quad L^{2}(\Omega, d\mu_{\varepsilon}).$$

The following convergence properties in variable spaces hold:

- (a) Compactness : if a sequence is bounded in  $L^2(\Omega, d\mu_{\varepsilon})$  then this sequence is relatively compact in the sense of the weak convergence;
- (b) Lower semicontinuity : if  $v_{\varepsilon} \rightharpoonup v$  in  $L^2(\Omega, d\mu_{\varepsilon})$  then

$$\liminf_{\varepsilon \to 0} \int_{\Omega} |v_{\varepsilon}|^2 \, d\mu_{\varepsilon} \ge \int_{\Omega} |v|^2 \, d\mu;$$

(c) Strong convergence :  $v_{\varepsilon} \to v$  if and only if  $v_{\varepsilon} \rightharpoonup v$  in  $L^2(\Omega, d\mu_{\varepsilon})$  and

$$\lim_{\varepsilon \to 0} \int_{\Omega} |v_{\varepsilon}|^2 \, d\mu_{\varepsilon} = \int_{\Omega} |v|^2 \, d\mu.$$

## 6. Asymptotic Analysis of Parametrized OCP (2.1)-(2.3)

In order to study the asymptotic behaviour of the OCP (2.1)–(2.3) as  $\varepsilon \to 0$ , the passage to the limit in (2.14) as  $\varepsilon \to 0$  has to be realized. It is worth to notice that the expression

"passing to the limit" means that we have to find a kind of "limit cost functional" I and the "limit set of constraints"  $\Xi$  with a clearly defined structure such that the limit object

$$\left\langle \inf_{(u,y)\in\Xi_{\varepsilon}} I_{\varepsilon}(u,y) \right\rangle \stackrel{\varepsilon \to 0}{\to} \left\langle \inf_{(u,y)\in\Xi} I(u,y) \right\rangle$$
(6.1)

can be interpreted as some OCP.

Since for each  $\varepsilon > 0$  the OCP (2.1)–(2.3) lives in the corresponding functional space (see (4.5))  $L^2(D, d\mu_{\varepsilon}) \times H^1_0(\Omega_{\varepsilon}; \Gamma^D)$ , we begin with the definition of the convergence (6.1).

**Definition 6.1.** We say that a bounded sequence of feasible pairs  $\{(u_{\varepsilon}, y_{\varepsilon}) \in \widehat{\Xi}_{\varepsilon}\}$   $\sigma$ -converges to a pair (u, y) as  $\varepsilon \to 0$ , if

- 1.  $u \in L^2(0,L); \quad y \in H^1_0(\Omega_0;\Gamma^D);$
- 2.  $u_{\varepsilon} \rightharpoonup u$  weakly in  $L^2(\Gamma_{\varepsilon}^N, d\mu_{\varepsilon});$
- 3.  $\widehat{y}_{\varepsilon} \rightharpoonup \widehat{y}$  weakly in  $H_0^1(D)$  for any prototypes  $\widehat{y}_{\varepsilon} \in H_0^1(D)$  of  $y_{\varepsilon} \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$ ;
- 4.  $y = \hat{y}$  a.e. in  $\Omega_0$ .

Remark 6.1. Let us show that this definition is meaningful. Indeed, the boundedness of the sequence

$$\Big\{(u_\varepsilon,y_\varepsilon)\in\widehat{\Xi}_\varepsilon\subset L^2(D,d\mu_\varepsilon)\times H^1_0(\Omega_\varepsilon;\Gamma^D)\Big\}_{\varepsilon>0}$$

implies that

$$\sup_{\varepsilon>0} \|u_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} < +\infty, \qquad \sup_{\varepsilon>0} \|y_{\varepsilon}\|_{H^{1}_{0}(\Omega_{\varepsilon};\Gamma^{D})} < +\infty.$$
(6.2)

Hence, there exists an element  $u \in L^2(D, d\mu)$  and a sequence of prototypes  $\{\widehat{y}_{\varepsilon} \in H^1_0(D)\}$ such that (up to a subsequence)  $u_{\varepsilon} \rightharpoonup u$  in variable space  $L^2(D, d\mu_{\varepsilon})$  and

$$\sup_{\varepsilon > 0} \|\widehat{y}_{\varepsilon}\|_{H_0^1(D)} < +\infty.$$
(6.3)

Then the inclusion  $u \in L^2(0, L)$  is a direct consequence of the definition of the measure  $d\mu = \delta_{\{x_2=0\}} dx_1$ , whereas the condition (6.3) implies the existence of  $\hat{y} \in H^1_0(D)$  and a subsequence of  $\{\hat{y}_{\varepsilon}\}_{\varepsilon>0} \subset H^1_0(D)$  such that

$$\widehat{y}_{\varepsilon} \rightharpoonup \widehat{y}$$
 in  $H_0^1(D)$  as  $\varepsilon \to 0$ 

Hence,  $\widehat{y} \in H_0^1(\Omega_0; \Gamma^D)$ .

