Ukrainian Mathematical Bulletin
Volume 6 **(2009)**, № 4, 431 – 467

Suboptimal boundary control of the Bénard system in a cylindrically perforated domain

Vladimir V. Gotsulenko, Peter I. Kogut

(*Presented by A. E. Shishkov*)

Abstract. We study an optimal boundary control problem for the Bénard system in a cylindrically perforated domain. The control is the boundary velocity field supported on the "vertical" sides of thin cylinders. We show that an optimal solution to some limit problem in a nonperforated domain can be used as a basis for the construction of the so-called asymptotically suboptimal controls for the original control problem.

2000 MSC. 35Q30, 49J20, 35B27, 35B40.

Key words and phrases. Suboptimal control, Bénard system, variational convergence, convergence in variable spaces, homogenized problem.

Introduction

Optimal control problems for hydrodynamic systems have been widely studied by many authors (see, for example [2,18–20,22,23,33]). However, as was shown in [22], the numerical realization of optimality systems for such problems is very complicated even in the case of smooth admissible controls. This is caused by a nonlinearity of the control object, on the one hand, and by the geometry a domain where the hydrodynamic process is considered, on the other hand. Perhaps, mainly the above concerns the optimal control problems for such systems in densely perforated domains with quickly oscillating boundaries. Therefore, for such processes, it becomes actual to develop the methods of construction of suboptimal controls.

A Couette cell is a classical model object for the investigation of many hydrodynamical problems. However, the flow properties in a Couette cell with several inner cylinders have not been fully studied yet. Mainly this

Received 29.05.2009

ISSN 1812 – 3309.

concerns the incompressible viscous flow subject to a temperature field variation (the so-called Bénard problem). It is well known that an increase in the number of cylinders in a Couette cell makes the computer modeling of the controlled flow practically untenable. The high dimension of a system of nonlinear algebraic equations obtained through discretization of initial equations of the Navier–Stokes type makes it very difficult to solve even using modern computers. The above-mentioned fact provides even greater complexity when finding an optimal control for such objects. Therefore, the aim of this work is (in the case of the Bénard system) to find the control law providing a required closeness to optimal characteristics and also having a simple enough structure in view of its practical realization. The proposed approach is based on the investigation of the asymptotic behavior of the considered problem in the case where the number of inner cylinders in a generalized Couette cell increases infinitely while their diameters tend to zero.

1. Preliminaries and notation

To begin with, we give a formal description of the generalized Couette cell in $\mathbb{R}^3$. Let $\widetilde{\Omega}$ be an open bounded subset of $\mathbb{R}^2$ with a simple connected smooth boundary $\partial\widetilde{\Omega}$. Let $\widetilde{Y} = [-1/2, 1/2)^2$, and let Q be a compact subset of $\widetilde{Y}$ with a smooth boundary ∂Q such that $0 \in \text{int}Q$. Let $A = B(\mathbf{0}, r_0)$ be an open ball centered at the origin with a radius $r_0 < 1/2$ so that $Q \subset\subset A$. Let $\{\varepsilon\}$ be a sequence of positive numbers such that $\varepsilon = N^{-1}$, where $N \to \infty$. We introduce the following sets:

$$\begin{cases} \Theta_\varepsilon = \{\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 : \varepsilon(r_\varepsilon Q + \mathbf{k}) \subset\subset \widetilde{\Omega}\}; \\ \widetilde{\mathrm{T}}_\varepsilon^{\mathbf{k}} = \varepsilon(r_\varepsilon Q + \mathbf{k}), \quad \mathbf{k} \in \Theta_\varepsilon; \\ \widetilde{\mathrm{T}}_\varepsilon = \bigcup\limits_{\mathbf{k}\in\Theta_\varepsilon} \widetilde{\mathrm{T}}_\varepsilon^{\mathbf{k}}, \quad \mathrm{T}_\varepsilon = \widetilde{\mathrm{T}}_\varepsilon \times [0, \ell]; \\ \widetilde{\Omega}_\varepsilon = \widetilde{\Omega} \setminus \widetilde{\mathrm{T}}_\varepsilon, \quad \Omega_\varepsilon = \widetilde{\Omega}_\varepsilon \times (0, \ell). \end{cases}$$

It is clear that $\Omega_\varepsilon = \Omega \setminus \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon(r_\varepsilon Q + \mathbf{k}) \times [0, \ell]\right]$, where $\Omega = \widetilde{\Omega} \times (0, \ell)$. Here, ℓ denotes the Couette cell length, and r_ε is the cross-size of the thin cylinders. As a result, by a generalized Couette cell, we mean the open set Ω_ε in $\mathbb{R}^3$ which is a periodically perforated by thin cylinders $\mathrm{T}_\varepsilon^{\mathbf{k}} = \widetilde{\mathrm{T}}_\varepsilon^{\mathbf{k}} \times [0, \ell]$. Since each of the cylinders $\mathrm{T}_\varepsilon^{\mathbf{k}}$ can be created as an ε–homothetic transformation $\mathrm{T}_\varepsilon^{\mathbf{k}} = \{(x_1, x_2, x_3) : (x_1, x_2) \in \varepsilon(r_\varepsilon Q + \mathbf{k}), 0 \le x_3 \le \ell\}$, it follows that Ω_ε is the periodically perforated domain with the cell of periodicity $\Lambda = \varepsilon\widetilde{Y} \times [0, \ell]$. Let Γ_ε be the boundary of Ω_ε. Then

$$\Gamma_\varepsilon = \Gamma^1_\varepsilon \bigcup \Gamma^2_\varepsilon \bigcup \Gamma^3 \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \widetilde{\partial}\mathrm{T}^{\mathbf{k}}_\varepsilon,$$

where $\Gamma^1_\varepsilon = \widetilde{\Omega}_\varepsilon \times \{0\}$, $\Gamma^2_\varepsilon = \widetilde{\Omega}_\varepsilon \times \{\ell\}$, $\Gamma^3 = \partial\widetilde{\Omega} \times [0,\ell]$, $\widetilde{\partial}\mathrm{T}^{\mathbf{k}}_\varepsilon$ is the lateral surface of a thin cylinder $\mathrm{T}^{\mathbf{k}}_\varepsilon$.

Throughout the paper, we suppose that C and C_i (where i is any subscript) denote constants independent of ε. For any subset $E \subset \mathbb{R}^n$, we denote, by $|E|$, its n-dimensional Lebesgue measure $\mathcal{L}^n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of the manifold ∂E on $\mathbb{R}^n$. We will use the standard notations for the Lebesgue function space $L^p(\Omega)$ and the Sobolev spaces $H^m(\Omega)$ (see [1]). Let $L^2_0(\Omega) = \left\{q \in L^2(\Omega) \,:\, \int_\Omega q\,dx = 0\right\}$ with the norm

$$\|p\|^2_{L^2_0(\Omega)} = \int\limits_\Omega \left(p(x) - \int\limits_\Omega p(x)\,dx\right)^2 dx,$$

and let $L^2(\Omega, d\mu)$ be the Banach space of squared integrable functions in Ω with respect to the measure μ. If the measure μ is the Lebesgue one, we abbreviate the notation using $L^2(\Omega)$.

By the (harmonic) capacity of a set $E \subset \Omega$, we call a value $\mathrm{cap}(E,\Omega)$ which is defined as the infimum of $\int_\Omega |\nabla y|^2\,dx$ over the set of all functions $y \in H^1_0(\Omega)$ such that $y \geq 1$ a.e. in E. Let $\mathcal{M}_b(\Omega)$ be the space of bounded Borel measures on Ω. Let $\mathcal{M}^+_0(\Omega)$ be the cone of all nonnegative Borel measures μ on Ω such that $\mu(B) = 0$ for every set $B \subseteq \Omega$ with $\mathrm{cap}(B,\Omega) = 0$, and $\mu(B) = \inf\{\mu(U) \,:\, U - \text{quasi open},\ B \subseteq U\}$ for every Borel set $B \subseteq \Omega$. The space $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C^\infty_0(\Omega)$. For $m \geq 0$, we introduce the subspaces $H^m_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $H^m(\Omega)$, and the dual spaces $H^{-m}(\Omega) = (H^m_0(\Omega))^*$. The canonical pairing between a Sobolev space $H^s(\Omega)$ $(s > 0)$ and its dual space is denoted by $\langle\cdot,\cdot\rangle_{H^{-s}(\Omega);H^s(\Omega)}$. The space of traces $H^l(\partial\Omega)$ is a restriction of $H^{l+1/2}(\Omega)$ to the boundary (see [36]). The vector-valued counterparts of these spaces are denoted by boldface symbols, e.g. $\mathbf{L}^r(\Omega)$, $\mathbf{H}^m(\Omega)$, $\mathbf{H}^l(\partial\Omega)$, $\mathbf{H}^m_0(\Omega)$, and $\mathbf{C}^\infty_0(\Omega)$. If $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^N$ is a vector-valued function, then the gradient of $\mathbf{u}$ is a $N \times N$ tensor: $\nabla\mathbf{u} = (\partial u_i/\partial x_j)_{1\leq i,j\leq N}$, and the inner product of two $N \times N$ tensors $A = (a_{ij})$ and $B = (b_{ij})$ is denoted by $A : B = \mathrm{tr}\left({}^tAB\right) = \sum_{1\leq i,j\leq N} a_{ij}b_{ij}$.

We will also use the spaces of solenoidal vector fields

$$\mathbf{C}^\infty_{0,sol}(\Omega) = \left\{\mathbf{y} \in \mathbf{C}^\infty_0(\Omega) \;:\; \mathrm{div}\ \mathbf{y} = \sum_{i=1}^3 \frac{\partial y_i}{\partial x_i} = 0 \ \text{ in } \ \Omega\right\},$$

$$\mathbf{H}^m_{sol}(\Omega)=\left\{\mathbf{y}\in\mathbf{H}^m(\Omega)\ :\ \nabla\cdot\mathbf{y}=0,\ \int\limits_{\partial\Omega}\mathbf{y}\cdot\mathbf{n}\,d\mathcal{H}^2=0\right\},$$

$$\mathbf{H}^m_{0,sol}(\Omega)=\ \text{the closure of}\ \ \mathbf{C}^\infty_{0,sol}(\Omega)\ \text{in}\ \mathbf{H}^m(\Omega)\text{-norm},$$

where, when $m=0$, $\int_{\partial\Omega}\mathbf{y}\cdot\mathbf{n}\,d\mathcal{H}^2$ is the $\left\langle\mathbf{H}^{-1/2}(\partial\Omega);\mathbf{H}^{1/2}(\partial\Omega)\right\rangle$ duality pairing between the function $\mathbf{y}\cdot\mathbf{n}\in\mathbf{H}^{-1/2}(\partial\Omega)$ and the constant scalar function $\varphi\equiv 1\in\mathbf{H}^{1/2}(\partial\Omega)$.

