On Existence of Optimal Solutions to Boundary Control Problem for an Elastic Body with Quasistatic Evolution of Damage

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Abstract We study an optimal control problem for the mixed boundary value problem for an elastic body with quasistatic evolution of an internal damage variable. We use the damage field $\zeta = \zeta(t,x)$ as an internal variable which measures the fractional decrease in the stress-strain response. When $\zeta = 1$ the material is damage-free, when $\zeta = 0$ the material is completely damaged, and for $0 < \zeta < 1$ it is partially damaged. We suppose that the evolution of microscopic cracks and cavities responsible for the damage is described by a nonlinear parabolic equation, whereas the model for the stress in elastic body is given as $\sigma = \zeta(t,x)A\mathbf{e}(\mathbf{u})$. The optimal control problem we consider in this paper is to minimize the appearance of micro-cracks and micro-cavities as a result of the tensile or compressive stresses in the elastic body.

1 Introduction

The damage modeling in the context of industrial applications is in its infancy– corrosion, multi-micro cracking etc. This makes this problem extremely complex. The main idea of a novel approach to modeling material damage is to use the so-called damage field $\zeta = \zeta(t,x)$ as an internal variable which measures the fractional decrease in the stress-strain response. The evolution of the damage field is derived from the principle of virtual work under appropriate assumptions on the system's free energy, the dissipation pseudopotential, and the spatial interactions of the microcracks. In this approach the damage field ζ varies between one and zero at each point in the body. When $\zeta = 1$ the material is damage-free, when $\zeta = 0$ the material

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is completely damaged, and for $0 < \zeta < 1$ it is partially damaged. The evolution of the damage field is usually described by a parabolic inclusion or equation with a damage source function ϕ which depends on the mechanical compression or tension [7]. At the same time, the model for the stress is given as $\sigma = \zeta(t,x)A\mathbf{e}(\mathbf{u})$. Without the damage parameter ζ, this is the classical model of elastic material. However, if parameter ζ varies in the interval $[0,1]$, the corresponding elasticity system

$$-\text{div}\,(\zeta A\mathbf{e}(\mathbf{u})) = \mathbf{f}$$

becomes degenerate.

In this paper we assume that the elastic body under consideration occupies the domain Ω and is clamped on the part S of its boundary, and the rest part of the boundary $\Gamma = \partial\Omega \setminus S$ is the influence zone of a Neumann control. Therefore, the control variable is the density of a surface traction $\mathbf{p}$ acting on Γ. The optimal control problem we consider in this paper aims at two objectives. On the one hand we try to minimize the discrepancy between a given displacement field $\mathbf{u}_d$ and the solution of the initial-boundary value problem by choosing an appropriate surface traction $\mathbf{p} \in \mathscr{P}_{ad}$. On the other hand, we wish to minimize the appearance of micro-cracks and micro-cavities as a result of the tensile or compressive stresses in the elastic body. To the best knowledge of authors the existence of optimal solutions for the above problem is an open question. Moreover, only few papers deal with optimal control problems for degenerate partial differential equations (see for example [1, 2, 3, 5, 6]).

2 Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary. We assume that Ω is occupied by some elastic body and its outer surface $\partial\Omega$ is divided into two disjoint measurable parts $\partial\Omega = \Gamma \cup S$. Let the sets S and Γ have positive $(N-1)$-dimensional measures and let S be closed.

For any subset $E \subset \mathbb{R}^N$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathscr{L}^N(E)$. Let χ_E be the characteristic function of a subset $E \subset \mathbb{R}^N$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

We will often use the Lebesgue spaces of vector-value functions. For example, for the L^2-space of vector-valued functions $\mathbf{u}(x) = (u_1(x),\dots,n_N(x))^t \in \mathbb{R}^N$ we use the notation $L^2(\Omega)^N = L^2(\Omega,\mathbb{R}^N)$. At the same time, $L^2(\Omega)^{\frac{N(N-1)}{2}} = L^2\big(\Omega;\mathbb{R}^{\frac{N(N-1)}{2}}\big)$ is the space of square-summable functions whose values are symmetric matrices. We denote by $\mathbb{S}^N := \mathbb{R}^{\frac{N(N-1)}{2}}$ the set of all symmetric matrices $\xi = [\xi_{ij}]_{i,j=1}^N$, $(\xi_{ij} = \xi_{ji})$. We suppose that $\mathbb{S}^N$ is endowed with the euclidian scalar product $\xi \cdot \eta = \text{tr}(\xi\,\eta) = \xi_{ij}\eta_{ij}$ and with the corresponding euclidian norm $\|\xi\|_{\mathbb{S}^N} = (\xi\cdot\xi)^{1/2}$. Hereinafter, we adopt the convention regarding summation with respect to repeating indices. In particular, $\xi^2 = \xi_{ij}\xi_{ij}$.

We denote by $A(x) = \left[A^{kl}(x)\right]_{k,l=1}^{N} = \left\{a_{ij}^{kl}(x)\right\}$ an elasticity tensor at a material point $x \in \Omega$. The action of the elasticity tensor $A(x)$ on the matrix $\xi \in \mathbb{S}^N$ is defined by $A(x)\xi = \left\{a_{ij}^{kl}(x)\xi_{kl}\right\}$. Then, $A(x)\xi \cdot \xi = a_{ij}^{kl}(x)\xi_{kl}\xi_{ij}$ is the elastic energy density. It is assumed that $A(x)$ satisfies the usual symmetry conditions:

$$a_{ij}^{kl}(x) = a_{ji}^{lk}(x) = a_{il}^{kj}(x), \quad \forall i,j,k,l = 1,2,\dots,N. \tag{1}$$

Let κ_1 and κ_2 be two fixed constants such that $\kappa_2 > \kappa_1$. We define $\mathscr{A}_{\kappa_1}^{\kappa_2}(\Omega)$ as the set of all symmetric elasticity tensors $A(x) = \left\{a_{ij}^{kl}(x)\right\}$ such that the positive definiteness condition holds:

$$\kappa_1\xi^2 \le \mathscr{A}(x)\xi \cdot \xi \le \kappa_2\xi^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{S}^N. \tag{2}$$

In order to describe a quasistatic evolution of damage in the elastic body Ω, we denote by $\mathbf{u}(x) = (u_1(x),\dots,u_N(x))$ the displacement field, $\sigma(x) = \left\{\sigma_{ij}(x)\right\}$ the stress tensor, and $\mathbf{e}(\mathbf{u}) = \left\{\mathbf{e}_{ij}(\mathbf{u})\right\}$ the strain tensor. We assume that for every smooth vector $\mathbf{u}(x) = (u_1(x),\dots,u_N(x))$ the formula for the strain tensor $\mathbf{e}_{ij}(\mathbf{u})$ is provided by the Cauchy law of small deformations

$$\mathbf{e}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \forall i,j = 1,\dots,N. \tag{3}$$

It is clear that $\mathbf{e}(\mathbf{u}) \in \mathbb{S}^N$ and $\mathbf{e}(\mathbf{u})$ is the symmetric part of the gradient of a displacement $\mathbf{u}$. Thus $\mathbf{e}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$, where the gradient of a displacement $\mathbf{u} \in \mathbb{R}^N$ is the $(N \times N)$ matrix $\nabla\mathbf{u}$ the entries of which are defined by $(\nabla\mathbf{u})_{ij} := \frac{\partial u_i}{\partial x_j}$.

Hence, for any symmetric tensor $A \in \mathscr{A}_{\kappa_1}^{\kappa_2}(\Omega)$, we have $A\mathbf{e}(\mathbf{u}) = A\nabla\mathbf{u}$. Therefore, we will use indifferently both expressions. Note also that the divergence of a smooth matrix $\sigma(x)$ is the vector $\operatorname{div}(\sigma) \in \mathbb{R}^N$ the components of which are defined by $(\operatorname{div}(\sigma))_i := \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j}$.

Let $\Omega_T = (0,T) \times \Omega$ for some $T > 0$. Let ζ denote a damage field in Ω_T and measures the fractional decrease in the strength of the material. Usually, for an isotropic material, the damage field $\zeta = \zeta(t,x)$ is defined as the ratio $\zeta = \zeta(t,x) = \frac{E_{eff}}{E}$ between the effective modulus of elasticity E_{eff} and that of the damage-free material E. It follows from this definition that the damage field should only have values between 0 and 1. Since every damage $\zeta : \Omega_T \to [0,1]$ gives rise to a measure on the measurable subsets of Ω_T through integration, we will denote this measure by ζ. Thus $\zeta(E) = \int_E \zeta\, dz$ for measurable sets $E \subset \Omega_T$. We will use the standard notation $L^2(\Omega_T, \zeta\, dz)$ for the set of measurable functions f on Ω_T such that

$$\|f\|_{L^2(\Omega_T,\zeta\,dz)} = \left(\int_{\Omega_T} f^2\,\zeta\,dz\right)^{1/2} = \left(\int_0^T \int_\Omega f^2\,\zeta\,dxdt\right)^{1/2} < +\infty.$$

Let $C_0^\infty(\mathbb{R}^N;S) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N)\ :\ \varphi = 0 \text{ on } S\right\}$ be the set of smooth damages in Ω. We define the space $H^1(\Omega;S)$ as the closure of $C_0^\infty(\mathbb{R}^N;S)$ with respect to the norm

$$\left(\int_\Omega \left[y^2 + |\nabla y|^2_{\mathbb{R}^N}\right] dx\right)^{1/2}.$$

Let $\mathscr{X} = L^2(0,T;H^1(\Omega))$, $\mathscr{V} = L^2(0,T;H^1(\Omega;S))$. Let $\mathscr{X}' = L^2(0,T;H^1(\Omega)')$ and $\mathscr{V}' = L^2(0,T;H^1(\Omega;S)')$ be their dual. The following theorem plays an important role in the study of an quasistatic evolution of damage in an elastic bodies (see Simon [10]).