Let us show that the  $\sigma$ -limit pair  $(u, y) \in L^2(0, L) \times H^1_0(\Omega_0; \Gamma^D)$  is unique for any bounded sequence  $\{(u_{\varepsilon}, y_{\varepsilon}) \in \widehat{\Xi}_{\varepsilon}\}_{\varepsilon > 0}$ . To begin with, we note that the property  $\nu_{\varepsilon} \stackrel{*}{\rightharpoonup} \nu = 0$  weakly in  $\mathcal{M}(\mathbb{R}^2)$  leads to the relation (see (3.18) and (4.3))

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \varphi \, d\nu_{\varepsilon} = \frac{1}{\alpha^2(\Delta_F)} \lim_{\varepsilon \to 0} \int_D \varphi(x) \chi_{\Omega \smallsetminus \Omega_0}(x) \, dx = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2) \tag{6.4}$$

i.e., in other words, we can deduce that

$$\chi_{\Omega \smallsetminus \Omega_0} \rightharpoonup \chi_{\emptyset} \quad \text{in} \quad L^2(D). \tag{6.5}$$

(Here, we use the fact that the set  $C_0^{\infty}(\mathbb{R}^2)$  is dense in  $L^2(D)$ ).

On the other hand, we see that

$$\begin{aligned} \|\chi_{\Omega_{\varepsilon} \smallsetminus \Omega_{0}}\|_{L^{2}(D)}^{2} &:= \int_{D} |\chi_{\Omega_{\varepsilon} \smallsetminus \Omega_{0}}(x)|^{2} dx = \int_{D} \chi_{\Omega_{\varepsilon} \smallsetminus \Omega_{0}}(x) dx \xrightarrow{\text{as } \varepsilon \to 0} 0 \\ \{ \text{by } (6.4) \text{ for } \varphi \equiv 1 \text{ in } D \} \\ &= \|\chi_{\emptyset}\|_{L^{2}(D)}^{2}. \end{aligned}$$

$$(6.6)$$

Since the weak convergence in  $L^2(D)$  and the convergence of norm (6.6) imply the strong convergence in  $L^2(D)$ , we obtain

$$\chi_{\Omega_{\varepsilon} \smallsetminus \Omega_0} \to 0 \quad \text{strongly in} \quad L^2(D).$$
 (6.7)

In order to prove the uniqueness of the  $\sigma$ -limit, it is enough to show that this limit does not depend on the choice of prototypes sequence. Indeed, let  $\{(u_{\varepsilon}, y_{\varepsilon}) \in \widehat{\Xi}_{\varepsilon}\}$  be given bounded sequence. Let  $\{\widehat{y}_{\varepsilon}\}_{\varepsilon>0}$  and  $\{\widehat{g}_{\varepsilon}\}_{\varepsilon>0}$  be two differen sequences of prototypes for elements  $\{y_{\varepsilon} \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)\}$ . So it is plausible to assume that

$$\widehat{y}_{\varepsilon} \rightarrow \widehat{y} \quad \text{in} \quad H_0^1(D) \widehat{g}_{\varepsilon} \rightarrow \widehat{g} \quad \text{in} \quad H_0^1(D).$$

$$(6.8)$$

Our aim is to show that  $\widehat{y}(x) = \widehat{g}(x)$  almost everywhere in  $\Omega_0$ . With that in mind, we set

$$\xi = \nabla \widehat{y} - \nabla \widehat{g} \in L^2(D)^2.$$

Then, for every  $\varepsilon > 0$  we have the following chain of equalities

$$\begin{aligned} \|\widehat{y} - \widehat{g}\|_{H_0^1(\Omega_0;\Gamma^D)}^2 &= \int_{\Omega_0} |\nabla \widehat{y} - \nabla \widehat{g}|^2 \, dx \\ &= \int_{\Omega_0} (\nabla \widehat{y} - \nabla \widehat{g}, \xi)_{\mathbb{R}^2} \, dx = \int_D (\nabla \widehat{y} - \nabla \widehat{y}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx \\ &+ \int_D (\nabla \widehat{g}_{\varepsilon} - \nabla \widehat{y}, \xi)_{\mathbb{R}^2} \, dx + \int_{\Omega_1} (\nabla \widehat{y}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx \\ &- \int_{\Omega_1} (\nabla \widehat{g}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx - \int_{\Omega_1} (\nabla \widehat{y}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx \\ &+ \int_{\Omega_1} (\nabla \widehat{g}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx + \int_{\Omega_0} (\nabla \widehat{y}_{\varepsilon} - \nabla \widehat{g}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) - I_4(\varepsilon) - I_5 + I_6 + I_0(\varepsilon), \end{aligned}$$
(6.9)

where

$$I_0(\varepsilon) := \int_{\Omega_0} (\nabla \widehat{y}_{\varepsilon} - \nabla \widehat{g}_{\varepsilon}, \xi)_{\mathbb{R}^2} \, dx = 0$$

by definition of prototypes.

Hence, passing to the limit in (6.9) as  $\varepsilon \to 0$ , we get

$$\|\widehat{y} - \widehat{g}\|_{H_0^1(\Omega_0;\Gamma^D)}^2 = \lim_{\varepsilon \to 0} I_1(\varepsilon) + \lim_{\varepsilon \to 0} I_2(\varepsilon) + \lim_{\varepsilon \to 0} (I_3(\varepsilon) - I_5) - \lim_{\varepsilon \to 0} (I_4(\varepsilon) - I_6) = 0$$

because of the weak convergence (6.8). Thus,  $\hat{y} = \hat{g}$  as elements of  $H_0^1(\Omega_0; \Gamma^D)$ .

As a result, following the scheme of the direct variational convergence (see P.Kogut & G. Lengering [11]), we adopt the following definition for the convergence of minimization problems (6.1) in variable spaces.