The norm on $\mathbf{H}^m_{sol}(\Omega)$ and $\mathbf{H}^m_{0,sol}(\Omega)$ is chosen to be that of $\mathbf{H}^m(\Omega)$. We define the divergence operator as follows:

$$\langle\operatorname{div}\mathbf{y},\varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}=-\int\limits_\Omega\mathbf{y}\cdot\nabla\varphi\,dx,\quad\forall\,\varphi\in H^1_0(\Omega).\tag{1.1}$$

Let $\mathbf{H}(\operatorname{div},\Omega)=\left\{\mathbf{y}\in\mathbf{L}^2(\Omega):\operatorname{div}\mathbf{y}\in L^2(\Omega)\right\}$. We need the following lemma on integration by parts for functions in the space $\mathbf{H}(\operatorname{div},\Omega)$ (for the proof see [36]).

Lemma 1.1. *Let* $\mathbf{w}\in\mathbf{H}(\operatorname{div},\Omega)$. *Then* $(\mathbf{w}\cdot\mathbf{n})|_{\partial\Omega}\in H^{-1/2}(\partial\Omega)$ *and*

$$\langle\mathbf{w}\cdot\mathbf{n},v\rangle_{H^{-1/2}(\partial\Omega);H^{1/2}(\partial\Omega)}=\int\limits_\Omega v\ \operatorname{div}\ \mathbf{w}\,dx+\int\limits_\Omega\mathbf{w}\cdot\nabla v\,dx\quad\forall\,v\in H^1(\Omega).\tag{1.2}$$

To the end of this section, we define the standard bi- and trilinear forms associated with the Navier–Stokes equations

$$a_\varepsilon(\mathbf{y},\mathbf{v})=\int\limits_{\Omega_\varepsilon}\nabla\mathbf{y}:\nabla\mathbf{v}\,dx\quad\forall\,\mathbf{y},\mathbf{v}\in\mathbf{H}^1(\Omega_\varepsilon),$$

$$b_\varepsilon(\mathbf{y},q)=-\int\limits_{\Omega_\varepsilon}q\ \operatorname{div}\ \mathbf{y}\,dx\quad\forall\,\mathbf{y}\in\mathbf{H}^1(\Omega_\varepsilon),\quad\forall\,q\in L^2(\Omega_\varepsilon),$$

$$c_\varepsilon(\mathbf{y},\mathbf{v},\mathbf{w})=\int\limits_{\Omega_\varepsilon}(\mathbf{y}\cdot\nabla)\,\mathbf{v}\cdot\mathbf{w}\,dx\quad\forall\,\mathbf{y},\mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega_\varepsilon).$$

2. Statement of the optimal control problem

Let $\mathbf{z}^\partial_\varepsilon\in\mathbf{L}^2(\Omega)$ and $b\in H^1(\Omega)$ be given functions. We study a viscous incompressible flow in the cylindrically perforated domain Ω_ε with an adherence condition on the lateral side of Ω. Hereinafter, we use the following notation: $\mathbf{y}_\varepsilon$ is a velocity field, p_ε is the pressure in a liquid, and θ_ε is its temperature. We suppose that the boundary velocities on the lower and upper boundaries of Ω_ε are prescribed. With regard for the temperature field, we suppose that the flow obeys the Rayleigh–Bénard law [5]. The optimal control problem we consider is to find boundary velocities $\overline{\alpha}_\varepsilon = \left(\alpha_1^\varepsilon, \dots, \alpha_{J_\varepsilon}^\varepsilon\right)$ on the "vertical" sides of thin cylinders

$$\mathrm{T}_\varepsilon^{\mathbf{k}} = \{(x_1, x_2, x_3) : (x_1, x_2) \in \varepsilon(r_\varepsilon Q + \mathbf{k}),\ 0 \le x_3 \le \ell\} \quad \forall\, \mathbf{k} \in \Theta_\varepsilon$$

such that the velocity field in Ω_ε would be as close as possible to a given velocity field z_ε^∂.

Following [5], we adopt the following equations for the steady-state liquid motion in Ω_ε:

$$-\Delta \mathbf{y}_\varepsilon + (\mathbf{y}_\varepsilon \cdot \nabla)\, \mathbf{y}_\varepsilon = -\nabla p_\varepsilon + \theta_\varepsilon\, \overrightarrow{\mathbf{e}_3}, \tag{2.1}$$

$$\operatorname{div}\, \mathbf{y}_\varepsilon = 0, \tag{2.2}$$

$$-\Delta \theta_\varepsilon + (\mathbf{y}_\varepsilon \cdot \nabla)\, \theta_\varepsilon = \mathbf{y}_\varepsilon \cdot \overrightarrow{\mathbf{e}_3} \tag{2.3}$$

with boundary conditions

$$\mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1, \quad \mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2, \quad \mathbf{y}_\varepsilon|_{\Gamma^3} = 0, \tag{2.4}$$

$$\mathbf{y}_\varepsilon|_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}^\varepsilon, \quad \theta_\varepsilon|_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} = b|_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} \quad (j = \overline{1; J_\varepsilon}), \tag{2.5}$$

$$\theta_\varepsilon|_{\Gamma_\varepsilon^1} = 0, \quad \theta_\varepsilon|_{\Gamma_\varepsilon^2} = 0, \quad \theta_\varepsilon|_{\Gamma^3} = b|_{\Gamma^3}\,. \tag{2.6}$$

Definition 2.1. *A triplet* $(\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \mathbf{H}^1(\Omega_\varepsilon) \times L_0^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ *is said to be a solution of the boundary-value problem* (2.1)–(2.6) *if*

$$a_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{v}) + b_\varepsilon(\mathbf{y}_\varepsilon, p_\varepsilon) = \int_{\Omega_\varepsilon} \theta_\varepsilon \left(\overrightarrow{\mathbf{e}_3} \cdot \mathbf{v}\right) dx \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega_\varepsilon),$$

$$b_\varepsilon(\mathbf{y}_\varepsilon, q) = 0 \quad \forall\, q \in L_0^2(\Omega_\varepsilon),$$

$$\int_{\Omega_\varepsilon} \nabla\theta_\varepsilon \cdot \nabla\varphi\, dx + \int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon \cdot \nabla\theta_\varepsilon)\, \varphi\, dx = \int_{\Omega_\varepsilon} \left(\overrightarrow{\mathbf{e}_3} \cdot \mathbf{y}_\varepsilon\right) \varphi\, dx \quad \forall\, \varphi \in H_0^1(\Omega_\varepsilon)$$

and the boundary conditions (2.4)–(2.6) *hold true.*

By analogy with [36] and [5], it can be shown that the boundary-value problem (2.1)–(2.6) has a solution in the sense of Definition 2.1, if $b \in H^1(\Omega; \Gamma_1 \cup \Gamma_2) = \left\{\varphi \in H^1(\Omega) \ :\ \varphi|_{\Gamma_1 \cup \Gamma_2}\right\}$, where $\Gamma_1 = \widetilde{\Omega} \times 0$, $\Gamma_2 = \widetilde{\Omega} \times \ell$, and $\alpha_{\mathbf{k}_j}^\varepsilon \in \mathbf{H}^{1/2}(\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j})$ for all $j = 1, 2, \dots, J_\varepsilon$.

Let $\gamma > 0$ be an *a priori* given value. We say that the boundary velocity field $\overline{\alpha}_\varepsilon = (\alpha_{\mathbf{k}_1}^\varepsilon, \dots, \alpha_{\mathbf{k}_{J_\varepsilon}}^\varepsilon)$ is admissible, if there exists a function $\mathbf{u} \in \mathbf{H}_{sol}^1(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ (the so-called prototype of boundary controls) such that $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma$ and

$$\mathbf{u}|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1, \quad \mathbf{u}|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2, \quad \mathbf{u}|_{\Gamma^3} = 0, \quad \mathbf{u}|_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}^\varepsilon, \quad \forall\, j = 1, \dots, J_\varepsilon,$$

where

$$\mathbf{y}^1_\varepsilon = \mathbf{y}^*_\varepsilon \big|_{\Gamma^1_\varepsilon}, \quad \mathbf{y}^2_\varepsilon = \mathbf{y}^*_\varepsilon \big|_{\Gamma^2_\varepsilon}, \quad \mathbf{y}^*_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega_\varepsilon),$$
$$\mathbf{y}^*_\varepsilon \rightharpoonup \mathbf{y}^* \text{ weakly in } \mathbf{H}^2(\Omega).$$

By $\mathbb{U}^\varepsilon_\partial$, we denote the set of all admissible controls for a fixed ε, i.e.

$$\mathbb{U}^\varepsilon_\partial = \left\{ \overline{\alpha}_\varepsilon = \left(\alpha^\varepsilon_{\mathbf{k}_1}, \dots, \alpha^\varepsilon_{\mathbf{k}_{J_\varepsilon}}\right) \;\middle|\; \begin{array}{l} \alpha^\varepsilon_{\mathbf{k}} = \mathbf{u}\big|_{\widetilde{\partial}\mathrm{T}^{\mathbf{k}}_\varepsilon} \quad \forall\, \mathbf{k} \in \Theta_\varepsilon, \\ \forall\, \mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega), \quad \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma, \\ \mathbf{u}\big|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon, \quad \mathbf{u}\big|_{\Gamma^2} = \mathbf{y}^2_\varepsilon, \quad \mathbf{u}\big|_{\Gamma^3_\varepsilon} = 0 \end{array} \right\}. \tag{2.7}$$

In this case, we have the following result concerning the solvability of the boundary-value problem (2.1)–(2.6) (see, e.g., [12]):

