# EFFICIENT CONTROLS FOR TRAFFIC FLOW ON NETWORKS

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ABSTRACT. We study traffic flow models for road networks in vector-valued optimization statement where the flow is controlled at the nodes of network. We consider the case when an objective mapping possesses a weakened property of upper semicontinuity and make no assumptions on the interior of the ordering cone. We derive sufficient conditions for the existence of efficient controls of the traffic problem and discuss the scalarization approach to its solution. We also prove the existence of the so-called generalized efficient controls.

## 1. Introduction

The main goal of this paper is to discuss macroscopic traffic flow models on road networks in the framework of the vector optimization statement. Modeling and simulation of traffic flow on the road networks has been investigated intensively during the last years; see for example [3, 4, 7, 10, 11, 14]. Since the support of the decision making in traffic management is a topical problem, the most investigations in this field deal with the optimal control problems of the flow in traffic networks. Typically, the main focus of such investigations is on the construction of optimal traffic parameters on the network and on the derivation of an optimality system and the evaluation of the gradient of the objective functions that appear in the corresponding optimization problems.

In this paper we focus on the approach based on Lighthill-Whitham-Richards (LWR) model. We suppose that the road networks consist of a finite set of roads, that meet at some junctions. To describe the dynamics on a network, represented by a directed topological graph, we use the hyperbolic system of conservation laws in one dimension and suppose that the flow is controlled at the nodes of network. Our prime interest is to consider the traffic optimization problems in new statement, which involves topological properties of an objective space, and discuss the problem of their solvability. We deal with the case when the objective mapping possesses a weakened property of lower semicontinuity and takes values in the space $L^2(\Omega)$, closely related with the geometry of road network and partially ordered by a cone $\Lambda$ of positive elements. We prove the existence of the so-called efficient controls and generalized solutions to the corresponding vector optimization problem on a network and study their main properties.

## 2. Notation and Preliminaries

In this section we recall some known basic notions about functions with bounded variation, networks, and introduce a few notation concerning vector-valued mappings and partially ordered functional spaces.

*Functions with Bounded Variation.* Let $J = (a, b)$ $(a < b)$ be a given interval in $\mathbb{R}$. Consider a function $f : J \to \mathbb{R}$ such that $f \in L^1(J)$. The total variation of $f$ on $J$ is

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defined as

$$\operatorname{Tot} V_J(f) = \sup\left\{\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| \ : \ m \in \mathbb{N}, \ a < x_0 < x_1 < \cdots < x_m < b\right\},$$

where $x_j$ are the points of approximal continuity of $f$ (see [6]).

**Definition 2.1.** We say that $f \in L^1(J)$ is a function of bounded variation on $J$ if there exists a constant $K > 0$ such that $\operatorname{Tot} V_J(f) \leq K$. We denote with $BV(J)$ the set of all real functions $f \in L^1(J)$ with bounded variation on $J$.

Note that the total variation of a function $f$ is a positive number. If $f \in BV(J)$, then it is clear that $f : J \to \mathbb{R}$ is bounded almost everywhere on $J$. The following statements are equivalent (see [8]):

**(i):** $f \in BV(J)$;

**(ii):** $f \in L^1(J)$ and

$$|Df|(J) := \sup\left\{\int_J f\, \varphi'\, dx \ : \ \varphi \in C_0^1(J), \ |\varphi| \leq 1\right\} < +\infty;$$

**(iii):** there exists a sequence of smooth functions $\{f_k\}_{k=1}^{\infty} \subset C_0^{\infty}(\mathbb{R})$ such that $f_k \to f$ strongly in $L^1(J)$ and $\limsup_{k\to\infty} \int_J |f_k'|\, dx < +\infty$,

where the distributional derivative $Df$ is a Radon measure and $|Df|(J)$ coincides with the total variation of $f$ on $J$. Moreover, for any $f \in BV(J)$ the right-hand side and left-hand side limits

$$f(x^+) = \lim_{h\to 0+} \frac{1}{h}\int_x^{x+h} f(s)\, ds, \quad f(x^-) = \lim_{h\to 0+} \frac{1}{h}\int_{x-h}^{x} f(s)\, ds$$

exist at all $x \in [a, b)$, and $x \in (a, b]$, respectively. Finally, $f(x^+) = f(x^-)$ if $|Df|(\{x\}) = 0$.

The following theorem holds.

**Theorem 2.1.** *(a) The space $BV(J)$ is a Banach space with respect to the norm*

$$\|f\|_{BV(J)} = \|f\|_{L^1(J)} + |Df|(J);$$

*(b) the mapping $f \to |Df|(J)$ is lower semicontinuous with respect to the $L^1(J)$ convergence, that is, if $f_k \to f$ in $L^1(J)$ then $|Df|(J) \leq \liminf_{k\to\infty} |Df_k|(J)$;*
*(c) if $\{f_k\}_{k=1}^{\infty} \subset BV(J)$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(J)} < +\infty$ then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ strongly converging in $L^1(J)$ to some $f \in BV(J)$.*

In what follows, we say that a sequence $\{f_k\}_{k=1}^{\infty} \subset BV(J)$ converges weakly in $BV(J)$ and we write $f_k \rightharpoonup f$ in $BV(J)$ if $f_k \to f$ strongly in $L^1(J)$ and $\sup_{k\in\mathbb{N}} |Df_k|(J) < +\infty$. Note that if $f_k \rightharpoonup f$ in $BV(J)$ then $f \in BV(J)$ and $Df_k \rightharpoonup Df$ as Radon measures.

*Networks of roads.* Let $\mathcal{O}$ be an open convex subset of $\mathbb{R}^2$ and let $\mathfrak{F}$ be a planar graph on $\mathbb{R}^2$.

**Definition 2.2.** We say that the set $\Omega = \mathcal{O} \cap \mathfrak{F}$ is a network of roads enclosed by the region $\Omega$ if it can be represented as a couple $(\mathcal{I}, \mathcal{J})$ where

**(a):** $\mathcal{I}$ is a finite collection of edges, which correspond to roads in the network and are parameterized by intervals $I_i = [a_i, b_i]$ in $\mathbb{R}$ with $i = 1, \ldots, N$;

**(b):** $\mathcal{J}$ is a finite collection of vertices, which correspond to junctions in the network. Each vertex $J$ is union of two nonempty subsets $\operatorname{Inc}(J)$ and $\operatorname{Out}(J)$ of $\{1, \ldots, N\}$,

such that:

**(i):** each vertex $J \in \mathcal{J}$ is an interior point of $\Omega$;
**(ii):** for every $J \neq J' \in \mathcal{J}$ we have $\mathrm{Inc}\,(J) \cap \mathrm{Inc}\,(J') = \emptyset$ and $\mathrm{Out}\,(J) \cap \mathrm{Out}\,(J') = \emptyset$;
**(iii):** if $i \notin \cup_{J \in \mathcal{J}} \mathrm{Inc}\,(J)$ then $b_i$ corresponds to some point on $\partial\Omega$ (an outgoing road from the network) and if $i \notin \cup_{J \in \mathcal{J}} \mathrm{Out}\,(J)$ then $a_i$ corresponds to some point on $\partial\Omega$ (an incoming road to the network).

Moreover, the two cases are mutually exclusive.

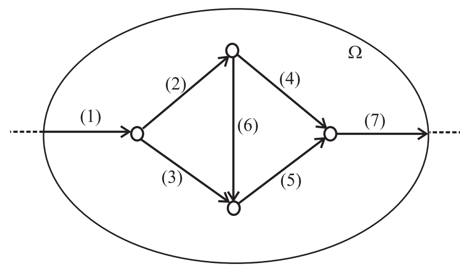


*Figure 2.1: Geometry of a sample network*

Thus a sample network might have a structure as indicated in figure 2.1.

*Optimality Notion in Partially Ordered Spaces.* Let $\Omega$ be a network. We associate with this set the objective space $L^2(\Omega)$. Hereinafter we adopt the notation $y \in L^2(\Omega)$ if and only if $y = (y_1, \dots, y_N)$ and $y_k \in L^2(I_k)$ for $k = 1, \dots, N$. By default suppose that $L^2(\Omega)$, as topological space, is endowed with the weak topology. For a subset $S \subset L^2(\Omega)$ we denote by $\mathrm{int}_w\, S$ and $\mathrm{cl}_w\, S$ its interior and closure with respect to the weak topology, respectively. We also assume that $L^2(\Omega)$ is partially ordered by the natural ordering cone of positive elements $\Lambda$, which is defined as

$$\Lambda = \left\{ f \in L^2(\Omega) \ : \ f(x) \geq 0 \text{ almost everywhere on } \ \Omega \right\}. \tag{2.1}$$

Then for elements $y, z \in L^2(\Omega)$, we will write $y \leq_\Lambda z$ whenever $z \in y + \Lambda$ and $y <_\Lambda z$ for $y, z \in L^2(\Omega)$, if $z - y \in \Lambda \setminus \{0\}$. We say that a sequence $\{y_k\}_{k=1}^\infty \subset L^2(\Omega)$ is non-increasing and we use the notation $y_k \searrow$ whenever, for all $k \in \mathbb{N}$, we have $y_{k+1} \leq_\Lambda y_k$. We also say that a sequence $\{y_k\}_{k=1}^\infty \subset L^2(\Omega)$ is bounded below if there exists an element $y^* \in L^2(\Omega)$ such that $y^* \leq_\Lambda y_k$ for all $k \in \mathbb{N}$.

For the study of “optimal”elements of a nonempty subset $S$ of the partially ordered space $L^2(\Omega)$ we are mainly interested in maximal elements of this set.

**Definition 2.3.** (see [12]) An element $y^* \in S \subset L^2(\Omega)$ is said to be maximal of the set $S$, if there is no $y \in S$ such that $y \geq_\Lambda y^*, \ y \neq y^*$, that is

$$S \cap (y^* + \Lambda) = \{y^*\}.$$

Let $\mathrm{Max}_\Lambda(S)$ denote the family of all maximal elements of $S$. Let us introduce two singular elements $-\infty_\Lambda$ and $+\infty_\Lambda$ in $L^2(\Omega)$. We assume that these elements satisfy the following conditions:

$$1) - \infty_\Lambda \preceq y \preceq +\infty_\Lambda, \ \forall y \in L^2(\Omega); \quad 2) + \infty_\Lambda + (-\infty_\Lambda) = 0.$$

Let $Y^\bullet$ denote the semi-extended Banach space: $Y^\bullet = L^2(\Omega) \cup \{-\infty_\Lambda\}$ assuming that

$$\| - \infty_\Lambda \|_{L^2(\Omega)} = +\infty \ \text{ and } \ y + \lambda(-\infty_\Lambda) = -\infty \ \forall\, y \in L^2(\Omega) \ \text{ and } \ \forall\, \lambda \in \mathbb{R}_+.$$

The following concept is a crucial point in this paper.

**Definition 2.4.** We say that a set $E$ is the efficient supremum of a set $S \subset L^2(\Omega)$ with respect to the weak topology of $L^2(\Omega)$ (or shortly $(\Lambda, w)$-supremum) if $E$ is the collection of all maximal elements of $\mathrm{cl}_w\, S$ in the case when this set is non-empty, and $E$ is equal to $\{+\infty_\Lambda\}$ in the opposite case.

Hereinafter the $(\Lambda, w)$-supremum for $S$ will be denoted by $\mathrm{Sup}^{\Lambda,w}\, S$. Thus, in view of the definition given above, we have

$$\mathrm{Sup}^{\Lambda,w}\, S := \begin{cases} \mathrm{Max}_\Lambda(\mathrm{cl}_w\, S), & \mathrm{Max}_\Lambda(\mathrm{cl}_w\, S) \neq \emptyset \\ +\infty_\Lambda, & \mathrm{Max}_\Lambda(\mathrm{cl}_w\, S) = \emptyset. \end{cases}$$

Let $X_\partial$ be a nonempty subset of a Banach space $X$, and $I : X_\partial \to L^2(\Omega)$ be some mapping. Note that the mapping $I : X_\partial \to L^2(\Omega)$ can be associated with its natural extension $\widehat{I} : X \to Y^\bullet$ to the whole space $X$, where

$$\widehat{I}(x) = \begin{cases} I(x), & x \in X_\partial, \\ -\infty_\Lambda, & x \notin X_\partial. \end{cases} \tag{2.2}$$

We say that a mapping $I : X_\partial \to Y^\bullet$ is bounded above if there exists an element $z \in L^2(\Omega)$ such that $z \geq_\Lambda I(x)$ for all $x \in X_\partial$.