**Definition 6.2.** The constrained minimization problem  $\langle \inf_{(u,y)\in\Xi} I(u,y) \rangle$  is the variational  $\sigma$ -limit of the sequence  $\{\langle \inf_{(u,y)\in\Xi\varepsilon} I_{\varepsilon}(u,y) \rangle; \varepsilon \to 0\}$  as  $\varepsilon \to 0$  if the following conditions hold true:

(a) If the sequences 
$$\{\varepsilon_k\}_{k\in\mathbb{N}}$$
 and  $\{(u_k, y_k)\}_{k\in\mathbb{N}}$  are such that  $\varepsilon_k \to 0$  as  $k \to \infty$ ,  
 $(u_k, y_k) \in \Xi_{\varepsilon_k}, \forall k \in \mathbb{N}, \text{ and } (u_k, y_k) \xrightarrow{\sigma} (u, y) \text{ in } L^2(D, d\mu_{\varepsilon_k}) \times H^1_0(\Omega_{\varepsilon_k}; \Gamma^D), \text{ then}$   
 $(u, y) \in \Xi \text{ and } I(u, y) \leq \liminf_{k \to \infty} I_{\varepsilon_k}(u_k, y_k).$ 
(6.10)

(aa) For every  $(u, y) \in \Xi$  there exists a sequence  $\{(u_{\varepsilon}, y_{\varepsilon})\}_{\varepsilon>0}$  (called a  $\Gamma$ -realizing sequence) such that

$$(u_{\varepsilon}, y_{\varepsilon}) \in \Xi_{\varepsilon} \quad \forall \varepsilon > 0, \quad (u_{\varepsilon}, y_{\varepsilon}) \xrightarrow{\sigma} (u, y) \quad \text{and} \quad I(u, y) \ge \limsup_{\varepsilon \to 0} I_{\varepsilon}(u_{\varepsilon}, y_{\varepsilon}).$$
 (6.11)

Taking into account this definition, let us show that the constrained minimization problem

$$\left\langle \inf_{(u,y)\in\Xi} I(u,y) \right\rangle,$$
 (6.12)

where

is the variational  $\sigma$ -limit of (2.14) as  $\varepsilon \to 0$ .

Verification of condition (a). Let  $\{\varepsilon_k\}_{k\in\mathbb{N}}$  and  $\{u_k, y_k\}_{k\in\mathbb{N}}$  be such that  $\varepsilon_k \to 0$  as  $k \to \infty$ ,  $(u_k, y_k) \in \Xi_{\varepsilon_k} \quad \forall k \in \mathbb{N}, \text{ and } (u_k, y_k) \xrightarrow{\sigma} (u, y) \text{ in variable space } L^2(D, d\mu_{\varepsilon_k}) \times H^1_0(\Omega_{\varepsilon_k}; \Gamma^D).$ Let us show that  $(u, y) \in \Xi$ . Indeed, the conditions  $u \in L^2(0, L)$  and  $y \in H^1_0(\Omega_0; \Gamma^D)$  are immediately follows from the definition of  $\sigma$ -convergence. Since  $(u_k, y_k) \in \Xi_{\varepsilon_k}$  for  $k \in \mathbb{N}$ , it follows that (see (4.4) and (4.6))

$$\|u_k\|_{L^2(D,d\mu_{\varepsilon_k})} \leq |\Delta_{\partial F}|^{-1/2}\beta^*, \quad \forall k \in \mathbb{N},$$

$$\int_{\Omega_0} (\nabla \widehat{y}_k, \nabla \varphi)_{\mathbb{R}^2} \, dx + \alpha^2 (\Delta_F) \int_{\Omega_1} (\nabla \widehat{y}_k, \nabla \varphi)_{\mathbb{R}^2} \, d\nu_{\varepsilon_k} = \int_{\Omega_0} f\varphi \, dx$$

$$+ \alpha^2 (\Delta_F) \int_{\Omega_1} f\varphi \, d\nu_{\varepsilon_k} + |\Delta_{\partial F}| \int_D u_k \varphi \, d\mu_{\varepsilon_k}, \quad \forall \varphi \in C_0^\infty(D), \forall k \in \mathbb{N}.$$

$$(6.13)$$

Then the lower semi-continuity property of the weak convergence  $u_k \rightarrow u$  in  $L^2(D, d\mu_{\varepsilon_k})$  implies

$$\|u\|_{L^2(0,L)} \le \liminf_{k \to \infty} \|u_k\|_{L^2(D,d\mu_{\varepsilon_k})} \le |\Delta_{\partial F}|^{-1/2}\beta^*$$

It remains to pass to the limit in (6.14) as  $k \to \infty$ . With that in mind, we note that

$$\lim_{k \to \infty} \int_{\Omega_0} (\nabla \widehat{y}_k, \nabla \varphi)_{\mathbb{R}^2} dx = \int_{\Omega_0} (\nabla y, \nabla \varphi)_{\mathbb{R}^2} dx \quad \text{by } \sigma\text{-convergence } (u_k, y_k) \xrightarrow{\sigma} (u, y);$$
$$\lim_{k \to \infty} \int_D u_k \varphi \, d\mu_{\varepsilon_k} = \int_D u\varphi \, d\mu = \int_0^L u(x)\varphi(x, 0) \, dx \quad \text{by } u_k \rightharpoonup u \text{ in } L^2(D, d\mu_{\varepsilon_k});$$
$$\alpha^2(\Delta_F) \lim_{k \to \infty} \int_{\Omega_1} (\nabla \widehat{y}_k, \nabla \varphi)_{\mathbb{R}^2} \, d\nu_{\varepsilon_k} = \lim_{k \to \infty} \int_{\Omega_1} (\nabla \widehat{y}_k, \chi_{\Omega_\varepsilon \setminus \Omega_0} \nabla \varphi)_{\mathbb{R}^2} \, dx$$
(as a product of weak and strong convergent sequences in  $L^2(\Omega_1)^2$ )

rgent

$$= \int_{\Omega_1} (\nabla \widehat{y}, \nabla \varphi)_{\mathbb{R}^2} \chi_{\emptyset} \, dx = 0.$$