Theorem 1. *Let us define the Banach space*

$$\mathscr{W} = \left\{\zeta\ :\ \zeta \in \mathscr{X},\quad \frac{\partial \zeta}{\partial t} \in \mathscr{X}'\right\},$$

equipped with the norm of the graph. Then, the following properties hold true:

1. *the embedding $\mathscr{W} \hookrightarrow L^2(0,T;L^2(\Omega))$ is compact;*
2. *one has the embedding*
$$\mathscr{W} \hookrightarrow C([0,T];L^2(\Omega)), \tag{4}$$
where, $C([0,T];L^2(\Omega))$ denotes the space of measurable functions on $[0,T]\times\Omega$ such that $\zeta(t,\cdot) \in L^2(\Omega)$ for any $t \in [0,T]$ and such that the map $t \in [0,T] \mapsto \zeta(t,\cdot) \in L^2(\Omega)$ is continuous;
3. *for any $\zeta, \nu \in \mathscr{W}$*

$$\frac{d}{dt}\int_\Omega \zeta(t,x)\nu(t,x)\,dx = \left\langle \zeta'(t,\cdot), \nu(t,\cdot)\right\rangle_{\mathscr{X}',\mathscr{X}} + \left\langle \nu'(t,\cdot), \zeta(t,\cdot)\right\rangle_{\mathscr{X}',\mathscr{X}}. \tag{5}$$

Definition 1. We say that a damage $\zeta : \Omega_T \to [0,1]$ is substantial in Ω, if

$$\zeta^{-1} \notin L^\infty(\Omega_T) \quad \text{and} \quad \zeta^{-1} \in L^1(\Omega_T). \tag{6}$$

Note that in this case the functions in $L^2(\Omega_T, \zeta\, dxdt)$ are Lebesgue integrable on Ω_T.

Let W be the closure of the set of pairs $\left\{(\mathbf{u}, \mathbf{e}(\mathbf{u}))\ :\ \mathbf{u} \in C_0^\infty(R^N;S)^N\right\}$ in the product of spaces $L^1(\Omega)^N \times L^1(\Omega)^{\frac{N(N+1)}{2}}$. Thus the elements of W are pairs $(\mathbf{u}, \mathbf{z})$, where $\mathbf{u}$ is a vector and $\mathbf{z} = \mathbf{e}(\mathbf{u})$ is the symmetric gradient of the vector $\mathbf{u}$. In what follows, we define the space $\mathscr{W}^{1,1}(\Omega;S)$ as the union of the first components $\mathbf{u}$ of W. Following standard technique, it is easy to show that $\mathscr{W}^{1,1}(\Omega;S)$ is a Banach space with respect to the norm $\|\mathbf{u}\|_{\mathscr{W}^{1,1}(\Omega;S)} = \int_\Omega \left[|\mathbf{u}|_{\mathbb{R}^N} + \|\mathbf{e}(\mathbf{u})\|_{\mathbb{S}^N}\right] dx$. To each damage field $\zeta(t,x)$ we may associate two weighted spaces

$$W_\zeta(\Omega\times(0,T);S) \ \text{ and } \ H_\zeta(\Omega\times(0,T);S),$$

where $W_\zeta(\Omega\times(0,T);S)$ is the set of vector-functions $\mathbf{u} \in L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$ for which the norm

$$\|\mathbf{u}\|_\zeta = \Big(\int_0^T \int_\Omega \big(\mathbf{u}^2 + \mathbf{e}^2(\mathbf{u})\zeta\big)\, dxdt\Big)^{1/2} \tag{7}$$

is finite, and $H_\zeta(\Omega \times (0,T);S)$ is the closure of the set

$$\big\{\psi(t)\varphi(x) \ : \ \psi \in C_0^\infty(0,T), \ \varphi \in C_0^\infty(\mathbb{R}^N;S)^N\big\} \tag{8}$$

in the $W_\zeta(\Omega \times (0,T);S)$-norm. Note that due to the estimates

$$\int_0^T \int_\Omega |\mathbf{u}|_{\mathbb{R}^N}\, dxdt \le \Big(\int_0^T \int_\Omega \mathbf{u}^2\, dxdt\Big)^{1/2} \sqrt{T|\Omega|} \le C\|\mathbf{u}\|_\zeta, \tag{9}$$

$$\begin{aligned}\int_0^T \int_\Omega \|\mathbf{e}(\mathbf{u})\|\, dxdt &:= \int_0^T \int_\Omega (\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\mathbf{u}))^{1/2}\, dxdt\\ &\le \Big(\int_0^T \int_\Omega \mathbf{e}^2(\mathbf{u})\zeta\, dxdt\Big)^{1/2} \Big(\int_0^T \int_\Omega \zeta^{-1}\, dxdt\Big)^{1/2} \le C\|\mathbf{u}\|_\zeta, \end{aligned} \tag{10}$$

the space $W_\zeta(\Omega \times (0,T);S)$ is complete with respect to the norm $\|\cdot\|_\zeta$. Moreover, it is clear that $H_\zeta(\Omega \times (0,T);S) \subseteq W_\zeta(\Omega \times (0,T);S)$, and $W_\zeta(\Omega \times (0,T);S)$, $H_\zeta(\Omega \times (0,T);S)$ are Hilbert spaces endowed with the scalar product

$$(\mathbf{u},\mathbf{v})_\zeta = \int_0^T \int_\Omega [\mathbf{u}\cdot\mathbf{v} + \mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\mathbf{v})\zeta]\, dxdt. \tag{11}$$

If the damage field $\zeta = \zeta(t,x)$ is bounded between two positive constants, then it is easy to verify that

$$W_\zeta(\Omega \times (0,T);S) = H_\zeta(\Omega \times (0,T);S). \tag{12}$$

However, for a “substantial” damage ζ in the sense of Definition 1, the set of smooth functions (8) is not dense in $W_\zeta(\Omega \times (0,T);S)$. Hence the identity (12) is not always valid.

3 Radon measures and convergence in variable spaces

By a nonnegative Radon measure on Ω_T we mean a nonnegative Borel measure which is finite on every compact subset of Ω_T. The space of all nonnegative Radon measures on Ω_T will be denoted by $\mathscr{M}_+(\Omega_T)$. According to the Riesz Representation Theorem, each Radon measure $\mu \in M_+(\Omega_T)$ can be interpreted as element of the dual of the space $C_0(\Omega_T)$ of all continuous functions vanishing at infinity. If μ is a nonnegative Radon measure on Ω_T, we will use $L^r(\Omega_T, d\mu)$, $1 \le r \le \infty$, to denote the usual Lebesgue space with respect to the measure μ with the corresponding norm $\|f\|_{L^r(\Omega_T,d\mu)} = \Big(\int_{\Omega_T} |f(x)|^r\, d\mu\Big)^{1/r}$.

Let $\{\mu_k\}_{k\in\mathbb{N}}$, μ be Radon measures such that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}_+(\Omega_T)$, i.e.,

$$\lim_{k\to\infty}\int_{\Omega_T}\psi\varphi\,d\mu_k = \int_{\Omega_T}\psi\varphi\,d\mu \quad \forall\,\psi\in C_0(\mathbb{R}),\ \forall\,\varphi\in C_0(\mathbb{R}^N), \tag{13}$$

where $C_0(\mathbb{R}^N)$ is the space of all compactly supported continuous functions. A typical example of such measures is

$$d\mu_k = \zeta_k(t,x)\,dxdt,\ \ d\mu = \zeta(t,x)\,dxdt, \text{ where } \ 0\le\zeta_k\rightharpoonup\zeta \text{ in } L^1(\Omega_T). \tag{14}$$

Let us recall the definition and main properties of convergence in the variable L^2-space.

1. A sequence $\left\{\mathbf{v}_k\in L^2(\Omega_T,d\mu_k)^N\right\}_{k\in\mathbb{N}}$ is called bounded if
$$\limsup_{k\to\infty}\int_{\Omega_T}|\mathbf{v}_k|^2_{\mathbb{R}^N}\,d\mu_k < +\infty.$$
2. A bounded sequence $\left\{\mathbf{v}_k\in L^2(\Omega_T,d\mu_k)^N\right\}_{k\in\mathbb{N}}$ converges weakly to $\mathbf{v}\in L^2(\Omega_T,d\mu)^N$ if $\lim_{k\to\infty}\int_{\Omega_T}\mathbf{v}_k\cdot\varphi\,d\mu_k = \int_{\Omega_T}\mathbf{v}\cdot\varphi\,d\mu$ for any $\varphi\in C_0^\infty(\Omega_T)^N$, which is denoted as $\mathbf{v}_k\rightharpoonup\mathbf{v}$ in $L^2(\Omega_T,d\mu_k)^N$.
3. The strong convergence $\mathbf{v}_k\to\mathbf{v}$ in $L^2(\Omega_T,d\mu_k)^N$ means that $\mathbf{v}\in L^2(\Omega_T,d\mu)^N$ and
$$\lim_{k\to\infty}\int_{\Omega_T}\mathbf{v}_k\cdot\mathbf{z}_k\,d\mu_k = \int_{\Omega_T}\mathbf{v}\cdot\mathbf{z}\,d\mu \ \text{ as } \ \mathbf{z}_k\rightharpoonup\mathbf{z} \ \text{ in } \ L^2(\Omega_T,d\mu_k)^N. \tag{15}$$

The following convergence properties in variable spaces hold:

(a) *Compactness:* if a sequence is bounded in $L^2(\Omega_T,d\mu_k)^N$, then this sequence is compact in the sense of the weak convergence in $L^2(\Omega,d\mu_k)^N$;

(b) *Lower semicontinuity:* if $\mathbf{v}_k\rightharpoonup\mathbf{v}$ in $L^2(\Omega_T,d\mu_k)^N$, then
$$\liminf_{k\to\infty}\int_{\Omega_T}|\mathbf{v}_k|^2_{\mathbb{R}^N}\,d\mu_k \ge \int_{\Omega_T}|\mathbf{v}|^2_{\mathbb{R}^N}\,d\mu; \tag{16}$$

(c) *Strong convergence:* $\mathbf{v}_k\to\mathbf{v}$ if and only if $\mathbf{v}_k\rightharpoonup\mathbf{v}$ in $L^2(\Omega_T,d\mu_k)^N$ and
$$\lim_{k\to\infty}\int_{\Omega_T}|\mathbf{v}_k|^2_{\mathbb{R}^N}\,d\mu_k = \int_{\Omega_T}|\mathbf{v}|^2_{\mathbb{R}^N}\,d\mu. \tag{17}$$

For our further analysis we make use the following concept.