**Definition 2.5.** A subset $A$ of $L^2(\Omega)$ is said to be the efficient supremum of a mapping

$$I : \ X_\partial \to L^2(\Omega)$$

with respect to the weak topology of $L^2(\Omega)$ and is denoted by $\mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x)$, if $A$ is the $(\Lambda, w)$-supremum of the image $I(X_\partial)$ of $X_\partial$ in $L^2(\Omega)$, that is,

$$\mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x) = \mathrm{Sup}^{\Lambda,w} \left\{ I(x) \ : \ x \in X_\partial \right\}.$$

*Remark* 2.1. It is clear now that if $a \in \mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x)$ then

$$\mathrm{cl}_w \ \{I(x) \ : \ \ x \in X_\partial\} \cap (a + \Lambda) = \{a\}$$

provided $\mathrm{Max}_\Lambda \left[\mathrm{cl}_w \ \{I(x) \ : \ x \in X_\partial\}\right] \neq \emptyset$.

Let $\{y_k\}_{k=1}^\infty$ be a sequence in $L^2(\Omega)$. Let $\mathrm{L}^w\{y_k\}$ denote the set of all its cluster points with respect to the weak topology of $L^2(\Omega)$, that is, $y \in \mathrm{L}^w\{y_k\}$ if there is a subsequence $\{y_{k_i}\}_{i=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $y_{k_i} \rightharpoonup y$ in $L^2(\Omega)$ as $i \to \infty$. If this set is upper unbounded, i.e., $\mathrm{Sup}^{\Lambda,w}\, \mathrm{L}^w\{y_k\} = +\infty_\Lambda$, we assume that $\{+\infty_\Lambda\} \in \mathrm{L}^w\{y_k\}$. Let $x_0 \in X_\partial$ be a fixed element. In what follows for an arbitrary mapping $I : \ X_\partial \to L^2(\Omega)$ we make use of the following sets:

$$\mathrm{L}^{\sigma\times w}(I, x_0) := \bigcup_{\{x_k\}_{k=1}^\infty \in \mathfrak{M}_\sigma(x_0)} \mathrm{L}^w\{\widehat{I}(x_k)\}, \tag{2.3}$$

$$\mathrm{L}^{\sigma\times w}_{\max}(I, x_0) := \mathrm{L}^{\sigma\times w}(I, x_0) \cap \mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x), \tag{2.4}$$

where $\mathfrak{M}_\sigma(x_0)$ is the set of all sequences $\{x_k\}_{k=1}^\infty \subset X$ such that $x_k \to x_0$ with respect to a $\sigma$-topology of $X$.

We are able to introduce the notion of the upper limit for the vector-valued mappings.

**Definition 2.6.** We say that a subset $A \subset L^2(\Omega) \cup \{\pm\infty_\Lambda\}$ is the $\Lambda$-lower sequential limit of the mapping $I : X_\partial \to L^2(\Omega)$ at the point $x_0 \in X_\partial$ with respect to the product topology $\sigma \times w$ of $X \times L^2(\Omega)$, and we use the notation $A = \limsup^{\Lambda,w}_{x \xrightarrow{\sigma} x_0} I(x)$, if

$$\limsup^{\Lambda,w}_{x \xrightarrow{\sigma} x_0} I(x) := \begin{cases} \mathrm{L}^{\sigma\times w}_{\max}(I, x_0), & \mathrm{L}^{\sigma\times w}_{\max}(I, x_0) \neq \emptyset, \\ \mathrm{Sup}^{\Lambda,w}\, \mathrm{L}^{\sigma\times w}(I, x_0), & \mathrm{L}^{\sigma\times w}_{\max}(I, x_0) = \emptyset. \end{cases} \tag{2.5}$$

*Remark* 2.2. Note that in the scalar case ($I : X_\partial \to \mathbb{R}$) the sets

$$\mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x) \quad \text{and} \quad \mathrm{Sup}^{\Lambda,w}\, \mathrm{L}^{\sigma\times w}(I, x_0)$$

are singletons. Therefore, if $\mathrm{L}^{\sigma\times w}_{\max}(I, x_0) \neq \emptyset$ then we have

$$\mathrm{L}^{\sigma\times w}_{\max}(I, x_0) = \mathrm{L}^{\sigma\times w}(I, x_0) \cap \mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x) =$$

$$= \mathrm{Sup}^{\Lambda,w}\, \mathrm{L}^{\sigma\times w}(I, x_0) \cap \mathrm{Sup}^{\Lambda,w}_{x\in X_\partial} I(x) = \mathrm{Sup}^{\Lambda,w}\, \mathrm{L}^{\sigma\times w}(I, x_0).$$

Hence the choice rules in (2.5) coincide and we come to the classical definition of the upper limit.

## 3. Continuous Model of Traffic Flow

In this section we give a brief review of a fluid dynamic model for traffic flow networks following Coclite & Piccoli [3] (see also [7]).

Let $\Omega = \mathcal{O} \cap \mathfrak{F}$ be a given network which has a total of $N$ roads. For $i \in \{1, \dots, N\}$ road $i$ is parameterized by an interval $[a_i, b_i]$. Let $\rho_i = \rho_i(t, x)$ denote the density of cars on road $i$ at the point $x \in [a_i, b_i]$ and at time $t \in [0, T]$. Further, the maximal allowable density on road $i$ describing the situation where cars stand bumper to bumper is denoted by $\rho_{\max,i}$. As usual, we assume that the roads correspond to the edges of a graph $\mathfrak{F}$ enclosed by the region $\Omega$, and the junctions where the roads are connected correspond to the nodes of this graph. The number of cars crossing per unit time is called the traffic flow $f(\rho) = \rho v(\rho)$, where $v(\rho)$ is a velocity. A reasonable property of $v$ is that $v$ is a decreasing function of the density. By analogy with [7, 11], we assume that there exists a family of flux-functions $f_i$ such that for each road $i \in \{1, \dots, N\}$

$$\text{(3.1)} \quad \begin{cases} f_i \ \text{ is a function of } \ \rho_i \ \text{ only,} \\ f_i \ \text{ is continuously differentiable on } \ [0, \rho_{\max,i}], \\ f_i(0) = f_i(\rho_{\max,i}) = 0, \\ f_i \ \text{ is strictly concave,} \\ \text{there exists } \ \sigma_i \in (0, \rho_{\max,i}) \ : \ f_i'(\sigma_i) = 0 \ \text{ and } \ (\rho - \sigma_i) f_i'(\rho) < 0, \quad \forall \rho \neq \sigma_i. \end{cases}$$

As follows from the above flux-function definition, there is no traffic flow at $\rho_i = 0$ and $\rho_i = \rho_{\max,i}$. For other values of density $0 < \rho_i < \rho_{\max,i}$ the traffic flow must be strictly positive. The value $\sigma_i$ is the optimal density at which a maximum traffic flow occurs. Moreover, these conditions imply that the positive direction of flow on each road of the network is fixed. As a result, for each $i \in \{1, \dots, N\}$, the macroscopic model for traffic flow on road $i$ can be given by the following nonlinear conservation law (the so-called LWR-equations, see [15]):

$$\text{(3.2)} \qquad \partial_t \rho_i(t, x) + \partial_x f_i\left(\rho_i(t, x)\right) = 0, \quad \forall\, x \in (a_i, b_i),\ \forall\, t \in (0, T],$$

$$\text{(3.3)} \qquad \rho_i(0, x) = \overline{\rho}_i(x), \quad \forall\, x \in [a_i, b_i],$$

with flux

$$f_i(\rho) = \rho v_i(\rho),$$

where, by assumption, the velocity $v_i$ is a continuously differential decreasing function of only the density.

*Remark* 3.1. The main feature of the nonlinear system (3.2)–(3.3) is the fact that the classical solution may not exist for some positive time, even if the initial datums are smooth. As for the initial boundary valued problem for the equation (3.2), it is ill-posed in general (which means that there may be no solution or one that does not depend in a continuous way on the initial and boundary data, or nonuniqueness). In other words, as soon as the initial condition is given, the solution cannot be prescribed arbitrary on the boundary. In view of this the boundary conditions for roads which income to or outcome from the network $\Omega$ can be given in the sense of Bardos, LeRoux, and Nedeles [1]. However, for simplicity, we suppose that $a_i = -\infty$ and $b_i = +\infty$ if $i \notin \cup_{J\in\mathcal{J}}\mathrm{Inc}\,(J)$ and $i \notin \cup_{J\in\mathcal{J}}\mathrm{Out}\,(J)$, respectively.

In order to complete the model (3.2)–(3.3), one needs to define the flow through each of junctions $J \in \mathcal{J}$ in the network. For this, at each junction we consider a so-called Riemann solver (see [7]) satisfying the conservation of cars and the following rules:

**(A):** there are some prescribed preferences of drivers, that is the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;
**(B):** respecting (A), drivers choose so as to maximize fluxes.

Let us consider a single junction $J$ with $n$ incoming roads, say $I_1, \dots, I_n$ with end $b_i$ $(i \in \{1, \dots, n\})$ at the junction, and $m$ outgoing roads, say $I_{n+1}, \dots, I_{n+m}$ with end $a_i$ $(i \in \{n+1, \dots, n+m\})$ at the junction. Then to guarantee the conservation of the number of cars at the junction $J$ the following condition must be prescribed:

$$\sum_{i=1}^{n} f_i\left(\rho_i(t, b_i)\right) = \sum_{i=n+1}^{n+m} f_i\left(\rho_i(t, a_i)\right), \quad \forall\, t \in [0, T],\ \forall\, J \in \mathcal{J}. \tag{3.4}$$

This is the so-called Rankine-Hugoniot conditions at the junctions. However, the problem is that the condition (3.3)–(3.4) are not sufficient to determine a unique solution of the system (3.2) on the network. Indeed, let $\widehat{\rho} = (\widehat{\rho}_1, \dots, \widehat{\rho}_{n+m})$ denote the solution at above fixed junction $J \in \mathcal{J}$. If $\widehat{\rho}$ is known, then a Riemann problem (3.2)–(3.4) is solved for each road with $\widehat{\rho}_i$ as the right state for incoming roads $(i \le n)$ and the left state for outgoing roads with $(i \in \{n+1, \dots, n+m\})$. As a result the solution may consist of shock waves or rarefaction waves emerging from the junction. However, we have $n+m$ unknowns and only one equation (3.4) at the junction. To define these additional conditions, we follow the approach of Coclite, Garavello & Piccoli [7] and introduce a traffic distribution matrix $A(J) \in \mathbb{R}^{m\times n}$ such that

$$A(J) = \left[\alpha_{ji}(J)\right], \quad j \in \{n+1, \dots, n+m\},\ i \in \{1, \dots, n\}, \tag{3.5}$$

$$\begin{cases} a_{ji}(J) \neq a_{ji'}(J), \quad \forall\, i \neq i', \quad 0 < \alpha_{ji}(J) < 1, \\ \sum\limits_{j=n+1}^{n+m} \alpha_{ji}(J) = 1 \text{ for every } \ i \in \{1, \dots, n\}. \end{cases} \tag{3.6}$$

Given a junction $J$ and an incoming road $I_i$, the $i$-th column of $A(J)$ describes how the traffic from $I_i$ distributes in percentages to the outgoing roads. This means that if $C$ is the quantity of the traffic coming from the road $I_i$ then $\alpha_{ji}(J)C$ traffic moves towards roads $I_j$. Thus $A(J) \in \mathbb{R}^{m\times n}$ describes the percentages of drivers who want to drive from road $I_i$ to road $I_j$. For simplicity we assume that for every junction $J \in \mathcal{J}$ the corresponding matrix $A(J) \in \mathbb{R}^{m\times n}$ is independent of the time, even if, in general, the matrices $A(J) \in \mathbb{R}^{m\times n}$ are time-dependent. For instance, in the case of car traffic on an urban network, the preferences of drivers may change depending on the period of the day.