By analogy with the previous case, we have

$$\alpha^2(\Delta_F) \int_{\Omega_1} f\varphi \, d\nu_{\varepsilon_k} \stackrel{k \to \infty}{\to} 0.$$

Gathering together relations given above, we arrive at the following limiting integral identity

$$\int_{\Omega_0} (\nabla y, \nabla \varphi)_{\mathbb{R}^2} \, dx = \int_{\Omega_0} f\varphi \, dx + |\Delta_{\partial F}| \int_0^L u\varphi(s, 0) \, ds, \quad \forall \varphi \in C_0^\infty(D).$$

As a result, we have  $(u, y) \in \Xi$ .

As for the inequality (6.10), it immediately follows from compactness of the embedding  $H_0^1(\Omega_0; \Gamma^D) \hookrightarrow L^2(\Omega_0)$  and the lower semi-continuity property of the norm  $\|\cdot\|_{L^2(D, d\mu_{\varepsilon})}^2$  with respect to the weak convergence in  $L^2(D; d\mu_{\varepsilon})$ . Indeed, in this case we have

$$\lim_{k \to \infty} \int_{\Omega_0} (y_k - y_{ad})^2 dx = \int_{\Omega_0} (y - y_{ad})^2 dx,$$
$$\lim_{k \to \infty} \int_D u_k^2 d\mu_{\varepsilon_k} \ge \int_D u^2 d\mu = \int_0^L u^2(s) ds.$$

$$\left\{v_{\varepsilon}\in L^2(D,d\mu_{\varepsilon})\right\}_{\varepsilon>a}$$

as follows

$$\int_{D} v_{\varepsilon} \varphi \, d\mu_{\varepsilon} = \int_{D} u(\varphi)_{\varepsilon} \, d\mu = \int_{0}^{L} u(s)(\varphi)_{\varepsilon}(s,0) \, ds \tag{6.15}$$

Here,  $(\varphi)_{\varepsilon}$  is determining by the rule

$$\int_D \varphi \, d\mu_{\varepsilon} = \int_D (\varphi)_{\varepsilon} \, d\mu \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}^2)$$

Taking into account the definition of measures  $\mu_{\varepsilon}$  and  $\mu$ , we see that

$$\int_{D} (\varphi)_{\varepsilon} d\mu = \int_{0}^{L} \varphi \left( x_{1} - \varepsilon \widehat{F} \left( \frac{x_{1}}{\varepsilon} \right) \right) dx_{1}$$
(6.16)

Let us show that

$$v_{\varepsilon} \to u$$
 strongly in  $L^2(D, d\mu_{\varepsilon})$ . (6.17)

With that in mind, we make use of the following estimate

$$\left(\int_{D} (\varphi)_{\varepsilon} u \, d\mu\right)^{2} \leq \left(\int_{D} u^{2} \, d\mu\right) \left(\int_{D} (\varphi)_{\varepsilon}^{2} \, d\mu\right) = \operatorname{const} \int_{D} (\varphi)_{\varepsilon}^{2} \, d\mu$$

In view of (6.16), we can rewrite this inequality as follows

$$\left(\int_D (\varphi)_{\varepsilon} u \, d\mu\right)^2 \le \operatorname{const} \int_D \varphi^2 \, d\mu_{\varepsilon}$$

Hence, by Rietz Representation Theorem, there exists a function  $v_{\varepsilon}$  such that

$$\int_{D} v_{\varepsilon} \varphi \, d\mu_{\varepsilon} = \int_{D} u(\varphi)_{\varepsilon} \, d\mu \quad \forall \in C_{0}^{\infty}(\mathbb{R}^{2})$$
  
and  $\left| \int_{D} v_{\varepsilon} \varphi \, d\mu_{\varepsilon} \right| \leq \sqrt{\text{const}} \|\varphi\|_{L^{2}(D, d\mu_{\varepsilon})}.$  (6.18)

Since

$$\int_{D} u(\varphi)_{\varepsilon} \, d\mu = \int_{0}^{L} u(s)\varphi\left(s, -\varepsilon \hat{F}\left(\frac{s}{\varepsilon}\right)\right) \, ds$$

and

$$\lim_{\varepsilon \to 0} \sup_{0 \le s \le L} |\varphi(s, -\varepsilon \widehat{F}\left(\frac{s}{\varepsilon}\right)) - \varphi(s, 0)| = 0,$$

it follows from (6.18) that

$$\lim_{\varepsilon \to 0} \int_D v_\varepsilon \varphi d\mu_\varepsilon = \int_0^L u(s)\varphi(s,0) \, ds = \int_D u\varphi \, d\mu,$$
  
i.e.  $v_\varepsilon \rightharpoonup u$  weakly in  $L^2(D, d\mu_\varepsilon).$  (6.19)

To conclude the proof of assertion (6.17), it is enough to recall that, by the Rietz Representation Theorem, we have

$$\int_{D} v_{\varepsilon}^{2} d\mu_{\varepsilon} \leq \int_{D} u^{2} d\mu \tag{6.20}$$

Since, by the lower semi-continuity property (b),

$$\int_D u^2 \, d\mu \le \liminf_{\varepsilon \to 0} \int_D v_\varepsilon^2 \, d\mu_\varepsilon,$$

it follows from (6.20) that

$$\lim_{\varepsilon \to 0} \int_D v_\varepsilon^2 \, d\mu_\varepsilon = \int_D u^2 \, d\mu \tag{6.21}$$

Thus, the strong convergence (6.17) is a direct consequence of (6.19) and (6.21) by property(c) of the weak convergence in variable spaces.