Definition 2. We say that a bounded sequence

$$\left\{(\zeta_n,\mathbf{u}_n)\in L^2(\Omega_T)\times W_{\zeta_n}(\Omega\times(0,T);S)\right\}_{n\in\mathbb{N}} \tag{18}$$

w-converges to $(\zeta,\mathbf{u})\in L^2(\Omega_T)\times L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$ as $n\to\infty$, if

$$\zeta_n \rightharpoonup \zeta \quad \text{in } L^2(\Omega_T), \tag{19}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(\Omega_T)^N, \tag{20}$$

$$\mathbf{e}(\mathbf{u}_n) \rightharpoonup \mathbf{e}(\mathbf{u}) \quad \text{in the variable space } L^2\big(0,T;L^2(\Omega,\zeta_n\,dx)^{\frac{N(N+1)}{2}}\big), \tag{21}$$

that is,

$$\lim_{n\to\infty}\int_0^T\int_\Omega \zeta_n\eta\,dxdt = \int_0^T\int_\Omega \zeta\eta\,dxdt \quad \forall\,\eta\in L^2(\Omega_T), \tag{22}$$

$$\lim_{n\to\infty}\int_0^T\int_\Omega \mathbf{u}_n\cdot\lambda\,dxdt = \int_0^T\int_\Omega \mathbf{u}\cdot\lambda\,dxdt \quad \forall\,\lambda\in L^2(\Omega_T)^N, \tag{23}$$

and

$$\lim_{n\to\infty}\int_0^T\int_\Omega \zeta_n\mathbf{e}(\mathbf{u}_n)\cdot\xi(x)\phi(t)\,dxdt = \int_0^T\int_\Omega \zeta\mathbf{e}(\mathbf{u})\cdot\xi(x)\phi(t)\,dxdt \quad \forall\,\psi\in C_0^\infty(0,T),\ \forall\,\xi\in C_0^\infty(\Omega;\mathbb{S}^N). \tag{24}$$

In order to verify the correctness of this definition, we give the following result.

Lemma 1. *Let* $\big\{(\zeta_n,\mathbf{u}_n)\in L^2(\Omega_T)\times W_{\zeta_n}(\Omega\times(0,T);S)\big\}_{n\in\mathbb{N}}$ *be a sequence such that*

(i) *this sequence is bounded, i.e.*

$$\sup_{n\in\mathbb{N}}\left[\int_0^T\int_\Omega\big(\zeta_n^2+\mathbf{u}_n^2+\mathbf{e}^2(\mathbf{u}_n)\zeta_n\big)\,dxdt\right]<+\infty; \tag{25}$$

(ii) *there exists an element* $\zeta\in L^1(\Omega_T)$ *such that*

$$\zeta_n\to\zeta \quad \textit{and} \quad \zeta_n^{-1}\to\zeta^{-1} \quad \textit{in } L^1(\Omega_T) \textit{ as } n\to\infty, \tag{26}$$

(iii) $\zeta_n:\Omega_T\to[0,1]$ *for all* $n\in\mathbb{N}$.

Then, this sequence is relatively compact with respect to w-convergence. Moreover, each w-limit pair $(\zeta,\mathbf{u})$ *belongs to the corresponding space* $L^2(\Omega_T)\times W_\zeta(\Omega\times(0,T);S)$.

Proof. To begin with, we note that the condition (25) and estimates (9)–(10) immediately imply the boundedness of the sequence in $L^2(\Omega_T)\times L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$. The uniform boundedness of $\{\zeta_n\}_{n\in\mathbb{N}}$ in $L^2(\Omega_T)$ and property (26) ensure that the limit damage field ζ belongs to $L^2(\Omega_T)$ as well. Moreover, we have (see the property (14)): $d\zeta_n := \zeta_n\,dxdt \overset{*}{\rightharpoonup} \zeta\,dxdt =: d\zeta$ in $\mathscr{M}_+(\Omega_T)$.

Then, the compactness criterium of the weak convergence in variable spaces (see property (a)) and condition (25) leads us to the existence of a pair $(\mathbf{u},\mathbf{v})\in L^2\big(0,T;L^2(\Omega)^N\big)\times L^2\big(0,T;L^2(\Omega,\zeta\,dx)^{\frac{N(N+1)}{2}}\big)$ such that, within a subsequence of $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$,

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(\Omega_T)^N, \tag{27}$$

$$\mathbf{e}(\mathbf{u}_n) \rightharpoonup \mathbf{v} \text{ in variable space } L^2\big(0,T;L^2(\Omega,\zeta_n\,dx)^{\frac{N(N+1)}{2}}\big). \tag{28}$$

Our aim is to show that $\mathbf{v} = \mathbf{e}(\mathbf{u})$ and $\mathbf{u} \in W_\zeta(\Omega \times (0,T);S)$. Indeed, for any $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(0,T)$, we have

$$\int_0^T \int_\Omega \zeta_n^{-1}\varphi\psi\zeta_n\,dxdt = \int_0^T \int_\Omega \varphi\psi\,dxdt = \int_0^T \int_\Omega \zeta^{-1}\varphi\psi\zeta\,dxdt,$$

i.e. $\zeta_n^{-1} \rightharpoonup \zeta^{-1}$ in $L^2(\Omega_T,d\zeta_n)$. Moreover, the strong convergence in $(26)_2$ implies the relation

$$\lim_{n\to\infty}\int_0^T \int_\Omega \zeta_n^{-2}\zeta_n\,dxdt = \lim_{n\to\infty}\int_0^T \int_\Omega \zeta_n^{-1}\,dxdt = \int_0^T \int_\Omega \zeta^{-2}\zeta\,dxdt.$$

Hence, $\zeta_n^{-1} \to \zeta^{-1}$ strongly in $L^2(\Omega_T,d\zeta_n)$ (see property (c)), and therefore,

$$\psi\xi\zeta_n^{-1} \to \psi\xi\zeta^{-1} \text{ strongly in } L^2\big(0,T;L^2(\Omega,\zeta_n\,dx)^{\frac{N(N+1)}{2}}\big) \tag{29}$$

for each $\psi \in C_0^\infty(0,T)$ and $\xi \in C_0^\infty(\Omega;\mathbb{S}^N)$. Further, we note that for every measurable subset $K \subset \Omega_T$, the estimate

$$\int_K \|\mathbf{e}(\mathbf{u}_n)\|_{\mathbb{S}^N}\,dz \le \left(\int_K \zeta_n^{-1}\,dz\right)^{1/2}\left(\int_K \|\mathbf{e}(\mathbf{u}_n)\|^2_{\mathbb{S}^N}\zeta_n\,dz\right)^{1/2} \le C\left(\int_K \zeta_n^{-1}\,dz\right)^{1/2}$$

implies the equi-integrability of the family $\{\|\mathbf{e}(\mathbf{u}_n)\|_{\mathbb{S}^N}\}_{n\in\mathbb{N}}$. Hence, the sequence $\{\|\mathbf{e}(\mathbf{u}_n)\|_{\mathbb{S}^N}\}_{n\in\mathbb{N}}$ is weakly compact in $L^1(\Omega_T)$, which means the weak compactness of the matrix-valued sequence $\{\mathbf{e}(\mathbf{u}_n)\}_{n\in\mathbb{N}}$ in $L^1(0,T;L^1(\Omega;\mathbb{S}^N))$. As a result, by the properties of the strong convergence in variable spaces, we obtain

$$\begin{aligned}\int_0^T \int_\Omega \mathbf{e}(\mathbf{u}_n)\cdot\xi(x)\phi(t)\,dxdt &= \int_0^T \int_\Omega \mathbf{e}(\mathbf{u}_n)\cdot\big(\xi(x)\phi(t)\zeta_n^{-1}\big)\,\zeta_n\,dxdt\\ &\overset{\text{by (15), (28), and (29)}}{\longrightarrow} \int_0^T \int_\Omega \mathbf{v}\cdot\big(\xi(x)\phi(t)\zeta^{-1}\big)\,\zeta\,dxdt\\ &= \int_0^T \int_\Omega \mathbf{v}\cdot\xi(x)\phi(t)\,dxdt \quad \forall\,\psi \in C_0^\infty(0,T),\ \forall\,\xi \in C_0^\infty(\Omega;\mathbb{S}^N).\end{aligned}$$

Thus, in view of the weak compactness property of the sequence $\{\mathbf{e}(\mathbf{u}_n)\}_{n\in\mathbb{N}}$ in $L^1(0,T;L^1(\Omega;\mathbb{S}^N))$, we conclude

$$\mathbf{e}(\mathbf{u}_n) \rightharpoonup \mathbf{v} \text{ in } L^1(0,T;L^1(\Omega;\mathbb{S}^N)) \quad \text{as } n\to\infty. \tag{30}$$

Since $\mathbf{u}_n \in L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$ for all $n \in \mathbb{N}$ and the space $L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$ is complete, the conditions (27) and (30) imply $\mathbf{e}(\mathbf{u}) = \mathbf{v}$, and consequently $\mathbf{u} \in L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))$. To end the proof, it remains to observe that the conditions

(27)–(28) guarantee the finiteness of the norm $\|\mathbf{u}\|_\zeta$ (see (7)). Hence $\mathbf{u} \in W_\zeta(\Omega \times (0,T);S)$ and this concludes the proof.