We introduce a technical condition on matrix $A\,(J)$. We say that the matrix $A$ satisfies hypothesis $(C)$ if the following holds:

**(C):** Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$ indicate by $V^\perp$ its orthogonal. Define for every $i = 1, ..., n$, $H_i = \{e_i\}^\perp$, i. e. the coordinate hyperplane orthogonal to $e_i$ and, for every $j = n+1, ..., n+m$, let $\alpha_j = (\alpha_{j1}, \alpha_{j2}, ..., \alpha_{jn}) \in \mathbb{R}^n$ and define $H_j = \{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $k = (k_1, ..., k_l)$, $1 \le l \le n-1$, such that $0 \le k_1 < k_2 < ... < k_l \le n+m$ and for every $k \in \mathcal{K}$ set $H_k = \bigcap_{h=1}^{l} H_{k_l}$.Letting $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$, then for every $k \in \mathcal{K}$,

$$\mathbf{1} \notin H_k^\perp.$$

*Remark* 3.2. Note that from $(C)$ we immediately derive $m \ge n$.

Condition $(C)$ cannot hold for crossings with $n$ incoming roads and one outgoing road. We thus introduce some further parameters whose meaning is the following. When not all cars can go through the junction, there is a yielding rule that describes the percentage of cars crossing the junction, which come from a particular incoming road. In particular we assume the rule:

**(P):** Assume not all cars can enter the outgoing road and let $C$ be the amount that can do it. Then $q_i C$ cars come from the road $i, i = 1, ..., n$ with $\sum_{i=1}^{n} q_i = 1$.

We are now ready to give the definition of solution of (3.2) at junctions $J \in \mathcal{J}$ and on the whole network $\Omega$, following [7].

**Definition 3.1.** Let $J$ be a junction with $n$ incoming roads, say $I_1, \ldots, I_n$( with end $b_i$), and $m$ outgoing roads, say $I_{n+1}, \ldots, I_{n+m}$ (with end $a_i$). We say that

$$\rho = (\rho_1, \ldots, \rho_{n+m}) : \prod_{l=1}^{n+m} ([0,T] \times I_l) \to \mathbb{R}^{n+m}$$

$$\rho(t, \cdot) \in \prod_{l=1}^{n+m} BV(I_l) \quad \text{for every} \quad t \in [0,T]$$

is a weak solution of (3.2) related to the matrix $A(J) \in \mathbb{R}^{m \times n}$ at the junction $J$ if it is a collection of functions $\rho_l : [0,T] \times I_l \to \mathbb{R}$ with $l \in \{1, \ldots, n+m\}$, such that:

**(i):**

$$\sum_{l=1}^{n+m} \left( \int_0^T \int_{a_l}^{b_l} (\rho_l \partial_t \varphi_l + f_l(\rho_l) \partial_x \varphi_l) \, dx \, dt \right) = 0, \tag{3.7}$$

for every collection of test functions $\varphi_l$, $l \in \{1, \ldots, n+m\}$, smooth having compact support in the set $(0,T) \times (a_l, b_l]$ for $l \in \{1, \ldots, n\}$ (incoming roads) and in $(0,T) \times [a_l, b_l)$ for $l \in \{n+1, \ldots, n+m\}$ (outgoing roads), and smooth across the junction, i.e.

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j),$$
$$\partial_x \varphi_i(\cdot, b_i) = \partial_x \varphi_j(\cdot, a_j), \quad i \in \{1, \ldots, n\}, \; j \in \{n+1, \ldots, n+m\};$$

**(ii):** $f_j(\rho_j(\cdot, a_j^+)) = \sum_{i=1}^{n} \alpha_{ji}(J) f_i(\rho_i(\cdot, b_i^-))$ for each $j \in \{n+1, \ldots, n+m\}$;

**(iii):** $L(J, A, \rho) := \sum_{i=1}^{n} f_i(\rho_i(\cdot, b_i^-))$ is maximum subject to (i) and (ii).

*Remark* 3.3. The condition (i) of this Definition is essentially the conservation of cars at junctions. Moreover, as is evident from foregoing, formula (3.7) implies the conservation condition (3.4) if the functions $\rho_l$ are sufficiently regular. As for the rest conditions (ii) and (iii), they describe the rules (A) and (B), i.e. the preferences of drivers and the maximization procedure.

In general, the Riemann problem (3.2)–(3.4) for given initial data $\overline{\rho}_i : [a_i, b_i] \to \mathbb{R}$ possesses a solution on the network $\Omega$ in the following sense (see [7]).

**Definition 3.2.** Let $\overline{\rho}_i \in L^\infty(I_i) \cap BV(I_i)$, $i \in \{1, \dots, N\}$, be given functions. We say that a collection of functions $\rho = (\rho_1, \dots, \rho_N) : \prod_{i=1}^N ([0,T] \times I_i) \to \mathbb{R}^N$ with

$$\rho_i \in C([0,T]; L^1_{loc}(I_i)), \quad i \in \{1, \dots, N\}$$

is an admissible solution to the problem (3.2)–(3.5) if:

**(a):** $\rho_i : [0,T] \times I_i \to \mathbb{R}$ is a weak entropic solution of (3.2) on $I_i$, i.e.

$$\int_0^T \int_{a_i}^{b_i} (\rho_i \partial_t \varphi + f_i(\rho_i) \partial_x \varphi) \, dx \, dt = 0, \tag{3.8}$$

$$\int_0^T \int_{a_i}^{b_i} (|\rho_i - k| \, \partial_t \widetilde{\varphi} + \operatorname{sgn}(\rho_i - k) \, (f_i(\rho_i) - f_i(k)) \, \partial_x \widetilde{\varphi}) \, dx \, dt \geq 0, \tag{3.9}$$

for every function $\varphi : [0,T] \times I_i \to \mathbb{R}$ smooth with compact support on $(0,T) \times (a_i, b_i)$, every $k \in \mathbb{R}$, and every $\widetilde{\varphi} : [0,T] \times I_i \to \mathbb{R}$ smooth, positive with compact support on $(0,T) \times (a_i, b_i)$;

**(b):** $\rho_i(0, \cdot) = \overline{\rho}_i(\cdot)$ on $I_i$ for every $i \in \{1, \dots, N\}$;

**(c):** at each junction $J \in \mathcal{J}$, the collection $\rho$ is a weak solution of (3.2) related to the matrix $A(J) \in \mathbb{R}^{m \times n}$ at the junction $J$ in the sense of Definition 3.1.

*Remark* 3.4. As shown in [7], in the presence of discontinuities in the initial data $\overline{\rho}_i(\cdot)$ on $I_i$, the Rankine–Hugoniot equation (3.8) may not be sufficient to isolate a unique solution to the corresponding Cauchy problem (3.2)–(3.3). Therefore, the notion of weak solution of (3.2) must be supplemented with admissibility conditions, motivated by physical considerations. An admissibility criterion, coming from physical considerations, is the so-called entropy-admissibility condition, which in this case takes the form of the Kruzkov entropy admissibility condition (3.9) (see [13]).

**Definition 3.3.** Let $J \in \mathcal{J}$ be a junction of the network $(\mathcal{I}, \mathcal{J})$ such that it exactly has two incoming roads and two outgoing ones. Then, following [7], we say that the traffic distributional matrix $A(J)$, taking the form

$$A(J) = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix}, \tag{3.10}$$

satisfies hypothesis $(C)$ if $\alpha, \beta \in (0,1)$ and $\alpha \neq \beta$.

*Remark* 3.5. Hypothesis $(C)$ is a rather technical condition, which is important to isolate a unique solution to the corresponding Riemann problems at junctions. However, as follows from this definition, if either parameter $\alpha$ or $\beta$ in (3.10) takes the value in $\{0, 1\}$, then the corresponding junction $J$ has one incoming and two outgoing roads. Hence, in this case it is reasonably to introduce minor modifications in the original network and redefine the associated Cauchy problem (3.2)–(3.5).

Taking this into account, we give the following well-known result concerning existence and uniqueness of the Cauchy problem (3.2)–(3.5) (see [3, 7, 9].

**Theorem 3.1.** *Let $\Omega = \mathcal{O} \cap \mathfrak{F} = (\mathcal{I}, \mathcal{J})$ be a given network, let $\{f_i : \mathbb{R} \to \mathbb{R}\}_{i=1}^N$ be a family of flux-functions satisfying the properties (3.1), and let $\overline{\rho} = \{\overline{\rho}_i \in L^\infty(I_i) \cap BV(I_i)\}_{i=1}^N$ be an initial datum. Assume that the road network $(\mathcal{I}, \mathcal{J})$ is such that all its junctions $J \in \mathcal{J}$ have at most two incoming roads and two outgoing roads, and every traffic distribution matrix $A(J)$ belongs to the class $C$. Then, there exists a unique admissible solution $\rho = (\rho_1, \dots, \rho_N) : \prod_{i=1}^N ([0,T] \times I_i) \to \mathbb{R}^N$ to the problem (3.2)–(3.5) such that*

$$\rho_i \in C([0,T]; L^\infty(I_i) \cap BV(I_i)), \quad i \in \{1, \dots, N\}, \tag{3.11}$$

$$\operatorname{Tot} V_{I_i}(\rho_i(t,\cdot)) \le \operatorname{Tot} V_{I_i}(\overline{\rho}_i) \quad i \in \{1, \dots, N\}. \tag{3.12}$$

## 4. Statement of Vector Optimization Problem

In what follows, since condition $(C)$ is not closed with respect to the convergence in space of matrices, we restrict our next consideration to the case of networks $\Omega = (\mathcal{I}, \mathcal{J})$ with junctions of degree three, in particular characterized by an incoming road $m$ with the end $b_m$ at the junction and two outgoing roads labeled $r$, $s$ with ends $a_r$, $a_s$ at the junction (see Figure 4.1 for the illustration). According to Coclite, Garavello & Piccoli [3, 7], at such dispersing junction the flux distribution matrix $A(J)$ takes the form $A(J) = [\alpha_m, 1-\alpha_m]^t$, where $0 < \alpha_m < 1$. Hence we can suppose that at this junction we have a real-valued control-factor $\alpha_m \in [\beta, 1-\beta]$ for some small parameter $\beta \in (0, 1/2)$.

*Remark* 4.1. Note that in this case condition $(C)$ holds true for every junction $J$. Moreover, this condition is closed with respect to the convergence in the space of matrices $A(J) = [\alpha_m, 1-\alpha_m]^t$, where $\alpha_m \in [\beta, 1-\beta]$.

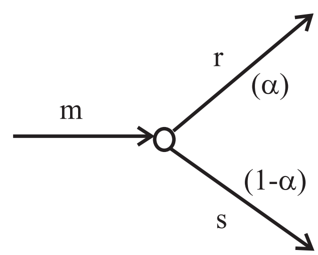


*Figure 4.1: Labeling of the roads connected to the dispersing junction.*

In what follows, we assume that the network $(\mathcal{I}, \mathcal{J})$, which has a total of $N$ roads, is such that the set $\mathcal{J}$ contains $K$ dispersing junctions of degree three. Roughly speaking, we have a network with $K$ controls $\alpha = (\alpha_1, \dots, \alpha_K)$. We also assume that on each road $I_i = [a_i, b_i] \in \mathcal{I}$ a velocity $v = v(\rho)$ is subjected to the following constraints:

$$v(\rho) \;\text{ is decreasing continuous on }\; [0, \max_{1 \le i \le N} \rho_{\max,i}], \tag{4.1}$$

$$0 \le v(\rho_i) \le v_{i,\max}, \quad \forall\, i \in \{1, \dots, N\}, \tag{4.2}$$

where $v_{i,\max} \in L^2(I_i)$ $(1 \le i \le N)$ are given functions.