Theorem 2.1. *Assume that $b \in H^1(\Omega)$, $\overline{\alpha}_\varepsilon \in \mathbb{U}^\varepsilon_\partial$, and $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ is a prototype of the boundary control. Then there exists a triplet of functions*

$$(\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times \left[H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon)\right] \times H^1(\Omega_\varepsilon)$$

such that $(\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ is a weak solution of (2.1)–(2.6), and

$$\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \quad \theta - b\big|_{\Omega_\varepsilon} \in H^1_0(\Omega_\varepsilon).$$

We define the set

$$\Xi_\varepsilon = \left\{ (\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \;\middle|\; \begin{array}{l} \overline{\alpha}_\varepsilon \in \mathbb{U}^\varepsilon_\partial, \quad \theta_\varepsilon - b \in H^1_0(\Omega_\varepsilon), \\ \mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \\ (\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \\ \text{is a weak solution of (2.1)–(2.6).} \end{array} \right\} \tag{2.8}$$

Then the optimal control problem we consider can be precisely stated as follows: seek a tuple $\left(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon, \theta^0_\varepsilon\right) \in \Xi_\varepsilon$ such that

$$\mathbb{P}_\varepsilon \;:\; \mathcal{I}_\varepsilon\left(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon, \theta^0_\varepsilon\right) = \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon} \mathcal{I}_\varepsilon\left(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon\right), \tag{2.9}$$

where the cost functional takes the form

$$\mathcal{I}_\varepsilon\left(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon\right) = \int\limits_{\Omega_\varepsilon} \left|\mathbf{y}_\varepsilon - \mathbf{z}_\varepsilon^\partial\right|^2 dx + \frac{\beta\,\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int\limits_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}^\varepsilon\right|^2 d\mathcal{H}^2. \quad (2.10)$$

Here, $\beta > 0$ is a weight coefficient. Hereinafter, we say that Ξ_ε is the set of admissible solutions to problem (2.9). As follows from the definition of Ξ_ε and *a priori* estimates for the weak solutions to the Bénard problem (2.1)–(2.6) (see [5]), the set Ξ_ε is uniformly bounded and closed with respect to the product of weak topologies of $\mathbf{H}^{1/2}(\widetilde{\partial}\mathrm{T}_\varepsilon) \times \mathbf{H}^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$, and the cost functional is lower semicontinuous with respect to the above topology. Hence (see [18]), the optimal control problem (2.9) has a solution if and only if $\Xi_\varepsilon \neq \emptyset$. In view of this, we assume that $\Xi_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$.

Since the boundary-value problem (2.1)–(2.6) may have a nonunique solution under a fixed boundary control, we define, in what follows, the binary relation $\langle L; \Xi_\varepsilon\rangle$ on each of the sets Ξ_ε by the rule

$$(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)\, L\, (\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon) \text{ if and only if } \overline{\alpha}_\varepsilon = \widehat{\overline{\alpha}}_\varepsilon.$$

It is easily seen that $\langle L; \Xi_\varepsilon\rangle$ is an equivalence relation. Hereinafter, we will not distinguish the solutions belonging to the same class of equivalence.

3. Formalism of singular measures

We begin this section with the description of a class of admissible solutions to the optimal control problem (2.8)–(2.10), using the approach of singular periodic Borel measures [38] (see also [11,12]). In order to do this, we introduce the set $Q^r = \{rx \in \mathbb{R}^2 : \forall\, x \in Q\}$, where $0 < r < 1$ is a fixed parameter. Let η_0^r be the normalized periodic Borel measure on $\mathbb{R}^2$ with the periodicity cell $\widetilde{Y} = [-1/2, 1/2)^2$ such that η_0^r is concentrated and uniformly distributed on the set ∂Q^r and is proportional to the one-dimensional Hausdorff measure $\mathcal{H}^1$ on $\mathbb{R}^1$. It is clear that, in this case, $\eta_0^r(\widetilde{Y} \backslash \partial Q^r) = 0$. We now consider the measure $d\eta^r = d\eta_0^r \times dx_3$ in $Y = [-1/2, 1/2)^2 \times [0, 1)$. It is easy to see that this measure is concentrated on the set $\partial Q^r \times [0, 1)$. For any smooth function g, we have

$$\int\limits_Y g\, d\eta^r = \int\limits_0^1 \int\limits_{\widetilde{Y}} g\, dx_3\, d\eta_0^r = \left[\mathcal{H}^2\left(\partial Q^r \times [0,1)\right)\right]^{-1} \int\limits_{\partial Q^r \times [0,1)} g\, d\mathcal{H}^2.$$

The properties of the Hausdorff measure yield

$$\mathcal{H}^2\left(\partial Q^r \times [0,1)\right) = \mathcal{H}^1\left(\partial Q^r\right) = r\mathcal{H}^1\left(\partial Q\right).$$

Using the notation $|\partial Q|_H = \mathcal{H}^1(\partial Q)$, the previous relation can be rewritten in the form

$$r\int_Y g d\eta^r = r\int_0^1 \int_{\widetilde{Y}} g\, dx_3 d\eta_0^r = |\partial Q|_H^{-1} \int_{\partial Q^r\times[0,1)} g\, d\mathcal{H}^2. \tag{3.1}$$

For every Borel set $B \subset \mathbb{R}^3$, we set $\eta_\varepsilon^r(B) = \varepsilon^3 \eta^r(\varepsilon^{-1}B)$, where $r = r_\varepsilon$ ($\lim_{\varepsilon\to 0} r_\varepsilon = 0$). Then

$$\int_{\varepsilon Y} d\eta_\varepsilon^r = \varepsilon^3 \int_Y d\eta^r = \varepsilon^3.$$

Hence, $d\eta_\varepsilon^r \rightharpoonup dx$ in the space of Borel measures [38]. In other words,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \varphi\, d\eta_\varepsilon^r = \int_{\mathbb{R}^3} \varphi\, dx \quad \forall \varphi \in C_0^\infty\left(\mathbb{R}^3\right). \tag{3.2}$$

Remark 3.1. It is easy to see that the scaling measure η_ε^r belongs to the class $\mathcal{M}_0^+(\Omega)$ of nonnegative Borel measures η on Ω such that, for any Borel set $B \subseteq \Omega$, we have $\eta(B) = 0\ \forall B \subset \Omega : \operatorname{cap}(B,\Omega) = 0$ and $\eta(B) = \inf\{\eta(U) : U \text{ is quasiopen}, B \subseteq U\}$. Note that if $\eta \in \mathcal{M}_0^+(\Omega)$, then the functions of $\mathbf{H}^1(\Omega)$ are defined η-almost everywhere and are η-measurable in Ω. Hence, the space $\mathbf{H}^1(\Omega) \cap \mathbf{L}^2(\Omega, d\eta)$ is well defined (see [14]).

In view of this, the term

$$\frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\widetilde{\partial} \mathrm{T}_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}^\varepsilon\right|^2 d\mathcal{H}^2$$

can be rewritten in the equivalent form

$$\begin{aligned}\frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int_{\widetilde{\partial} \mathrm{T}_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}^\varepsilon\right|^2 d\mathcal{H}^2 \\ &= \beta\varepsilon^2 |\partial Q|_H \sum_{j=1}^{J_\varepsilon} \int_0^1 \int_{\varepsilon(\widetilde{Y}+\mathbf{k}_j)} |\mathbf{u}_\varepsilon|^2\, d\eta_0^{r_\varepsilon}\left(\frac{x'}{\varepsilon}\right) dx_3 \\ &= \beta\varepsilon^3 |\partial Q|_H \sum_{j=1}^{J_\varepsilon} \int_0^1 \int_{\varepsilon(Y+\mathbf{k}_j)} |\mathbf{u}_\varepsilon|^2\, d\eta^{r_\varepsilon}\left(\frac{x}{\varepsilon}\right) \\ &= \beta |\partial Q|_H \int_\Omega |\mathbf{u}_\varepsilon|^2\, d\eta_\varepsilon^{r_\varepsilon},\end{aligned} \tag{3.3}$$

where $\mathbf{u}_\varepsilon$ is a prototype of the control function $\overline{\alpha}_\varepsilon = (\alpha^\varepsilon_{\mathbf{k}_1}, \dots, \alpha^\varepsilon_{\mathbf{k}_{J_\varepsilon}})$.

Remark 3.2. Note that any admissible control $\overline{\alpha}_\varepsilon = (\alpha^\varepsilon_{\mathbf{k}_1}, \dots, \alpha^\varepsilon_{\mathbf{k}_{J_\varepsilon}})$ can be obviously interpreted, due to (3.3), as an element of $\mathbf{L}^2(\Omega, d\eta^{r_\varepsilon}_\varepsilon)$. Indeed, let $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ be a prototype of $\overline{\alpha}_\varepsilon$. Then, using the imbedding result $\mathbf{H}^2(\Omega) \subset \mathbf{C}(\Omega)$, we get $\mathbf{u}_\varepsilon \in \mathbf{C}(\Omega)$. Hence, $\mathbf{u}_\varepsilon$ is an $\eta^{r_\varepsilon}_\varepsilon$−measurable function.

Definition 3.1. *Let $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon$ be any admissible solution to problem* (2.9)*. Then we say that a tuple $(\mathbf{u}_\varepsilon, \check{\mathbf{y}}_\varepsilon, \check{p}_\varepsilon, \check{\theta}_\varepsilon) \in \mathbb{X}_\varepsilon$ is a prototype to $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ for the problem $(\widehat{\mathbb{P}}_\varepsilon)$, if $\mathbf{u}_\varepsilon$ is a control prototype for $\overline{\alpha}_\varepsilon \in \mathbb{U}^\partial_\varepsilon$, and $(\check{\mathbf{y}}_\varepsilon, \check{p}_\varepsilon, \check{\theta}_\varepsilon)$ are some extensions of the functions $(\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ on the whole Ω, i.e. $(\check{\mathbf{y}}_\varepsilon, \check{p}_\varepsilon, \check{\theta}_\varepsilon)|_{\Omega_\varepsilon} = (\mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)$. Here, $\mathbb{X}_\varepsilon = [\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{L}^2(\Omega, d\eta^{r_\varepsilon}_\varepsilon)] \times [\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^1(\Omega))] \times [L^2_0(\Omega) \cap L^2_0(\Omega_\varepsilon)] \times H^1(\Omega)$.*

Remark 3.3. Since the perforated domain Ω_ε satisfies the so-called "condition of strong connectedness" (see [31]), it follows that there exist an extension operator $\mathbf{P}_\varepsilon : \mathbf{H}^1(\Omega_\varepsilon) \to \mathbf{H}^1(\Omega)$ and a constant $C > 0$ independent of ε such that $\|\mathbf{P}_\varepsilon \mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega)} \le C \|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}$ $\forall \mathbf{y}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)$. So, we can assume that $\check{\mathbf{y}}_\varepsilon = \mathbf{P}_\varepsilon \mathbf{y}_\varepsilon$.