Let  $B_{\varepsilon}$  be the ball in  $L^2(D, d\mu_{\varepsilon})$  centered at the origin and with radius  $|\Delta_{\partial F}|^{-1/2}\beta^*$ , i.e.

$$B_{\varepsilon} = \left\{ v \in L^2(D, d\mu_{\varepsilon}) : \|v\|_{L^2(D, d\mu_{\varepsilon})} \le |\Delta_{\partial F}|^{-1/2} \beta^* \right\}.$$

Let  $P_{\varepsilon}: L^2(D, d\mu_{\varepsilon}) \to B_{\varepsilon}$  be the orthogonal projection operator, which can be defined as follows

$$P(v) = \begin{cases} v, & \text{if } v \in B_{\varepsilon};\\ \operatorname{Argmin} \|v - z\|_{L^{2}(D, d\mu_{\varepsilon})}^{2}, & \text{otherwise.} \end{cases}$$

For every  $\varepsilon > 0$  we set  $u_{\varepsilon} = P_{\varepsilon}(v_{\varepsilon})$ , where  $v_{\varepsilon}$  is given by (6.15), and  $y_{\varepsilon} \in H_0^1(\Omega_{\varepsilon}; \Gamma^D)$  is a weak solution to the boundary value problem (2.2) with  $u = u_{\varepsilon}$ .

Let us show that  $\{(u_{\varepsilon}, y_{\varepsilon})\}_{\varepsilon>0}$  is a  $\Gamma$ -realizing sequence in the sense of (6.2). Indeed, as it was shown in Theorem 2.1, for every  $u_{\varepsilon} \in L^2(D, d\mu_{\varepsilon})$  (and, therefore,  $u \in L^2(\Gamma_{\varepsilon}^N)$ ) there exists a unique solution  $y_{\varepsilon} \in H_0^1(\Omega_{\varepsilon}, \Gamma^D)$  for which the a priori estimate (2.19) holds true. In particular, if  $\hat{y}_{\varepsilon} \in H_0^1(D)$  is a prototype for  $y_{\varepsilon}$ , then

$$\begin{aligned} \|\widehat{y}_{\varepsilon}\|_{H_{0}^{1}(D)} &\leq \widehat{C} \|y_{\varepsilon}\|_{H_{0}^{1}(\Omega_{\varepsilon};\Gamma^{D})} \\ &\leq & \widehat{C} \left[ \operatorname{diam} D \cdot \|f\|_{L^{2}(D)} + \beta^{*} \left( 1 + \sqrt{\delta_{0} \left( \frac{1}{2} + 3 \operatorname{diam}^{2} D \right)} \right) \right], \quad \forall \varepsilon > 0. \end{aligned}$$

$$(6.22)$$

As for the controls  $u_{\varepsilon} = P_{\varepsilon}(v_{\varepsilon}) \ \forall \varepsilon > 0$ , we see that  $u_{\varepsilon} \in B_{\varepsilon} \subset L^2(D, d\mu_{\varepsilon})$ , and, therefore,

$$\|u_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \leq |\Delta_{\partial F}|^{-1/2}\beta^{*}, \quad \forall \varepsilon > 0.$$
(6.23)

Thus, the sequence  $\{(u_{\varepsilon}, y_{\varepsilon})\}_{\varepsilon>0}$  is bounded in  $L^2(D, d\mu_{\varepsilon}) \times H^1_0(D)$ . Therefore, we can suppose that there exists a subsequence of  $\{(u_{\varepsilon}, y_{\varepsilon})\}_{\varepsilon>0}$  (still denoted by the same suffix  $\varepsilon$ ) and a pair  $(u^*, y^*) \in L^2(D, d\mu_{\varepsilon}) \times H^1_0(D)$  such that

$$u_{\varepsilon} 
ightarrow u^*$$
 in  $L^2(D, d\mu_{\varepsilon})$ , and  $\widehat{y}_{\varepsilon} 
ightarrow y^*$  in  $H^1_0(D)$  as  $\varepsilon 
ightarrow 0$ .

and  $(u^*, y^*)$  is subjected to the estimates (6.22) and

$$||u^*||_{L^2(D,d\mu)} \le |\Delta_{\partial F}|^{-1/2}\beta^*.$$

To begin with, let us show that  $u^* = u$ . With that in mind, we make use the following obvious properties:

- (i)  $(\varphi)_{\varepsilon} \to \varphi$  strongly in  $L^2(D, d\mu_{\varepsilon}) \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}^2)$ ;
- (*ii*)  $v_{\varepsilon} \to v$  strongly in  $L^2(D, d\mu_{\varepsilon})$ , hence,

$$\int_D v_{\varepsilon}^2 d\mu_{\varepsilon} \to \int_D u^2 d\mu \quad \text{(by definition of the strong convergence)};$$