As an obvious consequence of this lemma, we have the following result.

Corollary 1. *The main statement of Lemma 1 remains true if we replace the condition (ii) by the following one: there exists an element $\zeta \in L^1(\Omega_T)$ such that*

$$\zeta_n \to \zeta \text{ in } L^1(\Omega_T), \text{ and } \zeta_n^{-1} \to \zeta^{-1} \text{ in } L^2(\Omega_T, d\zeta_n).$$

4 The Model of Quasistatic Evolution of Damage in an Elastic Material

In this section we describe the model for the control process in an elastic body, present its variational formulation, and discuss the questions on existence and uniqueness of weak solution.

We consider an elastic body which occupies the domain Ω. We assume that the body is clamped on the surface S and so the displacement field vanishes there. We suppose that the remaining part of the boundary $\Gamma = \partial\Omega \setminus S$ is the influence zone of a Neumann control. So, the control variable is the density of surface traction $\mathbf{p}$ acting on Γ. Let $\mathbf{f}$ be a given density of volume forces acting in $\Omega_T = (0,T) \times \Omega$ for some $T > 0$.

For a simplicity, we assume that an initial displacement (when $t = 0$) and an initial stress tensor are equal to zero. Then, having assumed that the Hooke law $\sigma_{ij} = a_{ijkl}\mathbf{e}_{kl}$, $\forall i,j = 1,\dots,N$ holds true for the elastic body Ω, we adopt the following relation for the stress $\sigma : \Omega_T \to \mathbb{S}^N$ in the body with damage (see [4, 9] for the details):

$$\sigma(t,x) = \zeta(t,x)A\mathbf{e}(\mathbf{u}(t,x)) \quad \text{a.e. in } \Omega_T, \tag{31}$$

where $\zeta = \zeta(t,x)$ is a damage field in Ω_T.

Following the motivation in Kuttler [7], the evolution of the microscopic cracks and cavities responsible for the damage can be described by the equation

$$\zeta' - \kappa\Delta\zeta = \phi(x, \mathbf{e}(\mathbf{u}), \zeta).$$

Here the prime denotes the time derivative, Δ is the Laplace operator, $\kappa > 0$ is a damage diffusion constant, ϕ is the damage source function. Usually, it is assumed that the damage source term $\phi : \Omega \times \mathbb{S}^N \times \mathbb{R}$ satisfies some Lipschitz continuity property and is such that whenever $\zeta > 1$, $\phi(\mathbf{e}(\mathbf{u}), \zeta) \le 0$. This assumption makes sense because there should be no way that the source term for the damage produces damage greater than 1.

Let $\zeta_{ad} : \Omega \to [0,1]$ be a given $L^1(\Omega)$-function satisfying the properties

$$\zeta_{ad}^{-1} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \notin L^\infty(\Omega).$$

Let Ψ_* be a nonempty compact subset of $L^1(\Omega)$ such that the conditions

$$\zeta_{ad} \le \zeta \le 1 \text{ a.e. in } \Omega, \tag{32}$$

$$\zeta : \Omega \to [0,1] \text{ is smooth function on the surface } \Gamma, \tag{33}$$

$$\zeta = 1 \quad \text{on} \quad \Gamma. \tag{34}$$

hold true for every $\zeta \in \Psi_*$. So, each element $\zeta : \Omega \to [0,1]$ of Ψ_* can be interpreted as a substantial time-independent damage field in the sense of Definition 1.

The characteristic feature of this set is the following property.

Proposition 1. *Let $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ and ζ_* be such that $\zeta_{*,n} \to \zeta_*$ in $L^1(\Omega_T)$ as $n \to \infty$, and $\{\zeta_{*,n}(t,\cdot)\}_{n\in\mathbb{N}} \subset \Psi_*$ and $\zeta_*(t,\cdot) \in \Psi_*$ for all $t \in [0,T]$. Then*

$$\zeta_{*,n}^{-1} \to \zeta_*^{-1} \text{ in } L^1(\Omega_T), \text{ and } \zeta_{*,n}^{-1} \to \zeta_*^{-1} \text{ in } L^2(\Omega_T, d\zeta_{*,n}). \tag{35}$$

Proof. In view of the initial assumptions, we may assume that $\zeta_{*,n}^{-1} \to \zeta_*^{-1}$ almost everywhere in Ω_T. Since $\zeta_{*,n} \to \zeta_*$ in $L^1(\Omega_T)$ and $\zeta_*^{-1} \le \zeta_{ad}^{-1} \in L^1(\Omega)$, it follows that the sequence $\left\{\zeta_{*,n}^{-1}\right\}_{n\in\mathbb{N}}$ is equi-integrable on Ω_T. Hence the property $(35)_1$ is a direct consequence of Lebesgue's Theorem. As for the property $(35)_2$, it was proved in Lemma 1. The proof is complete.

As a result, we adopt the following model for the controlled process in Ω: for a given body force $\mathbf{f} \in L^2\left(0,T;L^2(\Omega)^N\right)$, a surface traction $\mathbf{p} \in \mathscr{P}_{ad}$, the set Ψ_*, and an initial damage field $\zeta_0 \in L^2(\Omega)$ for which

$$\exists\, \zeta_*^0 \in \Psi_* \text{ such that } \zeta_*^0 \le \zeta_0 \le 1 \quad \text{a.e. in } \Omega, \tag{36}$$

a displacement field $\mathbf{u} : \Omega_T \to \mathbb{R}^N$, a stress field $\sigma : \Omega_T \to \mathbb{S}^N$, and a damage field $\zeta : \Omega_T \to \mathbb{R}$ satisfy the relations

$$-\operatorname{div}\sigma = \mathbf{f} \quad \text{in} \quad \Omega_T, \tag{37}$$

$$\sigma = \zeta A \mathbf{e}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{38}$$

$$\sigma = 0 \quad \text{on} \quad (0,T)\times S, \tag{39}$$

$$\sigma\nu = \mathbf{p} \quad \text{on} \quad (0,T)\times\Gamma,\ \mathbf{p} \in \mathscr{P}_{ad}, \tag{40}$$

$$\zeta' - \kappa\Delta\zeta = \phi(\mathbf{e}(\mathbf{u}),\zeta) \quad \text{in} \quad \Omega_T, \tag{41}$$

$$\zeta(0,\cdot) = \zeta_0 \quad \text{in} \quad \Omega, \tag{42}$$

$$\zeta = 1 \quad \text{on } (0,T)\times\Gamma, \quad \partial\zeta/\partial n = 0 \quad \text{on } (0,T)\times S, \tag{43}$$

$$\exists\, \zeta_* \in \Psi_* \text{ such that } \zeta_* \le \zeta(t,x) \le 1 \quad \text{a.e. in} \quad \Omega_T. \tag{44}$$

Here ν is the outward unit normal to Γ, $\partial/\partial n = n_i \partial/\partial x_i$, n_i denotes i^{th}-component of the unit outward normal vector to S, and $\mathscr{P}_{ad}$ is the set of admissible controls to the process (37)–(44). For simplicity, we suppose that $\mathscr{P}_{ad}$ is defined as

$$\mathscr{P}_{ad} = \left\{ \mathbf{p} \in L^2\left(0,T;L^2(\Gamma)^N\right) \ :\ \|\mathbf{p}\|_{L^2\left(0,T;L^2(\Gamma)^N\right)} \le C_{\mathbf{p}} \right\}. \tag{45}$$

To begin with, we note that, to the best knowledge of the authors, the existence of a global weak solution to the initial-boundary value problem (37)–(44) in an open question. There are several reasons for this. First, this problem is restricted by the state constraints (44). It means that without the implication of the truncation operators in the model, the initial conditions (42) with properties (36) and parabolic equation (41) with boundary conditions (43), do not guarantee the fulfilment of the inequality (44). Secondly, even if a damage field is admissible, i.e. ζ remains between some $\zeta_* \in \Psi_*$ and 1, the properties (32)–(34), and (44) imply that the original problem (37)–(40) is a mixed boundary value problem for the degenerate elasticity system

$$-\text{div}\,(\zeta A\mathbf{e}(\mathbf{u})) = \mathbf{f} \quad \text{in} \quad \Omega_T,$$

This means that for some damage field $\zeta(t,x)$ this problem can exhibit non-uniqueness of weak solutions [11], the Lavrentieff phenomenon, and other surprising consequences.