In what follows, we use the following notations:

**(S1):** $\mathcal{A} = \{\alpha = (\alpha_1, \dots, \alpha_K) \mid \beta \le \alpha_i \le 1-\beta,\ i = 1, \dots, K\} \subset \mathbb{R}^K$ is the set of admissible control-factors, where $\beta \in (0, 1/2)$ is a given small parameter;

**(S2):** $X = R^K \times C(0,T; L^\infty(\Omega) \cap BV(\Omega))$ is the control-state space;

**(S3):** $P : \mathbb{R}^K \times C(0,T; L^\infty(\Omega) \cap BV(\Omega)) \to L^2(\Omega)$ $(1 < p < +\infty)$ is an objective mapping;

**(S4):** $\Lambda = \{g \in L^2(\Omega) \ : \ g(x) \geq 0$ almost everywhere on $\Omega\}$ is the ordering cone of positive elements in $L^2(\Omega)$.

From previous section we know that for every $\alpha = (\alpha_1, \dots, \alpha_K) \in \mathcal{A}$ the problem

$$\int_0^T \int_{a_i}^{b_i} (\rho_i \partial_t \varphi + f_i(\rho_i) \partial_x \varphi) \, dx \, dt = 0, \quad \forall \varphi \in C_0^\infty((0,T) \times (a_i, b_i)), \ \forall I_i \in \mathcal{I}, \tag{4.3}$$

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \int_{a_i}^{b_i} (|\rho_i - d| \, \partial_t \widetilde{\varphi} + \mathrm{sgn}\,(\rho_i - d)\, (f_i(\rho_i) - f_i(d))\, \partial_x \widetilde{\varphi}) \, dx \, dt \geq 0, \\ \quad \forall d \in \mathbb{R}, \ \forall \widetilde{\varphi} \in C_0^\infty((0,T) \times (a_i, b_i)), \ \widetilde{\varphi} \geq 0, \ \forall i \in \{1, \dots, N\}, \end{array} \right. \tag{4.4}$$

$$\rho_i(0, \cdot) = \overline{\rho}_i(\cdot) \ \text{ on } \ I_i \ \text{ for every } \ i \in \{1, \dots, N\}, \tag{4.5}$$

$$\left\{ \begin{array}{c} f_r(\rho_r(\cdot, a_r^+) = \alpha_k f_m(\rho_m(\cdot, b_m^-) \ \text{ and } \ f_s(\rho_s(\cdot, a_s^+) = (1 - \alpha_k) f_m(\rho_m(\cdot, b_m^-) \\ \text{for each junction } \ J_k \in \mathcal{J}, \ \text{which has} \\ \text{an incoming road } \ m \ \text{ with the end } \ b_m \ \text{ at } \ J_k \\ \text{and two outgoing roads } \ r, s \ \text{ with ends } \ a_r, a_s \ \text{ at } \ J_k, \ \forall k \in \{1, \dots, K\}, \end{array} \right. \tag{4.6}$$

has a unique solution

$$\rho = (\rho_1, \dots, \rho_N) : \prod_{i=1}^N ([0,T] \times I_i) \to \mathbb{R}^N \ \text{ in } \ C(0,T; L^\infty(\Omega) \cap BV(\Omega))$$

with properties (3.11)–(3.12).

We associate with (4.2)–(4.6) the vector optimization problem

$$\text{Realize} \quad \mathrm{Sup}^{\Lambda, w} \{P(\alpha, \rho)\} \tag{4.7}$$

over all $\alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}^K$ and $\rho = (\rho_1, \dots, \rho_N) \in C(0,T; L^\infty(\Omega) \cap BV(\Omega))$ given by (4.3)–(4.6), subject to the control constraints $\alpha \in \mathcal{A}$ and the state constraints (4.1)–(4.2).

**Definition 4.1.** We say that the problem (4.7) is regular if, for a given family of flux-functions $f = (f_1, \dots, f_N)$ with properties (3.1) there exists a pair

$$(\alpha, \rho) \in \mathcal{A} \times C(0,T; L^\infty(\Omega) \cap BV(\Omega)),$$

where $\rho = \rho(\alpha)$ is the corresponding solution of (4.3)–(4.6), such that $\rho$ satisfies the restrictions (4.1)–(4.2) and $P(\alpha, \rho) >_\Lambda z$ for some element $z$ of $L^2(\Omega)$. In this case the pair $(\alpha, \rho)$ is said to be admissible.

We denote by $\Xi$ the set of all admissible pairs to the problem (4.1)–(4.7). It is clear that $\Xi \subset \mathcal{A} \times C(0,T; L^\infty(\Omega) \cap BV(\Omega))$. In the sequel, we will associate this problem with the quaternary $\langle \Xi, P, \Lambda, w \rangle$, where $w$ is the weak topology of the objective space $L^2(\Omega)$.

*Remark* 4.2. Note that, in general, there is a principle difference between the problem (4.7) and the vector maximization problem in the classical statement:

$$\left. \begin{array}{c} \text{Maximize } \ P(\alpha, \rho) \ \text{ with respect to the cone } \ \Lambda \\ \text{subject to the constrains } (\alpha, \rho) \in \Xi. \end{array} \right\} \tag{4.8}$$

Indeed, let $(\alpha^{eff}, \rho^{eff}) \in \Xi$ be a $(\Lambda, w)$-efficient solution to the problem (4.7). Then $P(\alpha^{eff}, \rho^{eff}) \in \mathrm{Max}_\Lambda\,(\mathrm{cl}_w P(\Xi))$. Hence

$$P(\alpha^{eff}, \rho^{eff}) \in P(\Xi) \ \text{ and } \ P(\alpha^{eff}, \rho^{eff}) \in \mathrm{Max}_\Lambda\, P(\Xi).$$

Therefore, $(\alpha^{eff}, \rho^{eff})$ is a solution to the problem (4.8). However, the converse statement is not true in general. At the same time, this situation is atypical for the scalar case when we always have the implication

$$\text{if } \; P(\alpha^{eff}, \rho^{eff}) = \max_{(\alpha,\rho)\in\Xi} P(\alpha,\rho), \text{ then}$$
$$(\alpha^{eff}, \rho^{eff}) \in \Xi \text{ and } \; P(\alpha^{eff}, \rho^{eff}) = \sup_{(\alpha,\rho)\in\Xi} P(\alpha,\rho).$$

Note also that the vector optimizations problems (4.7) and (4.8) are identical in the case when $Y = \mathbb{R}$ and $\Lambda = \mathbb{R}_+$, and they lead us to the classical statement of a scalar constrained maximization problem.

We begin with the following concept:

**Definition 4.2.** An admissible pair $(\alpha^{eff}, \rho^{eff}) \in \Xi$ is said to be a $(\Lambda, w)$-efficient solution to the problem (4.1)–(4.7) if $(\alpha^{eff}, \rho^{eff})$ realizes the $(\Lambda, w)$-supremum of the mapping $P : \Xi \to L^2(\Omega)$, that is,

$$P(\alpha^{eff}, \rho^{eff}) \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho) = \mathrm{Sup}^{\Lambda,w} \left\{ P(\alpha,\rho) \,:\, \forall\, (\alpha,\rho) \in \Xi \right\}.$$

We denote by $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ the set of all $(\Lambda, w)$-efficient solutions to the vectorial problem (4.1)–(4.7), i.e.

$$\mathrm{Eff}_w(\Xi;\, P;\, \Lambda) = \left\{ (\alpha^{eff}, \rho^{eff}) \in \Xi \;:\; P(\alpha^{eff}, \rho^{eff}) \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho) \right\}. \tag{4.9}$$

To conclude this section we give the following observation concerning the topological properties of the set $\Xi$ of admissible pairs to the vector optimization problem (4.7). Let $\tau$ be the topology on

$$Y = R^K \times L^2(0,T; BV(\Omega))$$

defined as the product of the topology of pointwise convergence in $\mathbb{R}^K$ and the weak topology of $L^2(0,T; BV(\Omega))$.

**Lemma 4.1.** *Let $\left\{(\alpha^k, \rho^k) \in \Xi\right\}_{k=1}^{\infty}$ be a sequence of admissible pairs to the problem (4.7). Then there exists a subsequence of $\left\{(\alpha^k, \rho^k) \in \Xi\right\}_{k=1}^{\infty}$ (which will be still denoted by $k$ to simplify the notation), and a pair $(\alpha^*, \rho^*)$ satisfying*

$$(\alpha^*, \rho^*) \in \Xi, \quad (\alpha^k, \rho^k) \xrightarrow{\tau} (\alpha^*, \rho^*), \tag{4.10}$$

*that is, the set $\Xi$ is sequentially closed with respect to the $\tau$-convergence.*

*Proof.* Since $(\alpha^k, \rho^k) \in \Xi$ for each $k \in \mathbb{N}$, by Theorem 3.1 it follows that the sequence $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^{\infty}$ is uniformly bounded in $Y$. Then the Compactness Property of $BV$-functions (see Theorem 2.1) implies that this sequence is relatively $\tau$-compact. Hence, extracting, if necessary, a further subsequence, we can assume the existence of a pair $(\alpha^*, \rho^*)$ in $Y$ such that

$$\alpha^k \to \alpha^* \;\text{ in }\; \mathbb{R}^K, \;\text{ and }\; \rho^k \rightharpoonup \rho^* \;\text{ in }\; L^2(0,T; BV(\Omega)).$$

Our aim is to prove that $(\alpha^*, \rho^*) \in \Xi$. Since $\alpha^* \in \mathcal{A}$, it remains to show that the limit pair $(\alpha^*, \rho^*)$ satisfies relations (4.2)–(4.6). Then, in view of the Existence Theorem 3.1, $\rho^* = (\rho_1^*, \dots, \rho_N^*) : \prod_{i=1}^{N} ([0,T] \times I_i) \to \mathbb{R}^N$ is a unique admissible solution to the problem (3.2)–(3.5) in $C(0,T; L^\infty(\Omega) \cap BV(\Omega))$ under $\alpha = \alpha^*$.

Do to so, we consider relations (4.2)–(4.6) with $\alpha = \alpha^k$, $\rho = \rho^k$, and study their limit properties as $k \to \infty$. Since the family of flux-functions $f = (f_1, \dots, f_N)$ satisfies the properties (3.1), $v(\rho)$ is decreasing continuous on $[0, \max_{1\le i\le N} \rho_{\max,i}]$, $\rho^k \rightharpoonup \rho^*$ in

$L^2(0,T;BV(\Omega))$, and $\rho^k(t,\cdot) \to \rho(t,\cdot)$ strongly in $L^1(\Omega)$ $\forall t \in [0,T]$, then the passage to the limit in

$$0 \le v(\rho_i^k) \le v_{i,\max}, \quad \forall i \in \{1,\dots,N\}, \tag{4.11}$$

$$\int_0^T \int_{a_i}^{b_i} \left(\rho_i^k \partial_t \varphi + f_i(\rho_i^k)\partial_x \varphi\right) dx\,dt = 0, \quad \forall \varphi \in C_0^\infty((0,T)\times(a_i,b_i)),\ \forall I_i \in \mathcal{I}, \tag{4.12}$$

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \int_{a_i}^{b_i} \left(\left|\rho_i^k - d\right| \partial_t \widetilde{\varphi} + \operatorname{sgn}\left(\rho_i^k - d\right)\left(f_i(\rho_i^k) - f_i(d)\right)\partial_x \widetilde{\varphi}\right) dx\,dt \ge 0, \\ \quad \forall d \in \mathbb{R},\ \forall \widetilde{\varphi} \in C_0^\infty((0,T)\times(a_i,b_i)),\ \widetilde{\varphi} \ge 0,\ \forall i \in \{1,\dots,N\}, \end{array} \right. \tag{4.13}$$

$$\rho_i^k(0,\cdot) = \overline{\rho}_i(\cdot) \ \text{ on } \ I_i \ \text{ for every } \ i \in \{1,\dots,N\}, \tag{4.14}$$

$$\left\{ \begin{array}{c} f_r(\rho_r^k(\cdot,a_r^+) = \alpha_k f_m(\rho_m^k(\cdot,b_m^-) \ \text{ and } \ f_s(\rho_s^k(\cdot,a_s^+) = (1-\alpha_k) f_m(\rho_m^k(\cdot,b_m^-) \\ \text{for each junction } \ J_k \in \mathcal{J}, \ \text{which has} \\ \text{an incoming road } \ m \ \text{ with the end } \ b_m \ \text{ at } \ J_k \\ \text{and two outgoing roads } \ r,s \ \text{ with ends } \ a_r, a_s \ \text{ at } \ J_k,\ \forall k \in \{1,\dots,K\}, \end{array} \right. \tag{4.15}$$

as $k \to \infty$ gives the relations (4.2)–(4.6) with $\alpha = \alpha^*$ and $\rho = \rho^*$.