In fact, it is not obvious how to construct such extensions of the functions p_ε and θ_ε to $H^1(\Omega)$ and $L^2_0(\Omega)$, respectively. For that purpose, following [3, 10], we introduce an abstract framework of the hypotheses on cylindrical holes. By $\sim$, we denote the extension by zero onto the cylindrical holes $\mathrm{T}^{\mathbf{k}_j}_\varepsilon$. Let $\{\overrightarrow{\mathbf{e}_k}\}_{k=1,2,3}$ be the canonical basis in $\mathbb{R}^3$. We introduce a number of auxiliary hypotheses.

Let us assume that there exists a triplet $(\mathbf{w}^\varepsilon_k, q^\varepsilon_k, \mu_k)$ $(1 \le k \le 3)$, ω_ε, μ, and a linear operator $R_\varepsilon \in \mathcal{L}\left(\mathbf{H}^1(\Omega), \mathbf{H}^1(\Omega_\varepsilon)\right)$ such that

(H1) $\mathbf{w}^\varepsilon_k \in \mathbf{H}^1(\Omega)$, $q^\varepsilon_k \in L^2_0(\Omega)$, $\mu_k \in \mathbf{W}^{-1,\infty}(\Omega)$, $\mu \in W^{-1,\infty}(\Omega)$;

(H2) $\nabla \cdot \mathbf{w}^\varepsilon_k = 0$ in Ω and $\mathbf{w}^\varepsilon_k = 0$ on T_ε;

(H3) $\mathbf{w}^\varepsilon_k \rightharpoonup \overrightarrow{\mathbf{e}}_k$ in $\mathbf{H}^1(\Omega)$, and $q^\varepsilon_k \rightharpoonup 0$ in $L^2_0(\Omega)$;

(H4) $\forall \mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega)$ and $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ in $\mathbf{H}^1(\Omega)$ and $\mathbf{v}_\varepsilon = 0$ on T_ε, it follows that

$$\lim_{\varepsilon \to 0} \langle \nabla q^\varepsilon_k - \Delta \mathbf{w}^\varepsilon_k, \varphi\, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = \langle \mu_k, \varphi\, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}, \quad \forall \varphi \in C^\infty_0(\Omega);$$

(H5) if $\mathbf{y} \in \mathbf{H}^1_0(\Omega_\varepsilon)$, then $R_\varepsilon(\chi_\varepsilon \mathbf{y}) = \mathbf{y}$ in Ω_ε, if $\nabla \cdot \mathbf{y} = 0$ in Ω, then $\nabla \cdot (R_\varepsilon \mathbf{y}) = 0$ in Ω_ε, and $\|R_\varepsilon \mathbf{y}\|_{\mathbf{H}^1_0(\Omega_\varepsilon)} \le C \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)}$, where a constant C is independent of ε;

(H6) $\omega_\varepsilon \in \mathrm{H}^1(\Omega)$, $\omega_\varepsilon = 0$ on $\mathrm{T}_\varepsilon^{\mathbf{k}_j}$ $\left(j = \overline{1; J_\varepsilon}\right)$;

(H7) $\omega_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega)$;

(H8) $\forall\, \upsilon_\varepsilon \in H^1(\Omega)$ and $\forall\, \upsilon \in H^1(\Omega)$ such that $\upsilon_\varepsilon \rightharpoonup \upsilon$ in $H^1(\Omega)$ and $\upsilon_\varepsilon = 0$ on T_ε, it follows that

$$\lim_{\varepsilon\to 0} \langle -\Delta\omega_\varepsilon, \varphi\, \upsilon_\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle \mu, \varphi\, \upsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Remark 3.4. In what follows, we will show that the hypotheses mentioned above hold true for problem (2.8)–(2.10). Moreover, as an evident consequence of hypotheses (H1)–(H5), we have the following observation: there exists a linear operator $P_\varepsilon \in \mathcal{L}(L_0^2(\Omega_\varepsilon); L_0^2(\Omega))$ such that (see [35]):

$$\langle \nabla\left[P_\varepsilon q_\varepsilon\right], \mathbf{w}\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon \mathbf{w}\rangle_{H^{-1}(\Omega_\varepsilon);H_0^1(\Omega_\varepsilon)}, \quad \forall\, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{3.4}$$

(i) $P_\varepsilon q_\varepsilon = q_\varepsilon$ in $L_0^2(\Omega_\varepsilon)$,

(ii) $\|P_\varepsilon q_\varepsilon\|_{L_0^2(\Omega)} \le C\|q_\varepsilon\|_{L_0^2(\Omega_\varepsilon)}$,

(iii) $\|\nabla\left[P_\varepsilon q_\varepsilon\right]\|_{\mathbf{H}^{-1}(\Omega)} \le C\|\nabla q_\varepsilon\|_{\mathbf{H}^{-1}(\Omega_\varepsilon)}$, where a constant C does not depend on q_ε and ε.

Note that the identification of the measures μ_k for a wide class of boundary-value problems in perforated domains was done by many authors (see, for instance, [3, 9, 10, 31, 33]). However, this aspect was first studied in [31].

4. Convergence in the variable space $\mathbb{X}_\varepsilon$

The characteristic feature of the optimal control problem $(\widehat{\mathbb{P}}_\varepsilon)$ is the fact that the space of admissible solutions $\mathbb{X}_\varepsilon$ depends on ε. Since our main goal in the next sections is to study the asymptotic behavior of the $\widehat{\mathbb{P}}_\varepsilon$-problem as ε tends to zero, we recall the main types of convergence in variable spaces following [7, 38] (see also [30]). Let $\{\eta_\varepsilon^r\}_{\varepsilon>0}$ be a family of the periodic Borel measures. Let $\left\{\mathbf{u}_\varepsilon^r \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^r)\right\}$ be a bounded sequence, i.e., $\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon^r|^2\, d\eta_\varepsilon^r < +\infty$.

1. The weak convergence $\mathbf{u}_\varepsilon^r \rightharpoonup \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ means that $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and
$$\lim_{\varepsilon\to 0} \int_\Omega \mathbf{u}_\varepsilon^r \varphi\, d\eta_\varepsilon^r = \int_\Omega \mathbf{u}\varphi\, dx$$
for any $\varphi \in C_0^\infty(\Omega)$;

2. The strong convergence $\mathbf{u}_\varepsilon^r \to \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ means that $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and
$$\lim_{\varepsilon\to 0} \int_\Omega \mathbf{u}_\varepsilon^r \cdot \mathbf{w}_\varepsilon^r \, d\eta_\varepsilon^r = \int_\Omega \mathbf{u} \cdot \mathbf{w} \, dx$$
for every $\mathbf{w}_\varepsilon^r \rightharpoonup \mathbf{w}$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$.

The following properties of the convergence in variable spaces hold (see [38]):

(a) *Compactness criterion:* if a sequence is bounded in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, then this sequence is compact in the sense of weak convergence;

(b) *Property of lower semicontinuity:* if $\mathbf{u}_\varepsilon^r \rightharpoonup \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, then
$$\liminf_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon^r|^2 \, d\eta_\varepsilon^r \geq \int_\Omega |\mathbf{u}|^2 \, dx. \tag{4.1}$$

Let χ_ε be the characteristic function of Ω_ε. Then, by periodicity of Ω_ε, the representation
$$\chi_\varepsilon(x) = \chi_\varepsilon^r(x) = \chi^r\left(\frac{x}{\varepsilon}\right)$$
holds true, where
$$\chi^r(x) = \begin{cases} 1, & x \in Y \setminus [Q^r \times [0,1)], \\ 0, & x \in Q^r \times [0,1) = rQ \times [0,1). \end{cases}$$
This means that the Radon measure $d\eta_\varepsilon^r := \chi_\varepsilon(x)dx$ can be viewed as a scaling measure $d\eta_\varepsilon^r$ such that $\eta_\varepsilon^r(B) = \varepsilon^3\eta^r(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^3$, where η^r is a Y-periodic Borel measure in $\mathbb{R}^3$ concentrated on $Y \setminus [Q^r \times [0,1)]$ and proportional there to the Lebesgue measure $\mathcal{L}^3$, i.e., $\int_Y d\eta^r = |\widetilde{Y} \setminus Q^r|_{\mathcal{L}^2} = |\widetilde{Y}|_{\mathcal{L}^2} - r^2\,|Q|_{\mathcal{L}^2} = 1 - r^2\,|Q|_{\mathcal{L}^2}$. Therefore, $d\eta^r \rightharpoonup dx$ as $r \to 0$, and
$$\lim_{\varepsilon\to 0} \int_\Omega \varphi\chi_\varepsilon \, dx = \lim_{\varepsilon\to 0} \int_\Omega \varphi\chi^{r(\varepsilon)}\left(\frac{x}{\varepsilon}\right) dx = |Y| \int_\Omega \varphi \, dx = \int_\Omega \varphi \, dx \tag{4.2}$$
for every $\varphi \in C_0^\infty(\Omega)$.

Taking the definition of strong convergence in the space $L^2(\Omega, \chi_\varepsilon \, dx)$ and relation (4.2) into account, we get the following obvious result.