In view of this, we adopt the following concept:

Definition 3. We say that a vector-valued function $\mathbf{u} = \mathbf{u}(\mathbf{p},\mathbf{f},\zeta)$ is a weak solution to the boundary value problem (37)–(40) for a fixed control $\mathbf{p} \in \mathscr{P}_{ad}$, a given body force $\mathbf{f} \in L^2\big(0,T;L^2(\Omega)^N\big)$, and a given damage field $\zeta : \Omega_T \to [0,1]$ satisfying the condition (44), if $\mathbf{u} \in W_\zeta(\Omega\times(0,T);S)$ and the integral identity

$$\int_0^T \int_\Omega \big[\zeta(t,x)A(x)\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\varphi)\big]\,\psi\,dxdt = \int_0^T \int_\Omega \mathbf{f}\cdot\varphi\psi\,dxdt + \int_0^T \int_\Gamma \mathbf{p}\cdot\varphi\psi\,d\mathscr{H}^{N-1}dt \quad (46)$$

holds for any $\varphi \in C_0^\infty(\mathbb{R}^N;S)^N$ and $\psi \in C_0^\infty(0,T)$.

As was mentioned in Section 2, the set of smooth functions (8) is not dense in the weighted space $W_\zeta(\Omega\times(0,T);S)$. Hence, we can not assert that a weak solution to the degenerate elasticity problem (37)–(40) is unique. Further, we make use the following result:

Proposition 2. *Let Γ be a Lipschitz continuous part of the boundary $\partial\Omega$. Let $\zeta : \Omega_T \to [0,1]$ be a damage field satisfying the estimate* (44). *Then there exists a bounded linear operator*

$$\gamma_\Gamma : W_\zeta(\Omega\times(0,T);S) \to L^2\big(0,T;H^{1/2}(\Gamma)^N\big) \quad (47)$$

such that

(i) $\gamma_\Gamma(\mathbf{u}) = \mathbf{u}|_\Gamma$ *if* $\mathbf{u} \in W_\zeta(\Omega\times(0,T);S)\cap C\big([0,T];C(\overline{\Omega})^N\big)$,

(ii) $\|\gamma_\Gamma(\mathbf{u})\|_{L^2(0,T;H^{1/2}(\Gamma)^N)} \le C\|\mathbf{u}\|_{W_\zeta(\Omega\times(0,T);S)}$ *for each vector-valued function* $\mathbf{u} \in W_\zeta(\Omega\times(0,T);S)$ *with the constant C independent of Γ.*

Corollary 2. *Under the assumptions of Proposition 2, the space $W_\zeta(\Omega\times(0,T);S)$ does not contain rigid displacements. In other words, if $\widehat{\mathbf{u}} \neq 0$ is a vector-valued* *function for which there exists a sequence of smooth functions* $\left\{\varphi \in C_0^\infty\left(0,T;C_0^\infty(\mathbb{R}^N)^N\right)\right\}_{n\in\mathbb{N}}$ *such that*

$$\varphi_n \to \widehat{\mathbf{u}} \ \text{ in } \ L^2(\Omega_T)^N, \quad \mathbf{e}(\varphi_n) \to \mathbf{0} \ \text{ in } \ L^2\left(0,T;L^2(\Omega,\zeta\,dx)^{\frac{N(N+1)}{2}}\right),$$

then $\widehat{\mathbf{u}} \notin W_\zeta(\Omega\times(0,T);S)$.

We now give the variational formulation of the initial boundary value problem (41)–(43).

Definition 4. Let $\mathbf{p} \in \mathscr{P}_{ad}$, $\mathbf{f} \in L^2\left(0,T;L^2(\Omega)^N\right)$, and $\zeta_0 \in L^2(\Omega)$ be given functions. We say that a pair $(\zeta,\mathbf{u}) \in \mathscr{Z}\times W_\zeta(\Omega\times(0,T);S)$ is a corresponding weak variational solution to the initial-boundary value problem (37)–(44) with a nonlinear source for the damage $\phi : L^1(0,T;\mathscr{W}^{1,1}(\Omega;S))\times\mathscr{Z}\to L^2(\Omega_T)$, if

$$\frac{\partial\zeta}{\partial t} \in \mathscr{Z}', \quad \zeta - 1 \in \mathscr{V}, \tag{48}$$

and there is an element $\zeta_* \in \Psi_*$ such that the following relations hold true

$$\begin{aligned}\int_0^T\int_\Omega \left[\zeta(t,x)A(x)\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\varphi)\right]\psi\,dxdt = \int_0^T\int_\Omega \mathbf{f}\cdot\varphi\psi\,dxdt\\ + \int_0^T\int_\Gamma \mathbf{p}\cdot\varphi\psi\,d\mathscr{H}^{N-1}dt \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N;S)^N, \quad \forall\,\psi \in C_0^\infty(0,T),\end{aligned} \tag{49}$$

$$\begin{aligned}\left\langle\zeta',\varphi\psi\right\rangle_{\mathscr{Z}',\mathscr{Z}} + \kappa\int_0^T\int_\Omega \nabla\zeta\cdot\nabla\varphi\,\psi\,dxdt\\ = \int_0^T\int_\Omega \phi(\zeta,\mathbf{e}(\mathbf{u}))\varphi\psi\,dxdt \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma), \quad \forall\,\psi \in C_0^\infty(0,T),\end{aligned} \tag{50}$$

$$\zeta(0,\cdot) = \zeta_0(\cdot) \quad \text{in} \quad \Omega, \tag{51}$$

$$\zeta_* \le \zeta(t,x) \le 1 \quad \text{for all } \ t\in[0,T] \ \text{ and a.e. } \ x\in\Omega. \tag{52}$$

Remark 1. As follows from Theorem 1, the condition (52) is reasonable. It means that the initial damage field $\zeta_0 \in L^2(\Omega)$ must also be restricted by this inequality.

Remark 2. It is worth to notice that the original initial-boundary value problem (37)–(44) is ill-possed, in general. This means that there are no reasons to suppose that for every admissible initial data $\mathbf{p} \in \mathscr{P}_{ad}$, $\mathbf{f} \in L^2\left(0,T;L^2(\Omega)^N\right)$, $\zeta_0 \in L^2(\Omega)$, and $\zeta_* \in \Psi_*$ this system admits at least one weak variational solution $(\zeta,\mathbf{u}) \in \mathscr{Z}\times W_\zeta(\Omega\times(0,T);S)$ in the sense of Definition (4). At the same time, by analogy with [5, 6] it can be shown that this system may have an infinitely many weak solutions $(\zeta,\mathbf{u})$ for some fixed admissible control $\mathbf{p} \in \mathscr{P}_{ad}$.

5 Setting of the Optimal Control Problems and Existence Theorem for Optimal Traction

The optimal control problem we consider in this paper is twofold. On the one hand we try to minimize the discrepancy between a given displacement field $\mathbf{u}_d \in L^2\big(0,T;L^2(\Omega;\mathbb{R}^N)\big)$ and the solution of the problem (37)–(44) by choosing an appropriate surface traction $\mathbf{p} \in \mathscr{P}_{ad}$. On the other hand, we wish to minimize the appearance of micro-cracks and micro-cavities as a result of the tensile or compressive stresses in the elastic body. More precisely, we are concerned with the following optimal control problem

$$\text{Minimize} \quad \Big\{ I(\mathbf{p},\mathbf{u},\zeta) = \int_0^T \int_\Omega |\mathbf{u}-\mathbf{u}_d|^2_{\mathbb{R}^N}\, dxdt + \int_0^T \int_\Omega |\zeta - 1|\, dxdt + \int_0^T \int_\Omega \|\mathbf{e}(\mathbf{u})\|^2_{\mathbb{S}^N} \zeta \, dxdt \Big\} \tag{53}$$

subject to the constraints (37)–(45).

We introduce the set of admissible solutions to the original optimal control problem as follows:

$$\begin{aligned} \Xi := \big\{ (\mathbf{p},\zeta,\mathbf{u}) \,\big|\, & \mathbf{p} \in \mathscr{P}_{ad},\ \zeta \in \mathscr{Z},\ \mathbf{u} \in W_\zeta(\Omega \times (0,T);S), \\ & (\zeta,\mathbf{u}) \text{ is a weak variational solution to (37)–(44)} \\ & \text{in the sense of Definition 4} \big\}. \end{aligned} \tag{54}$$

We say that a triplet $(\mathbf{p}^0,\zeta^0,\mathbf{u}^0) \in L^2\big(0,T;L^2(\Gamma)^N\big) \times \mathscr{Z} \times W_{\zeta^0}(\Omega \times (0,T);S)$ is optimal for problem (53), (37)–(45) if

$$(\mathbf{p}^0,\zeta^0,\mathbf{u}^0) \in \Xi \ \text{ and } \ I(\mathbf{p}^0,\zeta^0,\mathbf{u}^0) = \inf_{(\mathbf{p},\zeta,\mathbf{u})\in\Xi} I(\mathbf{p},\zeta,\mathbf{u}). \tag{55}$$

Remark 3. Note that due to the estimates (9)–(10), we have the following obvious inclusion for the set of admissible solutions

$$\Xi \subset L^2\big(0,T;L^2(\Gamma)^N\big) \times L^2(0,T;H^1(\Omega)) \times L^1\big(0,T;\mathscr{W}^{1,1}(\Omega;S)\big).$$

However, the characteristic feature of this set is the fact that for different admissible controls $\mathbf{p} \in \mathscr{P}_{ad}$ and, therefore, for different admissible damage fields $\zeta : \Omega_T \to [0,1]$ with properties prescribed above, the corresponding admissible solutions $(\mathbf{p},\zeta,\mathbf{u})$ of optimal control problem (53), (37)–(45) belong to different weighted spaces. It is a non-typical situation from the point of view of the classical optimal control theory.