It remans to verify the relation

$$\lim_{k\to\infty} \left\{ \sum_{i=1}^n f_i(\rho_i^k(\cdot,b_i^-)) \text{ is maximum subject to (4.12)–(4.15)} \right\} \tag{4.16}$$
$$= \sum_{i=1}^n f_i(\rho_i^*(\cdot,b_i^-)) \text{ is maximum subject to (4.3)–(4.6) under } \alpha = \alpha^* \text{ and } \rho = \rho^*$$

at every junction $J \in \mathcal{J}$.

Let $J \in \mathcal{J}$ be a fixed junction. Since the function $\widetilde{L}(J,\alpha,\rho) := -\sum_{i=1}^n f_i(\rho_i(\cdot,b_i^-))$ is closed with respect to the $\tau$-convergence, i.e.

$$\lim_{l\to\infty} \widetilde{L}(J,\alpha^l,\rho^l) = \widetilde{L}(J,\alpha,\rho),$$

for every sequence $\left\{(\alpha^l,\rho^l)\right\}_{l=1}^\infty$ $\tau$-converging to $(\alpha,\rho)$, it follows that this function is closed with respect to the $\Gamma(\tau)$-convergence. Hence relation (4.16) is the direct consequence of variational properties of $\Gamma(\tau)$-limits (see Dal Maso [5]). Thus the $\tau$-limit pair $(\alpha^*,\rho^*)$ is an admissible solution to vector optimization problem (4.7) and this concludes the proof. □

As direct consequence of this lemma (see also Remark 4.1) we have:

**Corollary 4.2.** *If $\alpha \in \mathcal{A}$ (see condition $(S1)$ then the map $\alpha \mapsto \rho(\alpha)$ is continuous with respect to the topology of pointwise convergence in $\mathbb{R}^K$ and the weak topology of $L^2(0,T;BV(\Omega))$.*

*Remark* 4.3. Note that this conclusion is generally wrong if the control-factors $\alpha = (\alpha_1,\dots,\alpha_K)$ are such that $0 < \alpha_i < 1$, $i = 1,\dots,K$, that is, when controls at $J \in \mathcal{J}$ are restricted only by condition $(C)$.

## 5. Existence Theorem

Our main interest in this section is to obtain an existence theorem of the $(\Lambda, w)$-efficient solutions for the vector optimization problem (4.7). Let $\widehat{P} : \left[\mathbb{R}^K \times C(0,T;BV(\Omega))\right] \to Y^\bullet$ denote the natural extension of $P : \Xi \to L^2(\Omega)$ to the whole of $\mathbb{R}^K \times C(0,T;BV(\Omega))$ (see (2.2)). Here $Y^\bullet$ denotes the semi-extended Banach space $L^2(\Omega) \cup \{-\infty_\Lambda\}$. We begin with the following concept of upper semicontinuity for vector-valued mappings:

**Definition 5.1.** We say that a mapping $P : \Xi \to L^2(\Omega)$ is $(\Lambda, \tau\times w)$-upper semicontinuous ($(\Lambda, \tau \times w)$-usc) at the pair $(\alpha^0, \rho^0) \in \Xi$ if

$$P(\alpha^0, \rho^0) \in \limsup^{\Lambda,w}_{(\alpha,\rho) \xrightarrow{\tau} (\alpha^0,\rho^0)} \widehat{P}(\alpha, \rho).$$

A mapping $P$ is $(\Lambda, \tau \times w)$-usc on $\Xi$, if $P$ is $(\Lambda, \tau \times w)$-usc at each pair of $\Xi$.

The main motivation to introduce this concept is the following observation:

**Proposition 5.1.** *Assume that the objective space $L^2(\Omega)$ is partially ordered by the natural ordering cone of positive elements $\Lambda$ (see (2.1)). Let $\Xi$ be a nonempty subset of $\mathbb{R}^K \times C(0,T;BV(\Omega))$ and let $P : \Xi \to L^2(\Omega)$ be a given mapping. If $(\alpha^0, \rho^0) \in \Xi$ is any $(\Lambda, w)$-efficient solution to the problem (4.7), then the mapping $P : \Xi \to L^2(\Omega)$ is $(\Lambda, \tau \times w)$-usc at this pair.*

*Proof.* Let $(\alpha^0, \rho^0) \in \mathrm{Eff}_w(\Xi; P; \Lambda)$. Then $P(\alpha^0, \rho^0) \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$. On the other hand $P(\alpha^0, \rho^0) \in \mathrm{L}^{\tau\times w}(P, \alpha^0, \rho^0)$. Hence $P(\alpha^0, \rho^0) \in \mathrm{L}^{\tau\times w}_{\max}(P, \alpha^0, \rho^0)$. As a result, by Definition 2.6, we have $P(\alpha^0, \rho^0) \in \limsup^{\Lambda,w}_{(\alpha,\rho) \xrightarrow{\tau} (\alpha^0,\rho^0)} P(\alpha, \rho)$. This concludes the proof. □

Before proceeding further, we note that the cone of positive elements $\Lambda$ in $L^2(\Omega)$ satisfies the so-called Daniell property, which means that every increasing net (i.e. $i \le j \Longrightarrow y_i \le_\Lambda y_j$), which has an upper bound, weakly converges to its $(\Lambda, w)$-supremum.

**Definition 5.2.** We say that a nonempty subset $Y_0$ of $L^2(\Omega)$ with an ordering cone $\Lambda$ is upper semibounded, if every increasing net $\{y_i\} \subset Y_0$ is bounded from above.

As a direct consequence of Definition 5.2, we have the following observation:

*Remark* 5.1. Let $Y_0$ be a upper semibounded subset of a partially ordered linear space $\langle L^2(\Omega), \Lambda\rangle$. Then for any $z \in Y_0$ the section $Y_0^z = (\{z\} + \Lambda) \cap Y_0$ of $Y_0$ is bounded from above, that is, there exists an element $z^* \in L^2(\Omega)$ such that $z^* \le_\Lambda y$ for all $y \in Y_0^z$. Hence, the upper semiboundedness of the subset $Y_0$ implies the upper semiboundedness of its weak closure $\mathrm{cl}_w Y_0$. On the other hand, in contrast to the scalar case for vector optimization problem (4.7) with a sequentially $\tau$-compact subset $\Xi$ and $(\Lambda, \tau \times w)$-upper semicontinuous objective mapping $P : \Xi \to L^2(\Omega)$, the image set $P(\Xi)$ can be unbounded from above. It means that, in general, there does not exist an element $y^* \in L^2(\Omega)$ such that $P(\Xi) \subset \{y^*\} - \Lambda$.

We are able to prove the main result of this section.

**Theorem 5.2.** *Assume that the vector optimization problem (4.7) is regular. Let $P : \Xi \to L^2(\Omega)$ be a given $(\Lambda, \tau\times w)$-upper semicontinuous mapping. Then the vector optimization problem (4.7) has a non-empty set of $(\Lambda, w)$-efficient solutions.*

*Proof.* Since the proof of this theorem is rather technical, we divide it into several steps.

Step 1. First, we show that the image set $P(\Xi)$ is upper semibounded in the sense of Definition 5.2. Indeed, let us assume the converse. Then, there exists a sequence

$\left\{(\alpha^k,\rho^k)\right\}_{k=1}^\infty \subset \Xi$ such that the corresponding image sequence $\left\{y^k = P(\alpha^k,\rho^k)\right\}_{k=1}^\infty \subset P(\Xi)$ is increasing (i.e., $y_k \leq_\Lambda y_{k+1}\ \forall\, k \in \mathbb{N}$) and unbounded from above in $L^2(\Omega)$. Hence $\infty_\Lambda \in L^w\{y_k\}$, where $L^w\{y_k\}$ denotes the set of all its cluster points with respect to the weak topology of $L^2(\Omega)$. By Lemma 4.1, the sequence $\left\{(\alpha^k,\rho^k)\right\}_{k=1}^\infty \subset X_\partial$ is sequentially $\tau$-compact, so we may suppose that $(\alpha^k,\rho^k) \stackrel{\tau}{\to} (\alpha^*,\rho^*)$ in $Y = R^K \times L^2(0,T;BV(\Omega))$, where $(\alpha^*,\rho^*)$ is some pair of $\Xi$. Since the sequence $\left\{P(\alpha^k,\rho^k)\right\}_{k=1}^\infty$ is unbounded from above, we have $\{\infty_\Lambda\} \in \mathrm{L}^{\tau\times w}_{max}(P,\alpha^*,\rho^*)$. Hence, by Definition 2.6,

$$\limsup^{\Lambda,w}_{(\alpha,\rho)\stackrel{\tau}{\to}(\alpha^*,\rho^*)} P(\alpha,\rho) = \{\infty_\Lambda\}.$$

On the other hand, taking into account the $(\Lambda,\tau\times w)$-lower semicontinuity property of $P$, we obtain

$$P(\alpha^*,\rho^*) \in \limsup^{\Lambda,w}_{(\alpha,\rho)\stackrel{\tau}{\to}(\alpha^*,\rho^*)} P(\alpha,\rho)$$

which contradicts the previous conclusion. This concludes Step 1.

Step 2. In this part we prove that the set $\mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho)$ is nonempty. We show that there exists at least one increasing sequence $\{y_k\}_{k=1}^\infty \subset P(\Xi)$ such that

$$y_k \rightharpoonup y^* \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho) = \mathrm{Sup}^{\Lambda,w}\left\{P(\alpha,\rho)\,:\,\forall\,(\alpha,\rho)\in\Xi\right\}.$$

Let $y$ be an arbitrary element of $\mathrm{cl}_w\, P(\Xi)$. To begin with, we show that, for any neighbourhood of zero $\mathcal{V}_w$ in the weak topology of $L^2(\Omega)$, there exists an element $y^{\mathcal{V}} \in \mathrm{cl}_w\, P(\Xi)$ such that

$$y \leq_\Lambda y^{\mathcal{V}} \ \text{ and } \ \left(\left\{y^{\mathcal{V}}\right\} + \Lambda\setminus\{0\}\right) \cap \left(\mathrm{cl}_w\, P(\Xi) \setminus \left(\mathcal{V}_w + \left\{y^{\mathcal{V}}\right\}\right)\right) = \emptyset. \tag{5.1}$$

Having assumed the converse, we suppose the existence of a sequence $\{y_k\}_{k=1}^\infty \subset \mathrm{cl}_w\, P(\Xi)$ such that

$$y_1 \in P(\Xi), \quad y_{k+1} \in (\{y_k\} + \Lambda\setminus\{0\}) \cap (\mathrm{cl}_w\, P(\Xi) \setminus (\mathcal{V}_w + \{y_k\}))\ \ \forall\, k\in\mathbb{N}.$$

Since $y_{k+1} \in \{y_k\} + \Lambda\setminus\{0\}$, this sequence is increasing. Taking into account Remark 5.1, the set $\mathrm{cl}_w\, P(\Xi)$ is upper semibounded. Therefore, there exists an element $y^* \in L^2(\Omega)$ such that $y_k \leq_\Lambda y^*$ for all $k \in \mathbb{N}$. Hence, by Daniell property, this sequence weakly converges to its $(\Lambda,w)$-supremum: $y_k \rightharpoonup \widetilde{y} \in L^2(\Omega)$. However, this contradicts the condition $y_{k+1} \in \mathrm{cl}_w\, P(\Xi) \setminus (\mathcal{V}_w + \{y_k\})\ \forall\, k \in \mathbb{N}$. Thus, the choice by the rule (5.1) is possible for any neighbourhood $\mathcal{V}_w$.