(*iii*) Since  $\int_D u^2 d\mu \leq |\Delta_{\partial F}| (\beta^*)^2$ , it follows from (ii) that there exist a numerical sequence  $\{\delta_{\varepsilon}\}_{\varepsilon \to 0}$  such that

$$\|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \leq |\Delta_{\partial F}|^{-1/2}\beta^{*} + \delta_{\varepsilon}, \quad \forall \varepsilon > 0 \quad \text{and} \quad \delta_{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0;$$

 $(iv) \ \mbox{For each} \ \varphi \in C_0^\infty(\mathbb{R}^2),$  we have

$$\begin{aligned} \|(\varphi)_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})}^{2} &= \int_{D} (\varphi)_{\varepsilon}^{2} d\mu_{\varepsilon} \leq \int_{D} \left( \sup_{x \in D} |\varphi(x)| \right)^{2} d\mu_{\varepsilon} \\ &= \|\varphi\|_{C(\mathbb{R}^{2})}^{2} \cdot \int_{D} d\mu_{\varepsilon} = \mu_{\varepsilon}(D) \cdot \|\varphi\|_{C(\mathbb{R}^{2})}^{2} \leq \operatorname{Const} \cdot \|\varphi\|_{C(\mathbb{R}^{2})}^{2}; \end{aligned}$$

(v) By definition of the orthogonal projection operator  $P_{\varepsilon}: L^2(D, d\mu_{\varepsilon}) \to B_{\varepsilon}$ , we have

$$P_{\varepsilon}(v_{\varepsilon}) = \varphi_{\varepsilon} v_{\varepsilon}, \quad \text{where} \quad \varphi_{\varepsilon} = \min\left\{1, \frac{|\Delta_{\partial F}|^{-1/2} \beta^*}{\|v_{\varepsilon}\|_{L^2(D, d\mu_{\varepsilon})}}\right\}$$

(vi) In view of item(v),

$$\begin{split} \|P_{\varepsilon}(v_{\varepsilon}) - v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} &= |\gamma_{\varepsilon} - 1| \cdot \|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \\ \stackrel{\text{by (iii)}}{\leq} \left|1 - \frac{|\Delta_{\partial F}|^{-1/2}\beta^{*}}{\|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})}}\right| \cdot \|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \\ &= \left|\|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} - |\Delta_{\partial F}|^{-1/2}\beta^{*}\right| \\ &\leq |\Delta_{\partial F}|^{-1/2}\beta^{*} + \delta_{\varepsilon} - |\Delta_{\partial F}|^{-1/2}\beta^{*} = \delta_{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{split}$$

Taking these properties into account and choosing an arbitrary function  $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ , we get

$$\begin{split} \Delta &:= \int_D u^*(\varphi)_{\varepsilon} d\mu - \int_D u(\varphi)_{\varepsilon} d\mu \\ &= \int_D u^*(\varphi)_{\varepsilon} d\mu - \int_D u_{\varepsilon}(\varphi)_{\varepsilon} d\mu_{\varepsilon} + \int_D u_{\varepsilon}(\varphi)_{\varepsilon} d\mu_{\varepsilon} - \int_D v_{\varepsilon}(\varphi)_{\varepsilon} d\mu_{\varepsilon} \\ &+ \int_D v_{\varepsilon}(\varphi)_{\varepsilon} d\mu_{\varepsilon} - \int_D v_{\varepsilon} \varphi d\mu_{\varepsilon} = I_1^{\varepsilon} + I_2^{\varepsilon} + I_3^{\varepsilon}, \end{split}$$

where

$$\begin{split} I_1^{\varepsilon} &= \int_D u^*(\varphi)_{\varepsilon} \, d\mu - \int_D u_{\varepsilon}(u)_{\varepsilon} \, d\mu_{\varepsilon} \\ &= \int_D u^* \varphi \, d\mu - \int_D u_{\varepsilon} \varphi \, d\mu_{\varepsilon} + \int u^* \left( (\varphi)_{\varepsilon} - \varphi \right) \, d\mu + \int_D u_{\varepsilon} \left[ \varphi - (\varphi)_{\varepsilon} \right] \, d\mu_{\varepsilon} \\ &= J_1^{\varepsilon} + J_2^{\varepsilon} + J_3^{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0 \end{split}$$

because

- $J_1^{\varepsilon} \to 0$  by the weak convergence  $u_{\varepsilon} \rightharpoonup u^*$  in  $L^2(D, d\mu_{\varepsilon})$ ;
- $J_2^{\varepsilon} \leq \|u^*\|_{L^2(D,d\mu)}\|(\varphi)_{\varepsilon} \varphi\|_{L^2(D,d\mu)} \to 0$  by the properties of smoothing operator;

• 
$$J_3^{\varepsilon} \leq \underbrace{\left(\sup_{\varepsilon>0} \|u_{\varepsilon}\|_{L^2(D,d\mu_{\varepsilon})}\right)}_{\text{Const}} \cdot \|\varphi - (\varphi)_{\varepsilon})\|_{L^2(D,d\mu_{\varepsilon})} \leq \text{Const} \cdot \sup_{x \in D} |\varphi(x) - (\varphi)_{\varepsilon}(x)| \to 0$$

by the properties of smoothing operator.