Definition 5. We say that the mapping

$$\phi : L^1(0,T;\mathscr{W}^{1,1}(\Omega;S)) \times \mathscr{Z} \to L^2(\Omega_T) \tag{56}$$

possesses the property $(\mathfrak{M})$ on Ξ, if

(M$_1$) for any open bounded domain $Q \subset \mathbb{R}^N$ with a Lipschitz boundary such that $\Omega \subseteq Q$ and $S \subset \partial Q$, this mapping can be extended to the following one

$$\widetilde{\phi} : L^1(0,T;\mathscr{W}^{1,1}(Q;S)) \times L^2(0,T;H^1(Q)) \to L^2(0,T;L^2(Q))$$

which is weakly-$*$ continuous with respect to the w-convergence, i.e. for any sequence $\left\{(\mathbf{p}, \widetilde{\zeta}_n, \widetilde{\mathbf{u}}_n)\right\}_{n\in\mathbb{N}} \subset \Xi \subset L^2(0,T;H^1(Q)) \times L^1(0,T;\mathscr{W}^{1,1}(Q;S))$ such that

$$\widetilde{\mathbf{u}}_n \in W_{\widetilde{\zeta}_n}(Q \times (0,T);S) \quad \forall n \in \mathbb{N}, \tag{57}$$

$$(\widetilde{\zeta}_n, \widetilde{\mathbf{u}}_n) \stackrel{w}{\to} (\widetilde{\zeta}, \widetilde{\mathbf{u}}) \text{ as } n \to \infty \text{ in the sense of Definition 2} \tag{58}$$

(where instead of Ω we have to put Q), the equality

$$\lim_{n\to\infty} \left(\widetilde{\phi}\left(\mathbf{e}(\widetilde{\mathbf{u}}_n), \widetilde{\zeta}_n\right), \varphi\psi\right)_{L^2(0,T;L^2(Q))} = \left(\widetilde{\phi}\left(\mathbf{e}(\widetilde{\mathbf{u}}), \widetilde{\zeta}\right), \varphi\psi\right)_{L^2(0,T;L^2(Q))}$$

holds $\forall\, \varphi \in C_0^\infty(\mathbb{R}^N;\Gamma)$ and $\forall\, \psi \in C_0^\infty(0,T)$;

(M$_2$) the mapping (56) is locally bounded in the following sense: for any constants $C_1, C_2 > 0$ there is a constant $C_3 = C_3(C_1,C_2) > 0$ such that

$$\left|(\phi(\mathbf{e}(\mathbf{u}),\zeta), \zeta - 1)_{L^2(\Omega_T)}\right| \le C_3 \tag{59}$$

provided $(\mathbf{u},\zeta) \in W_\zeta(\Omega \times (0,T);S) \times \mathscr{Z}$, $\zeta - 1 \in \mathscr{V}$, and

$$(\|\mathbf{e}(\mathbf{u})\|_{L^2\left(0,T;L^2(\Omega,\zeta\,dx)^{\frac{N(N+1)}{2}}\right)} \le C_1, \quad \|\zeta\|_{L^2(\Omega_T)} \le C_2. \tag{60}$$

Remark 4. Note that for any admissible initial damage field $\zeta_0 \in L^2(\Omega)$, the verification of the regularity property $\Xi \neq \emptyset$ for the original optimal control problem (53), (37)–(45) is a non-trivial problem, in general. In the particular case, when the damage field $\zeta(t,x)$ is assumed to be strictly separated from 0, the regularity property follows from results of Kuttler & Shillor, where the solvability of a similar initial-boundary value problem with a fixed surface traction $\mathbf{p}$ is studied).

Since our prime interest in this section deals with the solvability of optimal control problem (53), (37)–(45), we begin with the study of the topological properties of the set of admissible solutions Ξ.

Definition 6. A sequence $\{(\mathbf{p}_n, \zeta_n, \mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ is called bounded if

$$\sup_{n\in\mathbb{N}} \left[\|\mathbf{p}_n\|_{L^2\left(0,T;L^2(\Gamma)^N\right)} + \|\zeta_n\|_{\mathscr{Z}} + \|\mathbf{u}_n\|_{\zeta_n}\right] < +\infty.$$

Definition 7. We say that a bounded sequence $\{(\mathbf{p}_n, \zeta_n, \mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ of admissible solutions τ-converges to a triplet $(\mathbf{p}, \zeta, \mathbf{u}) \in L^2\big(0,T;L^2(\Gamma)^N\big) \times L^2(0,T;H^1(\Omega)) \times L^1\big(0,T;\mathscr{W}^{1,1}(\Omega;S)\big)$ if

(S$_1$) $\mathbf{p}_n \rightharpoonup \mathbf{p}$ in $L^2\big(0,T;L^2(\Gamma)^N\big)$,
(S$_2$) $\zeta_n \rightharpoonup \zeta$ in $\mathscr{Z} := L^2(0,T;H^1(\Omega))$,
(S$_3$) $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $L^2\big(0,T;L^2(\Omega)^N\big)$,
(S$_4$) $\mathbf{e}(\mathbf{u}_n) \rightharpoonup \mathbf{e}(\mathbf{u})$ in the variable space $L^2\big(0,T;L^2(\Omega,\zeta_n\,dx)^{\frac{N(N+1)}{2}}\big)$.

Due to the estimates like (9)–(10), the inclusion $\mathbf{u} \in L^1\big(0,T;\mathscr{W}^{1,1}(\Omega;S)\big)$ is obvious.

Remark 5. As immediately follows from Definition 2 and Rellich-Kondrashov Theorem (see also Theorem 1), if $(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \xrightarrow{\tau} (\mathbf{p},\zeta,\mathbf{u})$ then $(\zeta_n,\mathbf{u}_n) \xrightarrow{w} (\zeta,\mathbf{u})$.

Lemma 2. *Let $\{(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ be a bounded sequence. Then there exists a triplet*

$$(\mathbf{p},\zeta,\mathbf{u}) \in L^2\big(0,T;L^2(\Gamma)^N\big) \times L^2(0,T;H^1(\Omega)) \times L^1\big(0,T;\mathscr{W}^{1,1}(\Omega;S)\big)$$

such that, up to a subsequence, $(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \xrightarrow{\tau} (\mathbf{p},\zeta,\mathbf{u})$ and $\mathbf{u} \in W_\zeta(\Omega\times(0,T);S)$.

Proof. To begin with, we note that by the compactness criterium of the weak convergence in Banach reflexive spaces, there exist a subsequence of $\{(\mathbf{p}_n,\zeta_n)\}_{n\in\mathbb{N}}$, still denoted by the same indices, and $\mathbf{p} \in L^2\big(0,T;L^2(\Gamma)^N\big)$, $\zeta \in L^2(0,T;H^1(\Omega))$ are such that the conditions (S$_1$)–(S$_2$) hold true. In order to check the rest conditions (S$_3$)–(S$_4$) of Definition 7, we make use the following observation.

Since $(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi$ for all $n\in\mathbb{N}$, it follows that there is a sequence $\{\zeta_{*,n}\}_{n\in\mathbb{N}}$ in Ψ_* such that (see Definition 4)

$$\zeta_{*,n}(x) \le \zeta_n(t,x) \le 1 \quad \text{for all } t\in[0,T] \text{ and a.e. } x\in\Omega. \tag{61}$$

Moreover, by L^1-compactness property of the set Ψ_*, there exists an element $\widehat{\zeta}_* \in \Psi_*$ such that $\zeta_{*,n} \to \widehat{\zeta}_*$ in $L^1(\Omega_T)$ as $n\to\infty$. Then Proposition 1 implies the strong convergence

$$\zeta_{*,n}^{-1} \to \widehat{\zeta}_*^{-1} \text{ in } L^1(\Omega_T). \tag{62}$$

Hence, in view of (61), we have: $\zeta_n \to \zeta$, $\zeta_n^{-1} \to \zeta^{-1}$ in $L^1(\Omega_T)$ as $n\to\infty$, and the inequality $\widehat{\zeta}_* \le \zeta \le 1$ holds a.e. in Ω_T. Thus, by Remark 5, all suppositions of Lemma 1 are fulfilled. As a result, the fulfilment of the rest conditions (S$_3$)–(S$_4$) and the inclusion $\mathbf{u} \in W_\zeta(\Omega\times(0,T);S)$ for w-limiting component of the sequence $\{(\zeta_n,\mathbf{u}_n)\}_{n\in\mathbb{N}}$, are ensured by Lemma 1. The proof is complete.

Our next step deals with the study of topological properties of the set of admissible solutions Ξ to the problem (53), (37)–(45).

Theorem 2. *Assume that $\Xi \neq \emptyset$ and the damage source term $\phi : \mathbb{S}^N \times \mathbb{R} \to \mathbb{R}$ possesses the property $(\mathfrak{M})$. Then for every force $\mathbf{f} \in L^2\big(0,T;L^2(\Omega)^N\big)$ and every initial damage field $\zeta_0 : \Omega \to [0,1]$ satisfying the condition* (44)*, the set of admissible solutions Ξ is sequentially closed with respect to the τ-convergence.*

Proof. Let $\{(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ be a bounded τ-convergent sequence of admissible solutions to the optimal control problem (53), (37)–(45). Let $(\widehat{\mathbf{p}},\widehat{\zeta},\widehat{\mathbf{u}})$ be its τ-limit. Our aim is to prove that $(\widehat{\mathbf{p}},\widehat{\zeta},\widehat{\mathbf{u}}) \in \Xi$.