Let $\{\mathcal{V}_k\}_{k=1}^\infty$ be a system of weak neighbourhoods of zero in $L^2(\Omega)$ such that $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ for every $k \in \mathbb{N}$, and for any weak neighbourhood $\mathcal{V}(0)$ in $L^2(\Omega)$ there is an integer $k^* \in \mathbb{N}$ such that $\mathcal{V}_{k^*} \subseteq \mathcal{V}(0)$. Then, using the choice rule (5.1), we can construct a sequence $\{u_k\}_{k=1}^\infty \subset \mathrm{cl}_w\, P(\Xi)$, where $u_1$ is an arbitrary element of $P(\Xi)$, as follows

$$u_{k-1} \leq_\Lambda u_k \ \text{ and } \ (\{u_k\} + \Lambda\setminus\{0\}) \cap (\mathrm{cl}_w\, P(\Xi)\setminus(\mathcal{V}_k + \{u_k\})) = \emptyset, \quad \forall\ k \geq 2. \tag{5.2}$$

Since $u_{k+1} \in \{u_k\} + \Lambda$ it follows that

$$u_{k+1} \in \mathrm{cl}_w\, P(\Xi) \ \text{ and } \ u_{k+1} \notin \mathrm{cl}_w\, P(\Xi)\setminus(\mathcal{V}_k + \{u_k\}).$$

Hence, in view of Daniell property, $\{u_k\}_{k=1}^\infty$ is the $\tau$-converging increasing sequence. As a result, there is an element

$$u^* \in \mathrm{Sup}^{\Lambda,w}\left\{u_k \in \mathrm{cl}_w\, P(\Xi)\,:\,\forall\, k\in\mathbb{N}\right\}$$

such that $u_k \rightharpoonup u^*$. It is clear that $u^* \in \mathrm{cl}_w\, P(\Xi)$. Our aim is to prove that $u^* \in \mathrm{Sup}^{\Lambda,w}\left\{P(\alpha,\rho)\,:\,\forall\,(\alpha,\rho)\in\Xi\right\}$. To do so, we assume that there exists an element

$$q \in \mathrm{Sup}^{\Lambda,w}\left\{P(\alpha,\rho)\,:\,\forall\,(\alpha,\rho)\in\Xi\right\}$$

such that $u^* \leq_\Lambda q$. Since $u_k \leq_\Lambda u^*$ for all $k \in N$, it follows that $u_k \leq_\Lambda q$ for all $k \in N$. Then (5.2) ensures that

$$(\{q\} + \Lambda \setminus \{0\}) \cap (\mathrm{cl}_w P(\Xi) \setminus (\mathcal{V}_k + \{u_k\})) = \emptyset, \quad \forall\ k \in \mathbb{N}. \tag{5.3}$$

Hence (5.3) and the fact that $q \in \mathrm{cl}_w P(\Xi)$ imply $q \in \mathcal{V}_k + \{u_k\}$ for every $k \in \mathbb{N}$, that is, $u_k \rightharpoonup q$ in $L^2(\Omega)$. Thus $u^* = q$ and then the Step 2 is finished.

Step 3: We show that the set $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ is nonempty. Let $\xi$ be any element of $\mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho)$. Then, by Definition 2.5, there exists a sequence $\{y_k\}_{k=1}^\infty \subset L^2(\Omega)$ such that $y_k \rightharpoonup \xi$ in $L^2(\Omega)$. We define a sequence $\left\{(\alpha^k,\rho^k)\right\}_{k=1}^\infty \subset \Xi$ as follows $(\alpha^k,\rho^k) = P^{-1}(y_k)$ for all $k \in \mathbb{N}$. Since the set $\Xi$ is sequentially $\tau$-compact (see Lemma 4.1), we may suppose that there exists a pair $(\alpha^0,\rho^0) \in \Xi$ such that $(\alpha^k,\rho^k) \stackrel{\tau}{\longrightarrow} (\alpha^0,\rho^0)$ in $Y$. Hence $\xi \in \mathrm{L}^{\tau\times w}(P,\alpha^0,\rho^0)$, and we get

$$\mathrm{L}^{\tau\times w}(P,\alpha^0,\rho^0) \cap \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho) \neq \emptyset.$$

Then, due to the $(\Lambda, \tau\times w)$-upper semicontinuity of the mapping $P$ on $\Xi$ and Definition 2.6, we obtain

$$P(\alpha^0,\rho^0) \in \limsup^{\Lambda,w}_{(\alpha,\rho)\stackrel{\tau}{\to}(\alpha^0,\rho^0)} P(\alpha,\rho) = \mathrm{L}^{\tau\times w}(P,\alpha^0,\rho^0) \cap \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho).$$

Thus, on the one hand, $P(\alpha^0,\rho^0) \in \mathrm{L}^{\tau\times w}(P,\alpha^0,\rho^0)$, which implies the equality

$$P(\alpha^0,\rho^0) = \xi = \text{weak}-\lim_{k\to\infty} y_k.$$

On the other hand, $\xi \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha,\rho)$. Hence, $(\alpha^0,\rho^0) \in \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ and we obtain the required result. The proof is complete. □

## 6. Scalarization of Traffic Optimization Problem

Typically, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem. It is a fundamental principle in vector optimization that optimal (minimal) elements of a subset of a partially ordered linear space can be characterized as optimal solutions of certain scalar optimization problems. Our prime interest in this section is to describe the set $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ of $(\Lambda, w)$-efficient solutions to the traffic optimization problem (4.7), which involves some topological properties of the objective mapping $P$ and the space $L^2(\Omega)$. In order to do it, we will consider the problem of the scalar representation of vector optimization problem (4.7) with a $(\Lambda, \tau\times w)$-upper semicontinuous mapping $P : \Xi \to L^2(\Omega)$, using the "simplest" method of the "weighted sum".

To begin with, we introduce some notation.

**Definition 6.1.** We say that $\lambda \in L^2(\Omega)$ is a quasi-interior point of the cone of positive elements $\Lambda$ if $\lambda(x) \geq 0$ almost everywhere in $\Omega$ and $\int_\Omega b(x)\lambda(x)\,dx > 0$ for all $b \in \Lambda \setminus \{0\}$.

We denote by $\Lambda^\sharp$ the set of all quasi-interior points to $\Lambda$. It is clear that

$$\Lambda^\sharp = \left\{\lambda \in L^2(\Omega) \;:\; \lambda(x) > 0 \text{ almost everywhere in } \Omega\right\}$$

(for more details we refer to [12]). In what follows, we associate with the vector optimization problem (4.7) the following scalar minimization problem

$$P_\lambda(\alpha,\rho) = \int_\Omega P(\alpha,\rho)\lambda(x)\,dx \to \sup \tag{6.1}$$
$$\text{subject to } \ (\alpha,\rho) \in \Xi \subset \mathbb{R}^K \times C(0,T;BV(\Omega)),$$

where $\lambda$ is an element of the cone (2.1).

The main property of this problem can be characterized as follows:

**Theorem 6.1.** *Let $P : \Xi \to L^2(\Omega)$ be a given objective mapping. Assume that there are a pair $(\alpha^0, \rho^0) \in \Xi$ and an element $\lambda \in \Lambda^\sharp$ such that*

$$(\alpha^0, \rho^0) \in \operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} \int_\Omega P(\alpha, \rho)\lambda(x)\, dx.$$

*Then $(\alpha^0, \rho^0)$ is a $(\Lambda, w)$-efficient solution to the problem (4.7).*

*Proof.* By the initial assumptions, we have

$$P_\lambda(\alpha^0, \rho^0) - P_\lambda(\alpha, \rho) = \int_\Omega \big(P_\lambda(\alpha^0, \rho^0) - P_\lambda(\alpha, \rho)\big)\, \lambda(x)\, dx \geq 0, \quad \forall\, (\alpha, \rho) \in \Xi. \tag{6.2}$$

Let $z$ be any element of the image set $\operatorname{cl}_w P(\Xi)$. Then there exists a sequence

$$\big\{(\alpha^k, \rho^k)\big\}_{k=1}^\infty \subset \Xi \ \text{ such that } \ P(\alpha^k, \rho^k) \rightharpoonup z \ \text{ in } \ L^2(\Omega) \ \text{ as } k \to \infty.$$

Hence, in view of (6.2), we get

$$\int_\Omega \big(P_\lambda(\alpha^0, \rho^0) - P_\lambda(\alpha^k, \rho^k)\big)\, \lambda(x)\, dx \geq 0, \quad \forall\, k \in \mathbb{N}. \tag{6.3}$$

Passing to the limit in (6.3) as $k \to \infty$, we obtain

$$\int_\Omega \big(P_\lambda(\alpha^0, \rho^0) - z\big)\, \lambda(x)\, dx \geq 0, \quad \forall\, z \in \operatorname{cl}_w P(\Xi). \tag{6.4}$$

Let us assume that $(\alpha^0, \rho^0) \not\in \operatorname{Eff}_w(\Xi;\, P;\, \Lambda)$. Then there exists an element $h \in \operatorname{cl}_w P(\Xi)$ such that $h >_\Lambda P(\alpha^0, \rho^0)$. So, $h - P(\alpha^0, \rho^0) \in \Lambda \setminus \{0\}$. Hence, by Definition 6.1,

$$\int_\Omega \big(h - P(\alpha^0, \rho^0)\big)\, \lambda(x)\, dx > 0,$$

and we come to a contradiction with (6.4). So, $(\alpha^0, \rho^0) \in \operatorname{Eff}_w(\Xi;\, P;\, \Lambda)$ and this concludes the proof. □

As an evident consequence of this result, we have the following corollary.

**Corollary 6.2.** *Under suppositions of Theorem 6.1, we have*

$$\bigcup_{\lambda \in \Lambda^\sharp} \operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} \int_\Omega P(\alpha, \rho)\lambda(x)\, dx \subseteq \operatorname{Eff}_w(\Xi;\, P;\, \Lambda). \tag{6.5}$$

We note that the objective mapping in Theorem 6.1 does not possesses $(\Lambda, \tau \times w)$-upper semicontinuity property, in general. So the question is about the solvability of the associated scalar minimization problems (6.1) with $\lambda \in \Lambda^\sharp$. Following the direct method in the Calculus of Variations, the constrained maximization problem (6.1) has a nonempty set of solutions provided $\Xi$ is a $\tau$-compact subset and

$$P_\lambda(\cdot, \cdot) = \int_\Omega P(\cdot, \cdot)\lambda(x)\, dx \ : \ \Xi \to \overline{\mathbb{R}}$$

is a proper upper $\tau$-semicontinuous function. However, the distinguishing feature of Vector optimization problems (4.7) is the fact that with any $(\Lambda, \tau \times w)$-upper semicontinuous mapping $P : \Xi \to L^2(\Omega)$, which is neither upper semicontinuous nor quasi-upper semicontinuous on $\Xi$, there can be always associated a scalar minimization problem (6.1) for which the corresponding cost functional $P_\lambda : \Xi \to \mathbb{R}$ is not upper $\tau$-semicontinuous on

$\Xi$. Indeed, let $(\alpha^0,\rho^0)$ be a pair of $\Xi$ where the quasi-upper semicontinuity of $P$ is failed. Then there exists at least one element $a^*\in \mathrm{cl}_w\,(P(\Xi))$ such that

$$a^*\in \limsup^{\Lambda,w}_{(\alpha,\rho)\xrightarrow{\tau}(\alpha^0,\rho^0)} P(\alpha,\rho),\ \ P(\alpha^0,\rho^0)\in \limsup^{\Lambda,w}_{(\alpha,\rho)\xrightarrow{\tau}(\alpha^0,\rho^0)} P(\alpha,\rho),\quad \text{and}\ \ a^*\neq P(\alpha^0,\rho^0). \tag{6.6}$$

Let $\left\{(\alpha^k,\rho^k)\right\}_{k=1}^{\infty}\subset\Xi$ be a sequence such that $(\alpha^k,\rho^k)\xrightarrow{\tau}(\alpha^0,\rho^0)$ in $Y$ and $P(\alpha^k,\rho^k)\rightharpoonup a^*$ in $L^2(\Omega)$. Since $a^*\not\leq_\Lambda P(\alpha^0,\rho^0)$ it follows that $P(\alpha^0,\rho^0)-a^*\not\in\Lambda$ and hence there exists a vector $\lambda^*\in\Lambda$ such that

$$\int_\Omega \left(P(\alpha^0,\rho^0)-a^*\right)\lambda^*(x)\,dx<0.$$

As a result, we have

$$\liminf_{k\to\infty} P_{\lambda^*}(\alpha^k,\rho^k)=\lim_{k\to\infty}\int_\Omega P(\alpha^k,\rho^k)\lambda^*(x)\,dx=$$

$$=\int_\Omega a^*(x)\lambda^*(x)\,dx>\int_\Omega P(\alpha^0,\rho^0)\lambda^*(x)\,dx=P_{\lambda^*}(\alpha^0,\rho^0).$$

Thus the upper $\tau$-semicontinuity property for $P_{\lambda^*}$ does not hold at the pair $(\alpha^0,\rho^0)$.