Lemma 4.1. *$\chi_\varepsilon \to 1$ as $\varepsilon \to 0$ both in $L^2(\Omega)$ and in the variable space $L^2(\Omega, \chi_\varepsilon \, dx)$.*

To introduce the convergence formalism for the sequences of tuples $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$, we begin with the following concepts:

Definition 4.1. *We say that a sequence of controls $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_a-converges to a function $\mathbf{a}_0$ if some sequence of its prototypes $\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\}_{\varepsilon>0}$, converges to $\mathbf{a}_0$ in $\mathbf{H}^2(\Omega)$.*

Definition 4.2. *We say that a sequence $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_b-converges to a function $\mathbf{b}_0 \in \mathbf{L}^2(\Omega)$ if some sequence of its prototypes $\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\}_{\varepsilon>0}$, converges to $\mathbf{b}_0$ weakly in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, i.e.,*

$$\limsup_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon\|_{\mathbf{L}^2(\Omega,d\eta_\varepsilon^r)} < +\infty \quad \textit{and} \quad \mathbf{u}_\varepsilon^r \rightharpoonup \mathbf{b}_0 \quad \textit{in} \quad \mathbf{L}^2(\Omega, d\eta_\varepsilon^r). \tag{4.3}$$

Lemma 4.2. *Any sequence of admissible controls $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ contains a subsequence, for which the w_a- and w_b-limits coincide almost everywhere in Ω.*

Proof. Let $\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\}_{\varepsilon>0}$ be a sequence of some control prototypes. Since this sequence is bounded in $\mathbf{H}^2(\Omega)$, we may suppose that there are an element $\mathbf{a}_0 \in \mathbf{H}^2(\Omega)$ and a subsequence of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index) such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{a}_0$ in $\mathbf{H}^2(\Omega)$ as $\varepsilon \to 0$.

By the Sobolev imbedding Theorem, we have that $\mathbf{a}_0 \in \mathbf{C}(\Omega)$ and $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ in $\mathbf{C}(\Omega)$. Then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon|^2 \, d\eta_\varepsilon^r \\
&\qquad \le 2\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{a}_0|^2 d\eta_\varepsilon^r + 2\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{a}_0|^2 d\eta_\varepsilon^r \\
&\qquad \le 2\limsup_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \eta_\varepsilon^r\left(\overline{\Omega}\right) + 2\|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \eta_\varepsilon^r\left(\overline{\Omega}\right) \\
&\qquad = 2\|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)} \eta_\varepsilon^r\left(\overline{\Omega}\right).
\end{aligned}$$

Hence, the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, and, by the compactness criterion of weak convergence in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, we may suppose that there is an element $\mathbf{b}_0 \in \mathbf{L}^2(\Omega)$ such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{b}_0$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ (passing to a subsequence, if it is necessary). On the other hand, for any function $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \varphi\left(\mathbf{a}_0 - \mathbf{b}_0\right) dx \\
&\qquad \le \left| \int_\Omega \varphi \mathbf{a}_0 \, dx - \int_\Omega \varphi \mathbf{a}_0 \, d\eta_\varepsilon^r \right| + \left| \int_\Omega \varphi\left(\mathbf{a}_0 - \mathbf{u}_\varepsilon\right) d\eta_\varepsilon^r \right|
\end{aligned}$$

$$+\left|\int_\Omega \varphi \mathbf{u}_\varepsilon\, d\eta_\varepsilon^r - \int_\Omega \varphi\, \mathbf{b}_0\, dx\right| \le \left|\int_\Omega \varphi \mathbf{a}_0\, dx - \int_\Omega \varphi \mathbf{a}_0\, d\eta_\varepsilon^r\right|$$

$$+ \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|_{\mathbf{C}(\Omega)} \int_\Omega |\varphi|\, d\eta_\varepsilon^r + \left|\int_\Omega \varphi \mathbf{u}_\varepsilon\, d\eta_\varepsilon^r - \int_\Omega \varphi\, \mathbf{b}_0\, dx\right|$$

$$= I_1 + I_2 + I_3. \quad (4.4)$$

Owing to the weak convergence $d\eta_\varepsilon^r \rightharpoonup dx$ and to the fact that $(\varphi\, \mathbf{a}_0) \in \mathbf{C}_0(\Omega)$, we obtain $I_1 \to 0$ as $\varepsilon \to 0$. By analogy, we also have that $I_2 \to 0$ and $I_3 \to 0$ as $\varepsilon \to 0$. Then, taking (4.3) into account, inequality (4.4) yields the following conclusion: $\int_\Omega \varphi\, (\mathbf{a}_0 - \mathbf{b}_0)\, dx = 0$ $\forall\, \varphi \in C_0^\infty(\Omega)$. Thus, $\mathbf{a}_0 = \mathbf{b}_0$ almost everywhere in Ω. The proof is complete. □

As a consequence, the following statements are readily true:

Lemma 4.3. *Let $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible controls. Then the weak limits in $\mathbf{H}^2(\Omega)$ of any weakly convergent sequences of prototypes*

$$\left\{\mathbf{u}_\varepsilon^{(1)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

and

$$\left\{\mathbf{u}_\varepsilon^{(2)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

are the same.

Lemma 4.4. *Any sequence of admissible controls $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the w_a-convergence. Moreover, its w_a-limit belongs to the set*

$$\mathbf{U} = \left\{\mathbf{u} \in \mathbf{H}^2(\Omega) \ : \ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma\right\}. \quad (4.5)$$

As follows from Remark 3.3, for any uniformly bounded sequence of functions $\left\{\mathbf{y}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$, there exist the extension operators $\mathbf{P}_\varepsilon$: $\mathbf{H}^1(\Omega_\varepsilon) \to \mathbf{H}^1(\Omega)$ and a constant C independent of ε such that $\|\breve{\mathbf{y}}_\varepsilon = (\mathbf{P}_\varepsilon \mathbf{y}_\varepsilon)\|_{\mathbf{H}^1(\Omega)} \le C\, \|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}$ for every $\varepsilon > 0$. Let us suppose that there are two different bounded sequences of prototypes

$$\{\breve{\mathbf{y}}_\varepsilon^{(1)} = \mathbf{P}_\varepsilon^{(1)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0} \ \text{ and } \ \{\breve{\mathbf{y}}_\varepsilon^{(2)} = \mathbf{P}_\varepsilon^{(2)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0}$$

such that $\breve{\mathbf{y}}_\varepsilon^{(1)} \rightharpoonup \mathbf{y}_1^*$ and $\breve{\mathbf{y}}_\varepsilon^{(2)} \rightharpoonup \mathbf{y}_2^*$ in $\mathbf{H}^1(\Omega)$. Then, using Lemma 4.1 and passing to the limit in the integral identity

$$\int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(1)} \varphi\, dx = \int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(2)} \varphi\, dx, \quad \forall\, \varphi \in \mathbf{H}^1(\Omega)$$

as $\varepsilon \to 0$, we obtain

$$\int_\Omega \mathbf{y}_1^* \varphi \, dx = \int_\Omega \mathbf{y}_2^* \varphi \, dx \quad \forall \varphi \in \mathbf{H}^1(\Omega).$$

Hence, $\mathbf{y}_1^* = \mathbf{y}_2^*$.

In view of this, we introduce the following notion:

Definition 4.3. *We say that a bounded sequence* $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *is w-convergent to a tuple* $(\mathbf{u}, \mathbf{y}, p, \theta) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ *in the variable space* $\mathbb{X}_\varepsilon$ *as* $\varepsilon \to 0$ *(in symbols,* $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \xrightarrow{w} (\mathbf{u}, \mathbf{y}, p, \theta)$*), if some bounded sequence of its prototypes* $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon, \breve{\theta}_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ *converges to* $(\mathbf{u}, \mathbf{y}, p, \theta)$ *in the following sense:*

(i) $\mathbf{u}_\varepsilon \xrightarrow{w_a} \mathbf{u}$ *in* $\mathbf{H}^2(\Omega)$*;*

(ii) $\breve{p}_\varepsilon \rightharpoonup p$ *in* $L^2_0(\Omega)$*;*

(iii) $\breve{\mathbf{y}}_\varepsilon \rightharpoonup \mathbf{y}$ *in* $\mathbf{H}^1(\Omega)$*;*

(iv) $\breve{\theta}_\varepsilon \rightharpoonup \theta$ *in* $H^1(\Omega)$*.*

As a consequence, we have the following result:

Theorem 4.1. *Let* $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *be a sequence of admissible solutions for the* $\mathbb{P}_\varepsilon$*-problem. Then there exist a subsequence* $\{(\overline{\alpha}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'}, \theta_{\varepsilon'})\}_{\varepsilon'>0}$ *and a tuple* $(\mathbf{u}, \mathbf{y}, p, \theta) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ *such that* $\mathbf{u} \in \mathbf{U}$*,* $(\overline{\alpha}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'}, \theta_{\varepsilon'}) \xrightarrow{w} (\mathbf{u}, \mathbf{y}, p, \theta)$*, and*

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega). \tag{4.6}$$

Proof. Let

$$\left\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon, \breve{\theta}^b_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}, \tag{4.7}$$

be a sequence of some prototypes, where $\breve{\mathbf{y}}_\varepsilon = \mathbf{y}_\varepsilon$ in Ω_ε and $\breve{\mathbf{y}}_\varepsilon = \mathbf{u}_\varepsilon$ on T_ε, $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon)$, and functions $\breve{\theta}^b_\varepsilon \in H^1(\Omega)$ are defined as $\breve{\theta}^b_\varepsilon = \theta_\varepsilon$ in Ω_ε and $\breve{\theta}^b_\varepsilon = b$ on T_ε. Here, $P_\varepsilon \in \mathcal{L}(L^2_0(\Omega_\varepsilon); L^2_0(\Omega))$ is an extension operator (see Remark 3.4). Since $\mathbf{y}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ and $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$, it follows that $\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon) \cap \mathbf{H}^1_0(\Omega)$.