As for the rest terms in  $\Delta$ , we have

$$I_3^{\varepsilon} \le \left( \sup_{\varepsilon > 0} \| v_{\varepsilon} \|_{L^2(D, d\mu_{\varepsilon})} \right) \cdot \| (\varphi)_{\varepsilon} - \varphi \|_{L^2(D, d\mu_{\varepsilon})} \to 0$$

(by analogy with the previous case);

$$I_{2}^{\varepsilon} = \int_{D} P_{\varepsilon}(v_{\varepsilon})(\varphi)_{\varepsilon} d\mu_{\varepsilon} - \int_{D} v_{\varepsilon}(\varphi_{\varepsilon}) d\mu_{\varepsilon} \leq \|P_{\varepsilon}(v_{\varepsilon}) - v_{\varepsilon}\|_{L^{2}(D, d\mu_{\varepsilon})]} \cdot \|\varphi\|_{C(\mathbb{R}^{2})} \cdot \text{Const}$$

$$\stackrel{\text{by (iv)}}{\leq} \delta_{\varepsilon} \|\varphi\|_{C(\mathbb{R}^{2})} \cdot \text{Const} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus, we finally arrived at the following relation

$$\int_D u^* \varphi \, d\mu = \int_D u\varphi \, d\mu \quad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2),$$

which implies

$$u^* = u \quad \text{in} \quad L^2(D, d\mu)$$

Since this inference is valid for any cluster point  $u^*$  of the sequence  $\{u_{\varepsilon}\}_{\varepsilon>0}$  with respect to the weak convergence in  $L^2(D, d\mu_{\varepsilon})$ , it follows that  $u \in L^2(D, d\mu)$  is the weak limit for the entire sequence  $\{u_{\varepsilon}\}_{\varepsilon>0}$ .

Our next step is to show that

$$\lim_{\varepsilon \to 0} \|u_{\varepsilon}\|_{L^{2}(D, d\mu_{\varepsilon})}^{2} = \|u\|_{L^{2}(D, d\mu)}^{2}, \qquad (6.24)$$

where

$$u_{\varepsilon} = P_{\varepsilon} v_{\varepsilon} = \gamma_{\varepsilon} v_{\varepsilon}, \quad \forall \varepsilon > 0,$$

and  $v_{\varepsilon}$  is related with u by (6.15) and, therefore,  $v_{\varepsilon} \to u$  strongly in  $L^2(D, d\mu_{\varepsilon})$ . Indeed,

$$\|u_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} = \|P_{\varepsilon}(v_{\varepsilon}) - v_{\varepsilon} + v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \le \|P_{\varepsilon}(v_{\varepsilon}) - v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} + \|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})}.$$

Then

$$\|P_{\varepsilon}(v_{\varepsilon}) - v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \leq \delta_{\varepsilon} \to 0 \quad \text{by property (vi)},$$
  
and  $\|v_{\varepsilon}\|_{L^{2}(D,d\mu_{\varepsilon})} \to \|u\|_{L^{2}(D,d\mu)} \quad \text{by property (ii)}.$ 

Hence, relation(6.24) is valid.

It remains to show that  $y^* = y$  as element of  $H_0^1(\Omega_0; \Gamma_D)$  and this equality holds for any cluster point  $y^*$  of the sequence  $\{\widehat{y}_{\varepsilon}\}_{\varepsilon>0} \subset H_0^1(D)$ .

Indeed, as follows from definition of the elements  $\hat{y}_{\varepsilon}$ , the following integral identity (see for comparison (6.14))

$$\int_{\Omega_0} (\nabla \widehat{y}_{\varepsilon}, \nabla \varphi)_{\mathbb{R}^2} dx + \alpha^2 (\Delta_F) \int_{\Omega_1} (\nabla \widehat{y}_{\varepsilon}, \nabla \varphi) d\nu_{\varepsilon} = \int_{\Omega_0} f\varphi \, dx + \alpha^2 (\Delta_F) \int_{\Omega_1} f\varphi \, d\nu_{\varepsilon} + |\Delta_{\partial F}| \int_D u_{\varepsilon} \varphi d\mu_{\varepsilon} \quad (6.25)$$

holds true for every test function  $\varphi \in C_0^{\infty}(D)$  and any  $\varepsilon > 0$ .

Using the fact that  $(u_{\varepsilon}, y_{\varepsilon}) \xrightarrow{\sigma} (u, y^*)$  and applying the similar argument as we did it before (see the substantiation of the limit passage in (6.14)), we can pass to the limit in (6.25) as  $\varepsilon \to 0$ . As a result, we arrive at the integral identity

$$\int_{\Omega_0} (\nabla y^*, \nabla \varphi)_{\mathbb{R}^2} \, dx = \int_{\Omega_0} f\varphi \, dx + |\Delta_{\partial F}| \int_0^L u(s)\varphi(s,0) \, ds,$$

which is valid for every  $\varphi \in C_0^{\infty}(D)$ .

Hence,  $y^*$  is a weak solution to the limit boundary value problem

$$-\nabla y = f \quad \text{in} \quad \Omega_0,$$
  

$$y = 0 \quad \text{on} \quad \Gamma_D = \partial \Omega_0 \setminus \Gamma^N,$$
  

$$\frac{\partial y}{\partial n} = u |\Delta_{\partial F}| \quad \text{on} \quad \Gamma^N = \left\{ (x_1, x_2) \middle| x_2 = 0, \ 0 < x_1 < L) \right\}.$$
(6.26)

Since, by Lax-Milgram Theorem, this problem admits a unique weak solution in  $H_0^1(\Omega_0, \Gamma^D)$ and the pair (u, y) belongs to the set  $\Xi$  (see (6.12)), it follows that for a given control  $u \in L^2(0, L)$ , we have

 $y^* = y$  as elements of  $H_0^1(\Omega_0, \Gamma_D)$ .