This fact motivates the introduction of the the following notion:

**Definition 6.2.** Let $P:\Xi\to L^2(\Omega)$ be a given mapping. The cone

$$\Lambda_P^\tau:=\{\lambda\in\Lambda\ :\ P_\lambda\ \text{ is upper }\ \tau\text{-semicontinuous on }\ \Xi\} \tag{6.7}$$

is called the cone of $\tau$-semicontinuity for the mapping $P$.

As a result, Theorem 6.1 can be sharped as follows:

**Theorem 6.3.** *Let $P:\Xi\to L^2(\Omega)$ be a $(\Lambda,\tau\times w)$-upper semicontinuous mapping. Assume that the vector optimization problem (4.7) is regular and $\Lambda_P^\tau\setminus 0\neq\emptyset$. Then*

$$\operatorname*{Argmax}_{(\alpha,\rho)\in\Xi}\int_\Omega P(\alpha,\rho)\lambda(x)\,dx\cap \mathrm{Eff}_w(\Xi;\,P;\,\Lambda)\neq\emptyset,\qquad\forall\ \lambda\in\Lambda_P^\tau\setminus 0. \tag{6.8}$$

*Proof.* As follows from Theorem 5.2, under above assumptions, we have

$$\mathrm{Eff}_w(\Xi;\,P;\,\Lambda)\neq\emptyset.$$

Let $\lambda$ be any element of $\Lambda_P^\tau\setminus 0$. Then, by the direct method in the Calculus of Variations, we obtain

$$\operatorname*{Argmax}_{(\alpha,\rho)\in\Xi}\int_\Omega P(\alpha,\rho)\lambda(x)\,dx\neq\emptyset.$$

If $\lambda\in\Lambda^\sharp$ then relation (6.8) is obvious by Theorem 6.1. So, we suppose that $\lambda\in\Lambda_P^\tau\setminus\left(\Lambda^\sharp\cup 0\right)$. Assume that

$$\operatorname*{Argmax}_{(\alpha,\rho)\in\Xi}\int_\Omega P(\alpha,\rho)\lambda(x)\,dx\not\subseteq \mathrm{Eff}_w(\Xi;\,P;\,\Lambda).$$

Then there exists a pair $(\alpha^*,\rho^*)\in\Xi$ such that

$$(\alpha^*,\rho^*)\in\operatorname*{Argmax}_{(\alpha,\rho)\in\Xi}\int_\Omega P(\alpha,\rho)\lambda(x)\,dx, \tag{6.9}$$

$$(\alpha^*,\rho^*)\not\in\mathrm{Eff}_w(\Xi;\,P;\,\Lambda). \tag{6.10}$$

Hence, by (6.9), there exists an element

$$y^* \in \mathrm{Max}_\Lambda \left(\mathrm{cl}_w P(\Xi)\right) \subseteq \mathrm{cl}_w P(\Xi) \ \text{ such that } \ y^* >_\Lambda P(\alpha^*, \rho^*).$$

However, in view of (6.10), this leads us to the equality

$$P_\lambda(\alpha^*, \rho^*) = \int_\Omega P(\alpha^*, \rho^*)\lambda(x)\, dx = \int_\Omega y^* \lambda(x)\, dx. \tag{6.11}$$

Let $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^\infty$ be a sequence in $\Xi$ such that

$$P(\alpha^k, \rho^k) \rightharpoonup y^* \ \text{ in } \ L^2(\Omega) \ \text{ as } \ k \to \infty. \tag{6.12}$$

Since the set $\Xi$ is sequentially $\tau$-compact (see Lemma 4.1), we may suppose that there exists a pair $(\alpha^0, \rho^0) \in \Xi$ such that $(\alpha^k, \rho^k) \stackrel{\tau}{\longrightarrow} (\alpha^0, \rho^0)$ in $Y$. On the other hand, $y^* \in \mathrm{Max}_\Lambda \left(\mathrm{cl}_w P(\Xi)\right)$. Hence $y^* \in \mathrm{Sup}^{\Lambda, w}_{(\alpha,\rho)\in \Xi} P(\alpha, \rho)$ by Definition 4.2. As a result, we have $(\alpha^0, \rho^0) \in \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$. Taking now into account the upper $\tau$-semicontinuity of the functional $P_\lambda : \Xi \to \mathbb{R}$, we get

$$\int_\Omega P(\alpha^0, \rho^0)\lambda(x)\, dx \geq \liminf_{k\to\infty} \int_\Omega P(\alpha^k, \rho^k)\lambda(x)\, dx \stackrel{\text{by (6.12)}}{=} \int_\Omega y^* \lambda(x)\, dx.$$

Then, combining this with (6.11), we obtain

$$\int_\Omega P(\alpha^0, \rho^0)\lambda(x)\, dx \geq \int_\Omega P(\alpha^*, \rho^*)\lambda(x)\, dx,$$

i.e.

$$(\alpha^0, \rho^0) \in \operatorname*{Argmax}_{(\alpha,\rho)\in \Xi} \int_\Omega P(\alpha, \rho)\lambda(x)\, dx.$$

Thus, we have shown that there exists at least one pair $(\alpha^0, \rho^0) \in \Xi$, which is a joint point of the sets

$$\operatorname*{Argmax}_{(\alpha,\rho)\in \Xi} \int_\Omega P(\alpha, \rho)\lambda(x)\, dx \ \text{ and } \ \mathrm{Eff}_w(\Xi;\, P;\, \Lambda),$$

respectively. This completes the proof. □

As an evident consequence of this theorem, we have the following corollary:

**Corollary 6.4.** *Assume that in addition to the conditions of Theorem 6.3 there exists an element* $\lambda \in \Lambda^\tau_P \setminus 0$ *such that the supremum in the scalar problem*

$$\textit{Maximize } P_\lambda(\alpha, \rho) = \int_\Omega P(\alpha, \rho)\lambda(x)\, dx \textit{ subject to } \ (\alpha, \rho) \in \Xi \tag{6.13}$$

*attains at a unique pair* $(\alpha^*, \rho^*) \in \Xi$. *Then* $(\alpha^*, \rho^*) \in \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$.

*Remark* 6.1. Typically in scalar optimization problems for traffic on road networks it is assumed that some cost functionals to measure the traffic behaviour are defined. A first functional $F_1$ measures the average velocity of drivers on the network, the second $F_2$ measures the expected mean traveling time on the network and finally the third $F_3$ is the total flux through the network. The analysis of the performances of the network through these functionals is a very delicate problem. However the goal of any driver is to find a fasten way through the network with respect to the traffic and road conditions. Having based on assumptions of the LWR model and used a linear density-velocity relation

$$v_i(\rho_i) = v_{max,i}\left(1 - \frac{\rho_i}{\rho_{max,i}}\right) \Longrightarrow f_i(\rho_i) = \rho_i v_{max,i}\left(1 - \frac{\rho_i}{\rho_{max,i}}\right),$$

this implies a low density on each road. Indeed, since the flux functions are concave, high densities are related to small velocities $v_i$, i.e. $\rho_i v_i = f_i(\rho_i)$. Therefore, in the scalar statement a well known measure for a better utilization of a single road $I_i = [a_i, b_i]$ of the network is the time and space averaged density given by the following expression

$$\int_0^T \int_{a_i}^{b_i} \rho_i(t,x)\, dxdt,$$

where $\rho_i \in C([0,T]; L^\infty(I_i) \cap BV(I_i))$ is the density approximation on each road $i \in \{1, \dots, N\}$. Hence, summing up for all roads in the network a cost functional can be defined as

$$F_4(\alpha, \rho) = \sum_{i=1}^N \int_0^T \int_{a_i}^{b_i} \rho_i(t,x)\, dxdt \longrightarrow \inf .$$

Thus the functional $F_4$ measures the average time and space densities in the whole network. So the "fasten" way through the network can be obtained by minimization $F_4$. Note that the functional $F_4$ is rather popular in the traffic engineering community [14]. Minimization of $F_4$ yields a traffic situation with a "large" average speed.

However, as follows from (6.1), the cost functional $F_4$ can be obtained as particular case of the associated scalar optimization problems (6.13) if

$$P(\alpha, \rho) = -\int_0^T \rho(t,x)\, dt, \quad \lambda(x) = 1 \text{ a.e. on } \Omega.$$

Then

$$P_\lambda(\alpha, \rho) := \int_\Omega P(\alpha, \rho)\lambda(x)\, dx = -\sum_{i=1}^N \int_0^T \int_{a_i}^{b_i} \rho_i(t,x)\, dxdt.$$

Note also that, due to the initial suppositions (3.1), we have the following property:

$$P(\alpha^k, \rho^k) \rightharpoonup P(\alpha, \rho) \text{ in } L^2(\Omega)$$

provided $(\alpha^k, \rho^k) \stackrel{\tau}{\longrightarrow} (\alpha^*, \rho^*)$ in $Y = R^K \times L^2(0,T; BV(\Omega))$, that is, the objective mapping $P : \Xi \to L^2(\Omega)$ is $(\Lambda, \tau \times w)$-upper semicontinuous. Hence $\Lambda_P^\tau \equiv \Lambda^\sharp$ and if the original vector optimization problem (4.7) is regular, then

$$\operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} P_\lambda(\alpha, \rho) \neq \emptyset \quad \text{and} \quad \operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} P_\lambda(\alpha, \rho) \subseteq \mathrm{Eff}_w(\Xi;\, P;\, \Lambda).$$

## 7. Generalized Solutions to Traffic Optimization Problem

Let $\lambda$ be an arbitrary element of the cone $\Lambda$. Denote by

$$\mathrm{Sol}(\Xi;\, P_\lambda) := \operatorname*{Argmax}_{(\alpha,\rho)\in\,\Xi} P_\lambda(\alpha, \rho)$$

the solution set to the scalar problem (6.13). We recall that the problem (6.13) is said to be well-posed in the generalized sense when every maximizing sequence $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^\infty \subset \Xi$ (i.e. such that $P_\lambda(\alpha^k, \rho^k) \to \sup_{(\alpha,\rho)\in\,\Xi} P_\lambda(\alpha, \rho)$) has a subsequence $\tau$-converging to some pair of $\mathrm{Sol}(\Xi;\, P_\lambda)$. We recall also a generalization of the above mentioned notion. The problem (6.13) is said to be well-set when every maximizing sequence contained in $\Xi \setminus \mathrm{Sol}(\Xi;\, P_\lambda)$ has a $\tau$-cluster point in $\mathrm{Sol}(\Xi;\, P_\lambda)$. However, as will follows from the arguments of this section, the problem (6.13) can be neither well-posed nor well-set, in general. The main reason is the $(\Lambda, \tau \times w)$-upper semicontinuity property of the objective mapping $P$ which is the weakened property of upper semicontinuity for vector-valued mappings in Banach spaces.

In many applications it has a sense to weaken the requirement on efficient solutions to the vector optimization problem (4.7). In particular, we may let that the objective mapping to attain its efficient supremum on the set $\Xi$ with some error. On the other hand, the set of $(\Lambda, w)$-efficient solutions to such problem can possibly be empty, i.e., the efficient supremum of the objective mapping is often unattainable on the given set $\Xi$. Nevertheless, the absence of its supremum does not mean that the vector optimization problem makes no sense, since its efficient supremum exists and hence can be approached with some accuracy.