By the definition of the set of admissible controls $\mathscr{P}_{ad}$, we have $\widehat{\mathbf{p}} \in \mathscr{P}_{ad}$, i.e. the limit function $\widehat{\mathbf{p}}$ is an admissible control. Closely following the proof arguments of Lemma 2, it can be shown there exists a compact in $L^1(\Omega_T)$ sequence of separating functions $\{\zeta_{*,n}\}_{n\in\mathbb{N}} \subset \Psi_*$ with properties (61)–(62). By Theorem 1 we have

$$\zeta_n \to \widehat{\zeta} \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)) \ \text{ and } \ \widehat{\zeta} \in C([0,T];L^2(\Omega)). \tag{63}$$

Hence, $\zeta_n(t,x) \to \widehat{\zeta}(t,x)$ for all $t \in [0,T]$ and a.e. $x \in \Omega$. Then passing to the limit in (61) and in the relation $\zeta_n(0,\cdot) = \zeta_0(\cdot)$, we deduce: $\widehat{\zeta}(0,\cdot) = \zeta_0(\cdot)$ in Ω, and the inequality $\widehat{\zeta}_*(x) \le \widehat{\zeta}(t,x) \le 1$ holds for all $t \in [0,T]$ and a.e. $x \in \Omega$. Thus the limit damage field $\widehat{\zeta} = \widehat{\zeta}(t,x)$ satisfies the conditions (51)–(51).

In what follows, we note that in view of the boundedness of the sequence $\{(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that the estimates (60) hold true for each pair $(\zeta_n,\mathbf{u}_n)$ with $n \in \mathbb{N}$. Hence, the (M_2)-property implies

$$\sup_{n\in\mathbb{N}} \Big| (\phi(\mathbf{e}(\mathbf{u}_n),\zeta_n),\zeta_n - 1)_{L^2(\Omega_T)} \Big| \le C_3.$$

Since the set $\big\{\varphi(x)\psi(t) \ \big|\ \forall\,\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma),\ \forall\,\psi \in C_0^\infty(0,T)\big\}$ is dense in $\mathscr{V} \subset \mathscr{X}$, by the completeness arguments and formula (5), we come to the energy identity

$$\begin{aligned}\|\zeta_n(t) - 1\|^2_{L^2(\Omega)} &+ \kappa \int_0^t \|\nabla(\zeta_n(s) - 1)\|^2_{L^2;\mathbb{R}^N)} ds \\ &= \|\zeta_0 - 1\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \phi(\zeta_n,\mathbf{e}(\mathbf{u}_n))(\zeta_n(s) - 1)\,dxds \quad \forall t \in [0,T].\end{aligned} \tag{64}$$

As a result, following a standard technique (see, for instance, Lions [8]) it can be shown that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{W} = \Big\{\zeta \ :\ \zeta \in \mathscr{X},\ \ \frac{\partial\zeta}{\partial t} \in \mathscr{X}'\Big\}$. Thus, without lost of generality, we may suppose that for the $\mathscr{X}$-weak limiting damage field ζ the conditions (48) are valid, and

$$\zeta_n' \rightharpoonup \widehat{\zeta}' \quad \text{in } \ \mathscr{X}'. \tag{65}$$

It remains to show that the triple $(\widehat{\mathbf{p}},\widehat{\zeta},\widehat{\mathbf{u}})$ is related by the integral identities (49)–(50) for all $\varphi \in C_0^\infty(\mathbb{R}^N;S)^N$, $\psi \in C_0^\infty(0,T)$, and $\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma)$. To do so, we note that for every $n \in \mathbb{N}$ the integral identities (50)–(51) (with $\mathbf{p}_n$, ζ_n, and $\mathbf{u}_n$ instead of $\mathbf{p}$, ζ, and $\mathbf{u}$, respectively), have to fulfil for the test functions $\varphi \in C_0^\infty(\mathbb{R}^N;S)^N$ and $\varphi \in$

$C_0^\infty(\mathbb{R}^N;\Gamma)$. In this case $\mathbf{e}(\varphi)\in C_0^\infty\big(\mathbb{R}^N;S\big)^{\frac{N(N+1)}{2}}$ and $\xi\varphi\in C_0^\infty(\mathbb{R}^N;\Gamma)^{\frac{N(N+1)}{2}}$ for any $\xi\in\mathbb{S}^N$. However, these classes are essentially wider than the space $C_0^\infty(\Omega)^{\frac{N(N+1)}{2}}$ in the definition of the weak convergence in variable space $L^2(\Omega,\zeta_n\,dx)^{\frac{N(N+1)}{2}}$ (see (24)). Therefore, in order to pass to the limit in that integral identities as $n\to\infty$, we make use the following trick (see Buttazzo & Kogut [2]).

Let $(\widetilde{\zeta}_n,\widetilde{\mathbf{u}}_n)\in L^2(0,T;H^1_{loc}(\mathbb{R}^N))\times L^1\big(0,T;\mathscr{W}^{1,1}_{loc}(\mathbb{R}^N;S)\big)$ be an extension of the functions $(\zeta_n,\mathbf{u}_n)$ to the whole of space $\mathbb{R}^N$ such that the sequence $\{(\widetilde{\zeta}_n,\widetilde{\mathbf{u}}_n)\}_{n\in\mathbb{N}}$ satisfies the properties:

$$\widetilde{\zeta}_n\in L^2(0,T;H^1(Q)),\quad \widetilde{\zeta}_n'\in L^2(0,T;(H^1(Q))') \tag{66}$$

$$\xi_*\le\widetilde{\zeta}_n\le 1 \text{ a.e. in } Q_T:=(0,T)\times Q, \tag{67}$$

$$\sup_{n\in\mathbb{N}}\Big[\|\widetilde{\zeta}_n\|_{L^2(0,T;H^1(Q))}+\|\widetilde{\mathbf{u}}_n\|_{L^2(0,T;L^2(Q)^N)} + \|\mathbf{e}(\widetilde{\mathbf{u}}_n)\|_{L^2\left(0,T;L^2(Q,\widetilde{\zeta}_n\,dx)^{\frac{N(N+1)}{2}}\right)}\Big]<+\infty \tag{68}$$

for any bounded domain Q in $\mathbb{R}^N$. Here $\xi_*\in L^1(Q_T)$ is a non negative function such that $\xi_*^{-1}\in L^1(Q_T)$ and $\xi_*|_{\Omega_T}\in\Psi_*$.

Then by analogy with Lemma 2 (see also the property (63)) it can be proved that for every bounded domain $Q\subset\mathbb{R}^N$ there exist functions $\widetilde{\zeta}\in L^2(0,T;H^1(Q))$ and $\widetilde{\mathbf{u}}\in W_{\widetilde{\zeta}}(Q\times(0,T);S)$ such that

$$\widetilde{\zeta}_n\rightharpoonup\widetilde{\zeta} \text{ in } L^2(0,T;H^1(Q)),\qquad \widetilde{\mathbf{u}}_n\rightharpoonup\widetilde{\mathbf{u}} \text{ in } L^2\big(0,T;L^2(Q)^N\big), \tag{69}$$

$$\zeta_n\to\widehat{\zeta} \text{ strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^N)), \tag{70}$$

$$\mathbf{e}(\widetilde{\mathbf{u}}_n)\rightharpoonup\mathbf{e}(\widetilde{\mathbf{u}})\in L^2\big(0,T;L^2(Q,\widetilde{\zeta}\,dx)^{\frac{N(N+1)}{2}}\big) \tag{71}$$

in the variable space $L^2\big(0,T;L^2(Q,\widetilde{\zeta}_n\,dx)^{\frac{N(N+1)}{2}}\big)$.

It is important to note that in this case we have

$$\widetilde{\mathbf{u}}=\widehat{\mathbf{u}} \text{ and } \widetilde{\zeta}=\widehat{\zeta}\quad \text{a.e. in } \Omega_T. \tag{72}$$

Taking this fact and (M_1)-property of the source term ϕ into account, we can rewrite the integral identities (49)–(50) in the equivalent form

$$\int_0^T\int_{\mathbb{R}^N}\big[\widetilde{\zeta}_n(t,x)A(x)\mathbf{e}(\widetilde{\mathbf{u}}_n)\cdot\mathbf{e}(\varphi)\big]\psi\chi_\Omega(x)\,dxdt=\int_0^T\int_{\mathbb{R}^N}\mathbf{f}\cdot\varphi\psi\chi_\Omega(x)\,dxdt + \int_0^T\int_\Gamma\mathbf{p}\cdot\varphi\psi\,d\mathscr{H}^{N-1}dt\quad \forall\varphi\in C_0^\infty(\mathbb{R}^N;S)^N,\quad \forall\psi\in C_0^\infty(0,T), \tag{73}$$

$$\left\langle \zeta_n', \varphi\psi \right\rangle_{\mathscr{X}',\mathscr{X}} + \kappa \int_0^T \int_{\mathbb{R}^N} \nabla \widetilde{\zeta}_n \cdot \nabla \varphi \, \psi \chi_\Omega(x)\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} \widetilde{\phi}(\widetilde{\zeta}_n, \mathbf{e}(\widetilde{\mathbf{u}}_n)) \varphi\psi\chi_\Omega \, dxdt \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma), \quad \forall \psi \in C_0^\infty(0,T). \tag{74}$$