As follows from the definition of the set Ξ_ε and *a priori* estimates for the weak solutions of the Bénard problem (2.1)–(2.6) (see [5]), the sequence of admissible solutions $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbb{X}_\varepsilon$. Hence, the sequence of its prototypes (4.7) is relatively compact in $\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$. Let $(\mathbf{u}, \mathbf{y}, p, \theta) \in \mathbf{H}^2(\Omega) \times$

$\mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ be its weak limit. We will show that $\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega)$. Since

$$\operatorname{div} : \mathbf{H}^1_0(\Omega_\varepsilon) \mapsto L^2(\Omega_\varepsilon)/\mathbb{R} = \left\{ g \in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} g(x)\, dx = 0 \right\},$$

it follows that $\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}(\operatorname{div}, \Omega_\varepsilon)\ \forall \varepsilon$. Therefore,

$$0 = \int_{\Omega_\varepsilon} \varphi \operatorname{div}(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\, dx = - \int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C^\infty_0(\mathbb{R}^3), \quad \forall \varepsilon > 0. \tag{4.8}$$

Taking Lemma 1.1 into account, we have

$$\int_\Omega \varphi\, \chi_\varepsilon \operatorname{div}(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx = - \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C^\infty_0(\mathbb{R}^3).$$

Since $\operatorname{div}[\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon] = \operatorname{div}[\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)] = \chi_\varepsilon \operatorname{div}(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)$, it follows that $\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}(\operatorname{div}, \Omega)$. As a result, passing to the limit in (4.8) as $\varepsilon \to 0$, we obtain, due to Lemma 4.1,

$$0 = \lim_{\varepsilon \to 0} \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx = \int_\Omega (\mathbf{y} - \mathbf{u}) \nabla \varphi \, dx, \quad \forall \varphi \in C^\infty_0(\mathbb{R}^3).$$

Hence, $(\mathbf{y} - \mathbf{u}) \in H^1_{0,sol}(\Omega)$, and this concludes the proof. □

5. Definition of suboptimal controls

The main question we are going to consider in this section concerns the approximation of the optimal solutions to the original problem (2.8)–(2.10) for ε small enough. We focus our attention on the possibility to define the so-called suboptimal solutions to the Bénard problem which have to guarantee the closeness of the corresponding value of the cost functional to its minimum. To do so, we introduce the following concept.

Definition 5.1. *We say that a function $\overline{\alpha}^{sub}_\varepsilon = (\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, \dots, \alpha^{sub}_{\mathbf{k}_{\mathrm{J}_\varepsilon}})$ is an asymptotically suboptimal control for the problem $(\mathbb{P}_\varepsilon)$ if*

$$\alpha^{sub}_{\mathbf{k}_j} \in \mathbf{H}^{1/2}(\widetilde{\partial}\mathrm{T}^{\mathbf{k}_j}_\varepsilon), \qquad \int_{\widetilde{\partial}\mathrm{T}^{\mathbf{k}_j}_\varepsilon} \mathbf{n} \cdot \alpha^{sub}_{\mathbf{k}_j}\, d\mathcal{H}^2 = 0, \quad \forall j = 1, \dots, \mathrm{J}_\varepsilon, \tag{5.1}$$

and, for every $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\Big| \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) - \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon^{sub}, \mathbf{y}_\varepsilon^{sub}, p_\varepsilon^{sub}, \theta_\varepsilon^{sub}) \Big| < \delta, \quad \forall \varepsilon < \varepsilon_0, \tag{5.2}$$

where

$$(\mathbf{y}_\varepsilon^{sub}, p_\varepsilon^{sub}, \theta_\varepsilon^{sub}) = \big(\mathbf{y}_\varepsilon(\overline{\alpha}_\varepsilon^{sub}), p_\varepsilon^{sub}(\overline{\alpha}_\varepsilon^{sub}), \theta_\varepsilon^{sub}(\overline{\alpha}_\varepsilon^{sub})\big)$$

denotes the corresponding solution of the boundary-value problem (2.1)–(2.6)*.*

To construct such controls, we use an approach coming from the variation convergence of constrained minimization problems (see [8, 15, 25–27, 33]). In view of this, we study the asymptotic behavior of the problem $(\mathbb{P}_\varepsilon)$ as $\varepsilon \to 0$. We represent this problem for various values of ε in the form of the sequence

$$\Big\{ \Big\langle \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon} \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \Big\rangle ; \, \varepsilon > 0 \Big\}. \tag{5.3}$$

Definition 5.2. *We say that a minimization problem*

$$\Big\langle \inf_{(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_0} \mathcal{I}_0(\mathbf{u}, \mathbf{y}, p, \theta) \Big\rangle \tag{5.4}$$

is the variational w-limit of the sequence (5.3) *as $\varepsilon \to 0$, if the following conditions are satisfied:*

(1) *if a sequence $\{(\overline{\alpha}_k, \mathbf{y}_k, p_k, \theta_k)\}_{k \in \mathbb{N}}$ w-converges to $(\mathbf{u}, \mathbf{y}, p, \theta)$ as $k \to \infty$, there exists a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and $(\overline{\alpha}_k, \mathbf{y}_k, p_k, \theta_k) \in \Xi_{\varepsilon_k}$ for all $k \in \mathbb{N}$, then*

$$(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_0, \quad \mathcal{I}_0((\mathbf{u}, \mathbf{y}, p, \theta) \leq \liminf_{k \to \infty} \mathcal{I}_{\varepsilon_k}(\overline{\alpha}_k, \mathbf{y}_k, p_k, \theta_k); \tag{5.5}$$

(2) *for every tuple of functions $(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_0$, there exists a sequence of admissible solutions $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon > 0}$ such that*

$$\begin{gathered}(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \xrightarrow{w} (\mathbf{u}, \mathbf{y}, p, \theta) \quad \text{and} \\ \mathcal{I}_0(\mathbf{u}, \mathbf{y}, p, \theta) \geq \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon).\end{gathered} \tag{5.6}$$

The following theorem deals with the main property of the variational w-limit problems (see [11, 12, 28, 29]):

Theorem 5.1. *Assume that* (5.4) *is a weak variational w-limit of sequence* (5.3). *Let* $\left\{(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon, \theta^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of optimal solutions for the* $\mathbb{P}_\varepsilon$*-problems. Then there exists a tuple* $(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0) \in \Xi_0$ *such that*

$$(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon, \theta^0_\varepsilon) \stackrel{w}{\longrightarrow} (\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0), \tag{5.7}$$

$$\begin{aligned} \inf_{(\mathbf{u},\mathbf{y},p,\theta)\in\Xi_0} \mathcal{I}_0(\mathbf{u},\mathbf{y},p,\theta) &= \mathcal{I}_0\left(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0\right) \\ &= \lim_{\varepsilon\to 0} \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)\in\Xi_\varepsilon} \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon). \end{aligned} \tag{5.8}$$

In what follows, we show that any solution of the w-limit problem (5.4) can be taken as a prototype for the construction of suboptimal controls in the sense of Definition 5.1.

6. Convergence theorem

The main question of this section is the study of the asymptotic behavior of the boundary-value problem (2.1)–(2.6) as $\varepsilon \to 0$. We suppose that hypotheses (H1)–(H8) are satisfied. Let μ_{kj} be the j-component of vector-functions $\mu_k \in \mathbf{W}^{-1,\infty}(\Omega)$ $(1 \le k \le 3)$. To begin with, we give some technical lemmas.

Lemma 6.1. $\mu_{ij} \in \mathcal{M}_b(\Omega), \quad \forall\, i,j\ :\ (1 \le i,j \le 3).$

Proof. We will prove that, for every compact set $K \subset \Omega$ of zero capacity, $\mu_{ij}(K) = 0$. By standard properties of Radon measures, it follows that $\mu_{ij}(D) = 0$ for any Borel set $D \subset \Omega$ of zero capacity.

Let K be a compact subset of Ω. Then, for any $k \in \mathbb{N}$, there exists $\varphi_k \in C^\infty_0(\Omega)$ such that $\varphi_k \ge \chi_K$, $0 \le \varphi_k \le 1$, $\|\varphi_k\|_{H^1_0(\Omega)} \le 1/k$. In view of hypothesis (H4), we have

$$\begin{gathered} \lim_{\varepsilon\to 0} \langle \nabla q_i^\varepsilon - \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} = 0, \\ \forall\, \mathbf{v}_\varepsilon \in \mathbf{H}^1_0(\Omega)\ :\ \mathbf{v}_\varepsilon \rightharpoonup 0 \ \text{ in }\ \mathbf{H}^1(\Omega) \ \text{ and }\ \mathbf{v}_\varepsilon = 0 \ \text{ on }\ \mathrm{T}_\varepsilon. \end{gathered} \tag{6.1}$$

Applying this to the sequence $\{\mathbf{v}_{\varepsilon,k} = \varphi_k \mathbf{w}^\varepsilon_j\}$, we obtain that, for any $\delta > 0$, there exist $\varepsilon_0(\delta)$ and $k_0(\delta)$ such that

$$\begin{aligned} \left|\, \langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} \,\right| + \left|\, \langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} \,\right| \le \delta, \\ \forall\, \varepsilon < \varepsilon_0(\delta),\ k > k_0(\delta). \end{aligned}$$

Then, following hypothesis (H2), we have

$$\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} = -\int_\Omega q_i^\varepsilon \mathbf{w}_j^\varepsilon \cdot \nabla\varphi_k \, dx.$$

In view of the estimates

$$\left|\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)}\right| \le \|q_i^\varepsilon\|_{L_0^2(\Omega)} \|\|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\mathrm{div},\Omega)} \|\varphi_k\|_{H_0^1(\Omega)} \le \frac{C}{k},$$

$$\langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = \int_\Omega \varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \, dx + \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, dx,$$

and

$$\left| \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, dx \right| \le \|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\nabla \varphi_k\|_{\mathbf{L}^2(\Omega)} \le \frac{C}{k} \ ,$$

we obtain

$$\int_\Omega \left|\varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| \, dx \le 2\delta, \quad \forall\, \varepsilon < \varepsilon_1(\delta), \ k > k_1(\delta). \tag{6.2}$$

Due to hypothesis (H3), each of the sequences $\{\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\}$ $(1 \le i, j \le 3)$ is bounded in $L^1(\Omega)$. So, extracting a subsequence, if necessary, we can suppose the existence of a symmetric matrix $\mathbf{M} = \{\mu_{ij}\}_{1\le i,j\le 3}$ of bounded Radon measures μ_{ij} such that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon$ converges to μ_{ij} in the weak-* sense of the space $\mathcal{M}_b(\Omega)$. Then, passing to the limit in inequality (6.2) as $\varepsilon \to 0$, we get $\int_\Omega \varphi_k \, d\mu_{ij} \le 2\delta$, $\forall\, k > k_1(\delta)$. Since $\varphi_k \ge \chi_K$, this yields $\mu_{ij}(K) \le 2\delta$, $\forall\, \delta > 0$, and we obtain the required result. □