**Definition 7.1.** We say that a sequence $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^{\infty} \subset \Xi$ is maximizing to the traffic optimization problem (4.7) if $P(\alpha^k, \rho^k) \rightharpoonup \xi$ in $L^2(\Omega)$, where $\xi$ is an element of $\mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$.

**Definition 7.2.** We say that the vector optimization problem (4.7) is well-posed in the Tikhonov sense with respect to the $\tau$-topology of $Y$if it is certainly solvable and every maximizing sequence $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^{\infty} \subset \Xi$ has a subsequence $\tau$-converging to some pair of $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$. In this case a maximizing sequence is called a Tikhonov maximizing sequence. We also say that the vector optimization problem (4.7) is well-set in the Tikhonov sense with respect to the $\tau$-topology of $Y$, if it is certainly solvable and every maximizing sequence contained in $\Xi \setminus \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ has a $\tau$-cluster pair in $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$.

Note that having a Tikhonov maximizing sequence, we can guarantee both the proximity of the corresponding values of the objective mapping to its efficient supremum and the proximity of the approximation itself to one of the $(\Lambda, w)$-efficient solutions of the problem. Nevertheless it should be stressed that even in simple applied problems the construction of Tikhonov maximizing sequences and corresponding Tikhonov approximate solutions usually turn out to be a very complicated and sometimes unsolvable problem. In view of this, it is reasonable to weaken the requirements on approximate solutions to the vector optimization problem (4.7).

**Definition 7.3.** We say that a pair $(\alpha^*, \rho^*) \in \Xi$ is a $(\tau, w)$-generalized solution to the traffic optimization problem (4.7) if there exist a sequence $\left\{(\alpha^k, \rho^k)\right\}_{k=1}^{\infty} \subset \Xi$ and an element $\xi \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$ such that $(\alpha^k, \rho^k) \stackrel{\tau}{\rightharpoonup} (\alpha^*, \rho^*)$ in $Y$ and $P(\alpha^k, \rho^k) \rightharpoonup \xi$ in $L^2(\Omega)$.

Thus, a vector optimization problem may have an approximate solution even in the absence of its solvability. It is clear that any Tikhonov approximate solution to the problem (4.7) is also a $(\tau, w)$-generalized solution. However, even if a $(\Lambda, w)$-efficient solution is available $((\alpha^{eff}, \rho^{eff}) \in \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$, we cannot guarantee the proximity of a $(\tau, w)$-generalized solution $(\alpha^*, \rho^*)$ to $\mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$ in the $\tau$-topology of $Y$.

We denote by $\mathrm{GenEff}_{\tau,w}(\Xi;\, P;\, \Lambda)$ the set of all $(\tau, w)$-generalized solutions to the problem (4.7). It is clear that

$$\mathrm{Eff}_w(\Xi;\, P;\, \Lambda) \subseteq \mathrm{GenEff}_{\tau,w}(\Xi;\, P;\, \Lambda).$$

However, the inverse inclusion

$$\mathrm{GenEff}_{\tau,w}(\Xi;\, P;\, \Lambda) \subset \mathrm{Eff}_w(\Xi;\, P;\, \Lambda)$$

does not generally hold. As an evident consequence of Theorem 5.2, we have the following obvious result:

**Proposition 7.1.** *Under suppositions of Theorem 5.2, the traffic optimization problem (4.7) is well-set in the Tikhonov sense with respect to $\tau$-topology of $Y$, and in addition*

$$\mathrm{GenEff}_{\tau,w}(\Xi;\ P;\ \Lambda) = \mathrm{Eff}_w(\Xi;\ P;\ \Lambda).$$

To obtain the sufficient conditions which would guarantee that the set of $(\tau, w)$-generalized solutions to the problem (4.7) is nonempty, we make use of the scalarization of this problem in the form (6.1).

Let $\mathrm{sc}_\tau^- P_\lambda : \Xi \to \mathbb{R}$ denote the upper $\tau$-semicontinuous envelope of the functional

$$P_\lambda(\alpha,\rho) = \int_\Omega P(\alpha,\rho)\lambda(x)\,dx \quad \text{with some} \quad \lambda \in \Lambda,$$

that is, $\mathrm{sc}_\tau^- P_\lambda$ is the smallest upper $\tau$-semicontinuous functional minorized by $P_\lambda$ on $\Xi$. Then, following the direct method in the Calculus of Variations, we get:

**Proposition 7.2.** *Let $\Xi$ be a nonempty subset of $\mathcal{A} \times C(0,T; L^\infty(\Omega) \cap BV(\Omega))$. Then for a fixed $\lambda \in \Lambda$ every maximizing sequence for the scalar problem*

$$\sup_{(\alpha,\rho)\in\Xi} \mathrm{sc}_\tau^- P_\lambda(\alpha,\rho)$$

*has a $\tau$-cluster pair which is a maximizer of $\mathrm{sc}_\tau^- P_\lambda$ on $\Xi$, i.e., $\mathrm{Sol}(\Xi;\ \mathrm{sc}_\tau^- P_\lambda) \neq \emptyset$.*

We are now ready to prove the main result of this section.

**Theorem 7.3.** *Assume that the vector optimization problem (4.7) is regular. Let $P : \Xi \to L^2(\Omega)$ be a given objective mapping (not necessary $(\Lambda, \tau\times w)$-upper semicontinuous on $\Xi$). Then the following inclusion is valid:*

$$\bigcup_{\lambda\in\Lambda^\sharp} \operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} \mathrm{sc}_\tau^- P_\lambda(\alpha,\rho) \subseteq \mathrm{GenEff}_{\tau,w}(\Xi;\ P;\ \Lambda). \tag{7.1}$$

*Proof.* To begin with, we note that for the cone of positive elements $\Lambda$ in $L^2(\Omega)$ we have that $\mathrm{cor}\,(\Lambda) \subset \Lambda^\sharp$ (see [12]). Hence, the quasi interior $\Lambda^\sharp$ of $\Lambda$ is nonempty. Let $\lambda$ be any element of $\Lambda^\sharp$. Then, by Proposition 7.2, there exists at least one pair $(\alpha^*,\rho^*) \in \Xi$ such that

$$(\alpha^*,\rho^*) \in \operatorname*{Argmax}_{(\alpha,\rho)\in\Xi} \mathrm{sc}_\tau^- P_\lambda(\alpha,\rho). \tag{7.2}$$

Since $\mathrm{sc}_\tau^- P_\lambda(\alpha,\rho)$ is the upper $\tau$-semicontinuous envelope of the functional

$$P_\lambda(\alpha,\rho) = \int_\Omega P(\alpha,\rho)\lambda(x)\,dx,$$

it follows that there exists a sequence $\left\{(\alpha^k,\rho^k)\right\}_{k=1}^\infty \subset \Xi$ such that $(\alpha^k,\rho^k) \xrightarrow{\tau} (\alpha^*,\rho^*)$ and

$$\lim_{k\to\infty} \int_\Omega P(\alpha^k,\rho^k)\lambda(x)\,dx = \mathrm{sc}_\tau^- P_\lambda(\alpha^*,\rho^*) \tag{7.3}$$

$$\overset{\text{by condition (7.2)}}{\geq} \mathrm{sc}_\tau^- P_\lambda(\alpha,\rho) \geq \int_\Omega P(\alpha,\rho)\lambda(x)\,dx, \quad \forall (\alpha,\rho)\in\Xi.$$

Since $\Lambda^\sharp \cup 0$ is a nontrivial convex cone in $L^2(\Omega)$ with nonempty algebraical interior, it follows that it is a reproducing cone in $L^2(\Omega)$, that is, $\left[\Lambda^\sharp \cup 0\right] - \left[\Lambda^\sharp \cup 0\right] = L^2(\Omega)$ (see

[12]). Then, following Peressini [16] and Borwein [2], we get that in $L^2(\Omega)$ the ordering cone $\Lambda$ is normal with respect to the norm topology of $L^2(\Omega)$, that is,

$$y <_\Lambda z \quad \Longrightarrow \quad \|y\|_{L^2(\Omega)} < \|z\|_{L^2(\Omega)}. \tag{7.4}$$

Now, turning back to the formula (7.3), we have that there exist an integer $\widehat{k} \in \mathbb{N}$ and an element $\widehat{y} \in L^2(\Omega)$ such that

$$\int_\Omega P(\alpha^k, \rho^k)\lambda(x)\, dx > \int_\Omega \widehat{y}(x)\lambda(x)\, dx, \quad \forall\, k > \widehat{k}.$$

Since $\lambda \in \Lambda^\sharp$, this implies $P(\alpha^k, \rho^k) >_\Lambda \widehat{y}$ for all $k > \widehat{k}$. Using the normality property (7.4) of the cone $\Lambda$ for the norm topology of $L^2(\Omega)$, we come to the conclusion: there exists a constant $C > 0$ such that $\|P(\alpha^k, \rho^k)\|_{L^2(\Omega)} \le C$ for all $k > \widehat{k}$. Hence, without loss of generality, we may suppose that the sequence $\left\{P(\alpha^k, \rho^k)\right\}_{k=1}^\infty$ is bounded in $L^2(\Omega)$. So, by Banach-Alaoglu Theorem, there exist an element $\eta \in L^2(\Omega)$ and a subsequence of $\left\{P(\alpha^k, \rho^k)\right\}_{k=1}^\infty$ (still denoted by suffix $k$) such that $P(\alpha^k, \rho^k) \rightharpoonup \eta$ in $L^2(\Omega)$ as $k \to \infty$.

For now we assume that

$$(\alpha^*, \rho^*) \notin \mathrm{GenEff}_{\tau,w}(\Xi;\, P;\, \Lambda). \tag{7.5}$$

Then, as follows from Definition 7.3, $\eta \notin \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$. Hence, there can be found an element $\xi \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$ such that $\xi >_\Lambda \eta$. Therefore $\xi - \eta \in \Lambda \setminus \{0\}$, and using the fact that $\lambda \in \Lambda^\sharp$, we just come to the inequality

$$\int_\Omega \eta(x)\lambda(x)\, dx < \int_\Omega \xi(x)\lambda(x)\, dx \tag{7.6}$$

which is equivalent to

$$\lim_{k\to\infty} \int_\Omega P(\alpha^k, \rho^k)\lambda(x)\, dx < \int_\Omega \xi(x)\lambda(x)\, dx.$$

On the other hand, for the element $\xi \in \mathrm{Sup}^{\Lambda,w}_{(\alpha,\rho)\in\Xi} P(\alpha, \rho)$ there exists a sequence

$$\left\{(\widetilde{\alpha}^k, \widetilde{\rho}^k)\right\}_{k=1}^\infty \subset \Xi \ \text{ such that } \ P(\widetilde{\alpha}^k, \widetilde{\rho}^k) \rightharpoonup \xi \ \text{ in } \ L^2(\Omega).$$

Since the set $\Xi$ is sequentially $\tau$-compact, we may suppose that $(\widetilde{\alpha}^k, \widetilde{\rho}^k) \xrightarrow{\tau} (\widetilde{\alpha}^*, \widetilde{\rho}^*) \in \Xi$. Then, by inequality (7.3), we deduce

$$\lim_{k\to\infty} \int_\Omega P(\alpha^k, \rho^k)\lambda(x)\, dx \ge \int_\Omega P(\widetilde{\alpha}^i, \widetilde{\rho}^i)\lambda(x)\, dx, \quad \forall\, i \in \mathbb{N}. \tag{7.7}$$

Passing to the limit in (7.7) as $i \to \infty$, we get

$$\lim_{k\to\infty} \int_\Omega P(\alpha^k, \rho^k)\lambda(x)\, dx \ge \int_\Omega \xi(x)\lambda(x)\, dx.$$

However, this contradicts (7.6) and hence (7.5). Thus $(\alpha^*, \rho^*)$ is the $(\tau, w)$-generalized solution to the traffic vector optimization problem (4.7). □

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