It remains to notice that this conclusion is valid for any cluster point  $y^*$  of the sequence  $\{\widehat{y}_{\varepsilon}\}_{\varepsilon>0} \subset H^1_0(D)$ . Thus,

$$(u_{\varepsilon}, y_{\varepsilon}) \xrightarrow{o} (u, y)$$
 as  $\varepsilon \to 0$  and  $(u, y) \in \Xi$ ,

i.e  $\{(u_{\varepsilon}, y_{\varepsilon})\}_{\varepsilon>0}$  is a  $\Gamma$ -realizing sequence.

To conclude the proof of item (aa), we make use of the following observation:

$$\frac{|\Delta_{\partial F}|}{2} \int_{D} u_{\varepsilon}^{2} d\mu_{\varepsilon} \stackrel{\varepsilon \to 0}{\to} \frac{|\Delta_{\partial F}|}{2} \int_{0}^{L} u^{2}(s) ds \quad \text{by}(6.24)$$

and

$$\frac{\alpha}{2} \int_{\Omega_0} (y_{\varepsilon} - y_{ad})^2 \ dx \xrightarrow{\varepsilon \to 0} \frac{\alpha}{2} \int_{\Omega_0} (y - y_{ad})^2 \ dx$$

by compactness of the embedding  $H^1_0(\Omega_0, \Gamma^D) \hookrightarrow L^2(\Omega_0)$ . Thus

$$\lim_{\varepsilon \to 0} I_{\varepsilon}(u_{\varepsilon}, y_{\varepsilon}) = I(u, y)$$

and this concludes the proof of (aa)-property of Definition 6.2.

It other words, we have shown that the constrained minimization problem (6.12) is the variational  $\sigma$ -limit of the sequence (2.14) as  $\varepsilon \to 0$ . Moreover, as immediately follows from the structure of the cost functional I(u, y) and the set of admissible pairs  $\Xi$ , the problem (6.12) can be recovered in the form of the following optimal control problem:

Minimize

$$I(u,y) = \frac{|\Delta_{\partial F}|}{2} \int_0^L u^2(s) \, ds + \frac{\alpha}{2} \int_{\Omega_0} (y - y_{ad})^2 \, dx \tag{6.27}$$

subjected to the constraints

$$-\nabla y = f \quad \text{in} \quad \Omega_0, y = 0 \quad \text{on} \quad \Gamma_D, \frac{\partial y}{\partial n} = u |\Delta_{\partial F}| \quad \text{on} \quad \Gamma^N, \\ ||u||_{L^2(0,L)} \le |\Delta_{\partial F}|^{-1/2} \beta^*, \end{cases}$$

$$(6.28)$$

where  $\Gamma^N = \{(x_1, 0) | 0 < x_1 < L\}, \ \Gamma^D = \partial \Omega_0 \setminus |\Gamma^N.$ 

To conclude this section, it is worth to note that the limit OCP (6.27)–(6.28) has a unique solution  $(u_0, y_0) \in L^2(0, L) \times H^1_0(\Omega_0; \Gamma^D)$  (see Fursikov [7]) and this solution possesses the following remarkable property:

**Theorem 6.1.** Let  $(u_{\varepsilon}^0, y_{\varepsilon}^0) \in L^2(\Gamma_{\varepsilon}^N) \times H^1_0(\Omega_{\varepsilon}, \Gamma^D)$  be an optimal pair to the original OCP (2.1)-(2.3). Then

$$(u_{\varepsilon}^{0}, y_{\varepsilon}^{0}) \xrightarrow{\sigma} (u^{0}, y^{0}) \in L^{2}(0, L) \times H^{1}_{0}(\Omega_{0}, \Gamma^{D})$$
$$I(u^{0}, y^{0}) = \lim_{\varepsilon \to 0} I_{\varepsilon}(u_{\varepsilon}^{0}, y_{\varepsilon}^{0}).$$

For the proof of this result, we refer to Kogut & Lengering (see Theorem Th 5.3, p.142 in [11]).

30

Properly speaking, this result reveals the way for the construction of suboptimal controls to OCPs in domains with rough boundaries. In particular, we can consider as a suboptimal control to the problem (2.1)-(2.3) the following one:

$$u_{\varepsilon}^{\mathrm{sub}} = P_{\varepsilon}(v_{\varepsilon}^{0}) = \min\left\{1, \frac{|\Delta_{\partial F}|^{-1/2}\beta^{*}}{\|v_{\varepsilon}^{0}\|_{L^{2}(D, d\mu_{\varepsilon})}}\right\} \cdot v_{\varepsilon}^{0},$$

where  $v_{\varepsilon}^0 \in L^2(D, d\mu_{\varepsilon})$  is the lift of optimal control to the limit problem (6.27)–(6.28), which can be defined by the rule

$$\int_D v_{\varepsilon}^0 \varphi \, d\mu_{\varepsilon} = \int_D u^0(\varphi)_{\varepsilon} \, d\mu = \int_0^L u^0(s)(\varphi)_{\varepsilon}(s,0) \, ds \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

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