Lemma 6.2. *For any function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and each i, j : $(1 \le i, j \le 3)$, we have*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) dx = \int_\Omega \varphi \, d\mu_{ij}. \tag{6.3}$$

Proof. Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. By analogy with work [9], we consider a sequence of functions $\{\varphi_k \in C_0^\infty(\Omega)\}$ satisfying the conditions

$$\sup_{k \in \mathbb{N}} \|\varphi_k\|_{L^\infty(\Omega)} < +\infty, \quad \varphi_k \to \varphi \ \text{ in } \ H_0^1(\Omega) \ \text{ and } \ \mu_{ij} \text{ — a.e. in } \ \Omega.$$

(the existence of such a sequence has been proved in [37]). Then

$$\left| \int_\Omega \varphi \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| dx - \int_\Omega \varphi \, d\mu_{ij} \right|$$
$$\le \int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) |\varphi - \varphi_k| \, dx$$

$$+\left|\int_\Omega \varphi_k \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)\, dx - \int_\Omega \varphi_k \, d\mu_{ij}\right| + \int_\Omega |\varphi_k - \varphi| \, d\mu_{ij}.$$

Passing to the limit in this relation for a fixed k and taking the weak-$*$ convergence of $(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon)$ to μ_{ij} in $\mathcal{M}_b(\Omega)$ into account, we obtain

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)\, dx - \int_\Omega \varphi \, d\mu_{ij}\right| \\ \le \limsup_{\varepsilon\to 0}\int_\Omega \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| |\varphi - \varphi_k| \, dx \\ + \limsup_{\varepsilon\to 0}\int_\Omega |\varphi_k - \varphi| \, d\mu_{ij}.$$

Passing now to the limit as $k \to \infty$, we find

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)\, dx - \int_\Omega \varphi \, d\mu_{ij}\right| \\ \le \lim_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| |\varphi - \varphi_k| \, dx.$$

We now show that the limit on the right-hand side is zero. We apply property (6.1) to the sequence $\{\mathbf{v}_{\varepsilon,k} = \pm|\varphi_k - \varphi|\mathbf{w}_j^\varepsilon\}$. We get

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\left[\pm\int_\Omega |\varphi_k - \varphi| \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)\, dx \right. \\ \left. \pm \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx\right] = 0.$$

Since $\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx \le 2\|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)}\|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)}\|\nabla|\varphi_k - \varphi|\|_{\mathbf{L}^2(\Omega)}$ and $\varphi_k \to \varphi$ strongly in $H_0^1(\Omega)$, it immediately follows that

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx\right| = 0.$$

Thus, $\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega |\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon||\varphi - \varphi_k| \, dx = 0$, and this concludes the proof. □

Lemma 6.3. *If a sequence* $\left\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\right\}$ *and* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ *are such that* $\mathbf{v}_\varepsilon = 0$ *on* T_ε *and* $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ *in* $\mathbf{H}_0^1(\Omega)$*, then* $\mathbf{v} \in \mathbf{L}^1(\Omega, d\mu_i)$ *for each* i : $(1 \le i \le 3)$.

Proof. For every $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = k$ if $s \ge k$; $T_k(s) = s$ if $-k \le s \le k$; and $T_k(s) = -k$ if $s \le -k$. We denote the vector-valued counterpart of this function with the bold symbol $\mathbf{T}_k : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\left\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\right\}$ and $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ be the above-given functions. We will show the fulfilment of the relation

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \left| \langle \nabla q_i^\varepsilon - \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i \right| = 0, \quad (6.4)$$

which implies that $\int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i$ is bounded independently of k, and hence, by the Beppo Levi's monotone convergence theorem, $\mathbf{v} \in \mathbf{L}^1(\Omega, d\mu_i)$.

For every $\varepsilon > 0$ and $k \in \mathbb{R}$, we define the functions $\mathbf{v}_{\varepsilon,k}$ by the rule $\mathbf{v}_\varepsilon = \mathbf{T}_k(\mathbf{v}_\varepsilon) + \mathbf{v}_{\varepsilon,k}$ and note that

$$\begin{aligned} &\left| \langle \nabla q_i^\varepsilon - \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i \right| \\ &\qquad \le \left| \langle \nabla q_i^\varepsilon - \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} \right| \\ &\qquad + \left| \langle \nabla q_i^\varepsilon - \Delta \mathbf{w}_i^\varepsilon, \mathbf{T}_k(\mathbf{v}_\varepsilon) \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i \right| \\ &\qquad = I_1 + I_2. \end{aligned} \quad (6.5)$$

Note that I_2 tends to zero as $\varepsilon \to 0$ for a fixed k by hypothesis (H4) (see the proof of Lemma 6.1). Then, in view of (6.1), we have $I_1 \to 0$ as $k \to \infty$ and $\varepsilon \to 0$. Hence, (6.4) holds true, and this concludes the proof. □

Our next step concerns the structural identification of the matrix $\mathbf{M} = \{\mu_{ij}\}_{1\le i,j\le 3}$, where μ_{ij} are the weak-$*$ limits of $\{\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\}$ in $\mathcal{M}_b(\Omega)$.

Lemma 6.4. *Assume the functions* $\mathbf{w}_j^\varepsilon = (w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon)^t \in \mathbf{H}^1(\Omega)$ $(1 \le j \le 3)$ *are such that*

$$w_{j,k}^\varepsilon \in H_0^1(\Omega) \quad \text{at} \quad j \ne k. \quad (6.6)$$

Then $\mathbf{M} = \operatorname{diag}(\mu_{11}, \mu_{22}, \mu_{33})$.

Proof. According to hypothesis (H3) and the Rellich-Kondrachov's compactness theorem, we conclude that $\mathbf{w}_j^\varepsilon = (w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon)^t$ converges to $\mathbf{e}_j$ strongly in $\mathbf{L}^2(\Omega)$. In addition, due to the initial assumptions (6.6), we also have $w_{j,k}^\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$, $w_{j,k}^\varepsilon \to 0$ in $L^2(\Omega)$, $\forall\, j \neq k$. Then, by the Poincaré inequality in Ω, we can deduce

$$\limsup_{\varepsilon\to 0} \|\nabla w_{j,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \le C \limsup_{\varepsilon\to 0} \|w_{j,k}^\varepsilon\|_{L^2(\Omega)} = 0, \quad \forall\, j \neq k. \tag{6.7}$$

Let (i,j) be a pair of indices such that $i \neq j$ and $1 \le i,j \le 3$. Extracting, if necessary, a subsequence, we can assume the existence of a bounded Radon measure μ_{ij} such that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \rightharpoonup \mu_{ij}$ in the weak-* sense of $\mathcal{M}_b(\Omega)$. This means that

$$\lim_{\varepsilon\to 0} \int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \varphi\, dx = \int_\Omega \varphi\, d\mu_{ij}, \quad \forall\, \varphi \in C_0(\Omega).$$

Observing that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon = \sum_{k=1}^3 \nabla w_{i,k}^\varepsilon \cdot \nabla w_{j,k}^\varepsilon$, we have the estimate

$$\int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \varphi\, dx \le \|\varphi\|_{C_0(\Omega)} \sum_{k=1}^3 \|\nabla w_{i,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \|\nabla w_{j,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)}. \tag{6.8}$$

Further, we can apply property (6.7). As a result, passing to the limit in (6.8) as $\varepsilon \to 0$, we obtain the required result: $\mu_{ij} = 0$. □

To check hypotheses (H1)–(H5) and to obtain a precise description for the measures μ_{ij}, we partition the set $\widetilde{\Omega}$ into squares $\varepsilon \widetilde{Y}_j$ with edges ε. The corresponding cylindrical cells $\varepsilon \widetilde{Y}_j \times (0,\ell)$ are denoted by Z_j^ε. Following the ideas of Cioranescu & Murat [10] and Allaire [3], we introduce the functions $\mathbf{w}_k^\varepsilon \in \mathbf{H}^1(Z_j^\varepsilon)$, $\omega_\varepsilon \in \mathrm{H}^1(Z_j^\varepsilon)$, and $q_k^\varepsilon \in L^2(Z_j^\varepsilon)$, where $\int_{Z_j^\varepsilon} q_k\, dx = 0$ and $(k = 1,2,3)$ as follows:

(1) For each cell Z_j^ε that meets the boundary Γ_3, we have

$$\{\mathbf{w}_k^\varepsilon = \mathbf{e}_k, \quad \omega_\varepsilon = 1, \quad q_k^\varepsilon = 0\} \quad \text{in} \quad Z_j^\varepsilon \cap \Omega. \tag{6.9}$$

(2) For each cell Z_j^ε entirely included in Ω (precisely, for each $\mathbf{k} \in \Theta_\varepsilon$), we have

$$\begin{gathered}\{\mathbf{w}_k^\varepsilon = \mathbf{e}_k,\ \omega_\varepsilon = 1,\ q_k^\varepsilon = 0\} \ \text{ in } \ Z_{\mathbf{k}}^\varepsilon \setminus \left[\varepsilon\,(A + \mathbf{k}) \times (0,\ell)\right], \\ \{-\Delta \mathbf{w}_k^\varepsilon + \nabla q_k^\varepsilon = 0,\ -\Delta\omega_\varepsilon = 0,\ \nabla\cdot \mathbf{w}_k^\varepsilon = 0\} \\ \text{in } \ \varepsilon\,(A \setminus Q^{r_\varepsilon} + \mathbf{k}) \times (0,\ell), \\ \{\mathbf{w}_k^\varepsilon = 0,\ \omega_\varepsilon = 0,\ q_k^\varepsilon = 0\} \ \text{ in } \ \varepsilon\,(Q^{r_\varepsilon} + \mathbf{k}) \times (0,\ell).\end{gathered} \tag{6.10}$$

It is now clear that the functions $\mathbf{w}_k^\varepsilon$, ω_ε, and q_k^ε are independent of x_3, that is,

$$\mathbf{w}_k^\varepsilon = \mathbf{w}_k^\varepsilon(x_1, x_2),\ \omega_\varepsilon = \omega_\varepsilon(x_1, x_2),\ q_k^\varepsilon = q_k^\varepsilon(x_1, x_2) \tag{6.11}$$

for all $1 \le k \le 3$. As a result, following closely [3,10], the following result can be proved.