In what follows, we note that due to the property (70) and the continuity of the embedding $L^2(Q_T) \hookrightarrow L^1(Q_T)$ for every bounded $Q \subset \mathbb{R}^N$, we have $\widetilde{\zeta}_n \to \widetilde{\zeta}$ strongly in $L^1(0,T; L^1_{loc}(\mathbb{R}^N))$. Hence

$$\int_0^T \int_{\mathbb{R}^N} \chi_\Omega^2 \widetilde{\zeta}_n \, dxdt = \int_0^T \int_{\mathbb{R}^N} \chi_\Omega \widetilde{\zeta}_n \, dxdt$$
$$\longrightarrow \int_0^T \int_{\mathbb{R}^N} \chi_\Omega \widetilde{\zeta} \, dxdt = \int_0^T \int_{\mathbb{R}^N} \chi_\Omega^2 \widetilde{\zeta} \, dxdt. \tag{75}$$

As follows from convergence properties (15) and (17), the equality (75) implies the strong convergence $\chi_\Omega \to \chi_\Omega$ in the variable space $L^2(0,T; L^2(\mathbb{R}^N, \widetilde{\zeta}_n\, dx))$. Taking this fact, properties (65), (69), (71), (M_1), and Remark 5 into account, we can pass to the limit in (73)–(74) as $n \to \infty$. As a result, we obtain

$$\int_0^T \int_{\mathbb{R}^N} \left[\widetilde{\zeta}(t,x) A(x) \mathbf{e}(\widetilde{\mathbf{u}}) \cdot \mathbf{e}(\varphi) \right] \psi \chi_\Omega(x)\, dxdt = \int_0^T \int_{\mathbb{R}^N} \mathbf{f} \cdot \varphi\psi\chi_\Omega(x)\, dxdt$$
$$+ \int_0^T \int_\Gamma \mathbf{p} \cdot \varphi\psi \, d\mathscr{H}^{N-1} dt \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; S)^N, \quad \forall \psi \in C_0^\infty(0,T),$$

$$\left\langle \widehat{\zeta}', \varphi\psi \right\rangle_{\mathscr{X}',\mathscr{X}} + \kappa \int_0^T \int_{\mathbb{R}^N} \nabla \widetilde{\zeta} \cdot \nabla \varphi \, \psi \chi_\Omega(x)\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} \widetilde{\phi}(\widetilde{\zeta}, \mathbf{e}(\widetilde{\mathbf{u}})) \varphi\psi\chi_\Omega \, dxdt \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma), \quad \forall \psi \in C_0^\infty(0,T)$$

which, due to the equalities (72), are equivalent to

$$\int_0^T \int_\Omega \left[\widehat{\zeta}(t,x) A(x) \mathbf{e}(\widehat{\mathbf{u}}) \cdot \mathbf{e}(\varphi) \right] \psi \, dxdt = \int_0^T \int_\Omega \mathbf{f} \cdot \varphi\psi \, dxdt$$
$$+ \int_0^T \int_\Gamma \mathbf{p} \cdot \varphi\psi \, d\mathscr{H}^{N-1} dt \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; S)^N, \quad \forall \psi \in C_0^\infty(0,T),$$

$$\left\langle \widehat{\zeta}', \varphi\psi \right\rangle_{\mathscr{X}',\mathscr{X}} + \kappa \int_0^T \int_\Omega \nabla \widehat{\zeta} \cdot \nabla \varphi \, \psi \, dxdt$$
$$= \int_0^T \int_\Omega \phi(\widehat{\zeta}, \mathbf{e}(\widehat{\mathbf{u}})) \varphi\psi \, dxdt \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma), \quad \forall \psi \in C_0^\infty(0,T).$$

Hence, the pair $(\widehat{\zeta},\widehat{\mathbf{u}}) \in \mathscr{X} \times W_{\widehat{\zeta}}(\Omega\times(0,T);S)$ is a weak solution to the initial-boundary value problem (37)–(44) under $\mathbf{p} = \widehat{\mathbf{p}}$ in the sense of Definition 4. Thus, the τ-limit triplet $(\widehat{\mathbf{p}},\widehat{\zeta},\widehat{\mathbf{u}})$ belongs to set Ξ, and this concludes the proof.

We are now in a position to state the existence of weak optimal solution to the problem (53), (37)–(45).

Theorem 3. *Let* $\mathbf{u}_d \in L^2\big(0,T;L^2(\Omega;\mathbb{R}^N)\big)$, $\mathbf{f} \in L^2\big(0,T;L^2(\Omega)^N\big)$, *and* $\zeta_0 \in L^2(\Omega)$ *be given functions. Assume that* $\Xi \neq \emptyset$, *the damage source term* $\phi : \mathbb{S}^N \times \mathbb{R} \to \mathbb{R}$ *possesses the property* $(\mathfrak{M})$, *and the initial damage field* $\zeta_0 : \Omega \to [0,1]$ *satisfies the condition* (44). *Then the optimal control problem* (53), (37)–(45) *admits at least one solution* $(\mathbf{p}^0,\zeta^0,\mathbf{u}^0) \in L^2(0,T;H^1(\Omega)) \times \mathscr{W} \times W_{\zeta^0}(\Omega\times(0,T);S)$.

Proof. Since the cost functional $I = I(\mathbf{p},\mathbf{u},\zeta)$ is bounded below and $\Xi \neq \emptyset$, it provides the existence of a minimizing sequence $\{(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ to the problem (53). From the inequality

$$\inf_{(\mathbf{p},\zeta,\mathbf{u})\in\Xi} I(\mathbf{p},\zeta,\mathbf{u}) = \lim_{n\to\infty}\Big[\int_0^T\int_\Omega |\mathbf{u}_n - \mathbf{u}_d|^2_{\mathbb{R}^N}\,dxdt + \int_0^T\int_\Omega |\zeta_n - 1|\,dxdt + \int_0^T\int_\Omega \|\mathbf{e}(\mathbf{u}_n)\|^2_{\mathbb{S}^N}\zeta\,dxdt\Big] < +\infty, \quad (76)$$

there is a constant $C>0$ such that

$$\sup_{n\in\mathbb{N}} \|\mathbf{e}(\mathbf{u}_n)\|_{L^2\big(0,T;L^2(\Omega,\zeta_n dx)^{\frac{N(N+1)}{2}}\big)} \le C, \quad (77)$$

$$\sup_{n\in\mathbb{N}} \|\mathbf{u}_n\|_{L^2(0,T;L^2(\Omega)^N)} \le C, \quad \sup_{n\in\mathbb{N}} \|\zeta_n\|_{L^1(0,T;L^1(\Omega))} \le C. \quad (78)$$

Since the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is restricted by inequalities (61), the estimate $(78)_2$ implies

$$\sup_{n\in\mathbb{N}} \|\zeta_n\|^2_{L^2(0,T;L^2(\Omega))} \le \sup_{n\in\mathbb{N}} \|\zeta_n\|_{L^1(0,T;L^1(\Omega))} \le C. \quad (79)$$

Then, by energy equality (64) and (M_2)-property of the source term ϕ, we arrive at the estimate

$$\begin{aligned}\kappa\|\nabla\zeta_n\|^2_{L^2(0,T;L^2(\Omega)^N)} &\le 2\kappa\int_0^T \|\nabla(\zeta_n - 1)\|^2_{L^2;\mathbb{R}^N)}dt + 2\kappa T|\Omega| \\ &= 2\|\zeta_0 - 1\|^2_{L^2(\Omega)} + 2\int_0^T\int_\Omega \phi(\zeta_n,\mathbf{e}(\mathbf{u}_n))(\zeta_n - 1)\,dxdt + 2\kappa T|\Omega| \\ &\qquad \text{(by (77), (79), and property } (M_2)) \\ &\le 2\|\zeta_0 - 1\|^2_{L^2(\Omega)} + 2C_3 + 2\kappa T|\Omega| < +\infty.\end{aligned}$$

Hence, $\sup_{n\in\mathbb{N}}\|\zeta_n\|_{\mathscr{X}} < +\infty$, and in view of the definition of the class of admissible controls $\mathscr{R}_{ad}$, the minimizing sequence $\{(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \in \Xi\}_{n\in\mathbb{N}}$ is bounded in the sense of Definition 6. Hence, by Lemma 2 there exist functions $\mathbf{p}^0 \in$

$L^2\big(0,T;L^2(\Gamma)^N\big)$, $\zeta^0 \in L^2(0,T;H^1(\Omega))$, and $\mathbf{u}^0 \in W_{\zeta^0}(\Omega\times(0,T);S)$ such that, up to a subsequence, $(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) \xrightarrow{\tau} (\mathbf{p}^0,\zeta^0,\mathbf{u}^0)$. Moreover, by Theorem 1 we have $\zeta_n \to \zeta^0$ strongly in $L^2(0,T;L^2(\Omega))$. Hence

$$\zeta_n \to \widehat{\zeta} \ \text{ strongly in } \ L^1(0,T;L^1(\Omega)). \tag{80}$$

Since the set Ξ is sequentially closed with respect to the τ-convergence (see Theorem 2), it follows that the τ-limit triplet $(\mathbf{p}^0,\zeta^0,\mathbf{u}^0)$ is an admissible solution to the optimal control problem (53), (37)–(45) (i.e. $(\mathbf{p}^0,\zeta^0,\mathbf{u}^0)\in\Xi$). To conclude the proof it is enough to observe that by properties (16) and (80), the cost functional I is sequentially lower τ-semicontinuous. Thus

$$I(\mathbf{p}^0,\zeta^0,\mathbf{u}^0) \le \liminf_{n\to\infty} I(\mathbf{p}_n,\zeta_n,\mathbf{u}_n) = \inf_{(\mathbf{p},\zeta,\mathbf{u})\in\Xi} I(\mathbf{p},\zeta,\mathbf{u}).$$

Hence $(\mathbf{p}^0,\zeta^0,\mathbf{u}^0)$ is an optimal solution, and we come to the required conclusion.

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