Theorem 6.1. *Assume that the size $\varepsilon r_\varepsilon$ of thin cylinders $\mathrm{T}_\varepsilon^{\mathbf{k}}$ satisfies the condition*

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = \lim_{\varepsilon \to 0} \varepsilon^2 \left(\log 1/r_\varepsilon\right) > 0, \tag{6.12}$$

and the functions $(\mathbf{w}_k, q_k^\varepsilon, \omega_\varepsilon)$ $(k = 1, 2, 3)$ are defined as in (6.9)–(6.10). Then, there exist distributions

$$\mathbf{M} = (\mu_1, \mu_2, \mu_2) \in \left[\mathcal{M}_0^+(\Omega)\right]^{3\times 3}, \quad \mu \in \mathcal{M}_0^+(\Omega)$$

and an operator $R_\varepsilon \in \mathcal{L}\left(\mathbf{H}^1(\Omega), \mathbf{H}^1(\Omega_\varepsilon)\right)$ such that hypotheses (H1)–(H8) are satisfied. Moreover, in this case, $\mathbf{M} = \mathrm{diag}(\mu_{11}, \mu_{22}, \mu_{33})$, and there exists a linear map R_ε connected by (3.4) with the extension operator $P_\varepsilon \in \mathcal{L}(L_0^2(\Omega_\varepsilon); L_0^2(\Omega))$ such that

$$P_\varepsilon(p_\varepsilon) = p_\varepsilon \quad in \quad \Omega_\varepsilon,$$

$$P_\varepsilon(p_\varepsilon) = \frac{1}{\ell |A^\varepsilon \setminus Q^{\varepsilon r_\varepsilon}|} \int\limits_{(A^\varepsilon \setminus Q^{\varepsilon r_\varepsilon} + \varepsilon \mathbf{k}) \times (0,\ell)} p_\varepsilon \, dx \quad \textit{in each cylinder} \quad \mathrm{T}_\varepsilon^{\mathbf{k}}.$$

In order to identify the matrix $\mathbf{M} = \mathrm{diag}(\mu_{11}, \mu_{22}, \mu_{33})$ which appears as the weak-$*$ limit of $\nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_i^\varepsilon$ in $\mathcal{M}_b(\Omega)$, we give the following result.

Lemma 6.5. *Let Q be a compact subset of $\widetilde{Y}$ with sufficiently smooth boundary ∂Q $(\partial Q \in C^\infty)$, and let its boundary contain the origin $(0, 0)$. Then, under condition (6.12) for a sequence $\left\{\mathbf{w}_i^\varepsilon \in \mathbf{H}^1(\Omega)\right\}$ $(1 \le i \le 3)$ defined by (6.9)–(6.10), we have $(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_i^\varepsilon) \rightharpoonup \mu_{ii}^*$ weakly-$*$ in $\mathcal{M}_b(\Omega)$, where*

$$\mu_{ii}^* = 2\pi \lim_{\varepsilon \to 0} \sigma_\varepsilon^{-1}, \quad \forall\, i : 1 \le i \le 3. \tag{6.13}$$

Proof. The proof follows standard techniques and, in some aspects, it is similar to the one given in [13]. To begin with, we use the notation $|\nabla \mathbf{w}_i^\varepsilon|^2 = (\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_i^\varepsilon)$. We partition the cylindrical domain Ω into cubes εY_j with edges ε. It is clear that $|\varepsilon Y_j| = \varepsilon |\varepsilon \widetilde{Y}_j| = \varepsilon^3$ and $|\nabla \mathbf{w}_i^\varepsilon|^2 = \sum_{k=1}^3 |\nabla w_{i,k}^\varepsilon|^2$, where $w_{i,i}^\varepsilon \rightharpoonup 1$ in $H^1(\Omega)$, and $w_{i,k}^\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$ for every $k \ne i$. Then

$$\int\limits_{\varepsilon Y_j} \varphi \left|\nabla \mathbf{w}_i^\varepsilon\right|^2 \, dx = \int\limits_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx + \sum_{k\neq i} \int\limits_{\varepsilon Y_j} \varphi \left|\nabla w_{ik}^\varepsilon\right|^2 \, dx = \int\limits_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx + S_j(\varepsilon)$$

for any $\varphi \in C_0(\Omega)$. Hence, using (6.11), we have the obvious relation

$$\varphi(x_j^\varepsilon)\varepsilon \int\limits_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx = \varphi(x_j^\varepsilon) \int\limits_{\varepsilon Y_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx \leq \int\limits_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx \leq \varphi(y_j^\varepsilon) \int\limits_{\varepsilon Y_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx = \varphi(y_j^\varepsilon)\varepsilon \int\limits_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx,$$

where $x_j^\varepsilon, y_j^\varepsilon \in \varepsilon Y_j$. Combining this relation with the previous one, we obtain

$$\varepsilon \int\limits_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx + S_j(\varepsilon) \leq \int\limits_{\varepsilon Y_j} \left|\nabla \mathbf{w}_i^\varepsilon\right|^2 \varphi \, dx \leq \varphi(y_j^\varepsilon)\varepsilon \int\limits_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx + S_j(\varepsilon). \quad (6.14)$$

From the definition of the capacity and its properties, it readily follows that

$$\int\limits_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 \, dx = \operatorname{cap}\big(Q^{\varepsilon r(\varepsilon)}, A^\varepsilon\big) = \operatorname{cap}\left(r_\varepsilon Q, A\right) = \operatorname{cap}\left(Q, r_\varepsilon^{-1} A\right). \quad (6.15)$$

In view of $0 \in \partial Q$ and using the arguments of work [13], we can conclude that

$$\operatorname{cap}\left(K, r_\varepsilon^{-1} A\right) = \frac{2\pi}{\log(1/r_\varepsilon)}\left(1 + c_\varepsilon\right) = 2\pi\varepsilon^2\sigma_\varepsilon^{-1}\left(1 + c_\varepsilon\right), \quad (6.16)$$

where $\lim_{\varepsilon\to 0} c_\varepsilon = 0$.

Then, summing up inequalities (6.14)–(6.16), we obtain

$$2\pi\varepsilon^2\sigma_\varepsilon^{-1}(1+c_\varepsilon)\sum_j \varepsilon^3\varphi(x_j^\varepsilon)+\sum_j S_j(\varepsilon)\le \sum_j \int\limits_{\varepsilon Y_j} |\nabla \mathbf{w}_i^\varepsilon|^2\varphi\,dx$$
$$\le 2\pi\varepsilon^2\sigma_\varepsilon^{-1}(1+c_\varepsilon)\sum_j \varepsilon^3\varphi(y_j^\varepsilon)+\sum_j S_j(\varepsilon). \quad (6.17)$$

Observing that

$$-\|\varphi\|_{C_0(\Omega)}\lim_{\varepsilon\to 0}\int\limits_\Omega |\nabla w_{ik}^\varepsilon|^2\,dx \le \lim_{\varepsilon\to 0}\sum_j \int\limits_{\varepsilon Y_j}\varphi\,|\nabla w_{ik}^\varepsilon|^2\,dx$$
$$\le \|\varphi\|_{C_0(\Omega)}\lim_{\varepsilon\to 0}\int\limits_\Omega |\nabla w_{ik}^\varepsilon|^2\,dx,$$

and with regard for Lemma 6.4 and the construction of the Riemann sum for the integral $\int_\Omega \varphi\,dx$, we can pass to the limit in (6.17) as $\varepsilon\to 0$. As a result, we obtain

$$2\pi\lim_{\varepsilon\to 0}\sigma_\varepsilon^{-1}\int\limits_\Omega \varphi\,dx \le \lim_{\varepsilon\to 0}\int\limits_\Omega |\nabla \mathbf{w}_i^\varepsilon|^2\varphi\,dx \le 2\pi\lim_{\varepsilon\to 0}\sigma_\varepsilon^{-1}\int\limits_\Omega \varphi\,dx,$$
$$\forall\,\varphi\in C_0(\Omega).$$

Hence, $\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon|^2\varphi\,dx = 2\pi\lim_{\varepsilon\to 0}\sigma_\varepsilon^{-1}$, and this concludes the proof. ☐

As a consequence of this result, we have the following one (see also [9, 10, 12]).

Lemma 6.6. *Assume that the assumptions of Lemma* 6.5 *hold true. Let* $\{w_\varepsilon\in H^1(\Omega)\}$ *be the functions defined by* (6.9)–(6.10). *Then* $|\nabla w_\varepsilon|^2\to \mu^*$ *weakly-∗ in* $\mathcal{M}_b(\Omega)$, *where* $\mu^* = 2\pi\lim_{\varepsilon\to 0}\sigma_\varepsilon^{-1}$, *i.e.,* $\int_\Omega \varphi\,|\nabla w_\varepsilon|^2\,dx\to \int_\Omega \varphi\mu^*\,dx$ *for every* $\varphi\in C_0^\infty(\Omega)$.

Now, we are able to state and prove the main result of this section concerning the passage to the limit as $\varepsilon\to 0$ in the following integral identities:

$$\begin{cases} \int\limits_{\Omega_\varepsilon} (\nabla \mathbf{y}_\varepsilon \, : \, \nabla \mathbf{v}) \, dx + \int\limits_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon \cdot \nabla) \mathbf{y}_\varepsilon \cdot \mathbf{v} \, dx - \int\limits_{\Omega_\varepsilon} p_\varepsilon \, \operatorname{div} \, \mathbf{v} \, dx \\ \qquad\qquad = \int\limits_{\Omega_\varepsilon} \theta_\varepsilon \overrightarrow{\mathbf{e}_3} \cdot \mathbf{v} \, dx, \ \forall \, \mathbf{v} \in \mathbf{H}^1_0(\Omega_\varepsilon), \\ \int\limits_{\Omega_\varepsilon} q \, \operatorname{div} \, \mathbf{y}_\varepsilon \, dx = 0, \quad \forall \, q \in L^2_0(\Omega_\varepsilon), \\ \int\limits_{\Omega_\varepsilon} \nabla \theta_\varepsilon \cdot \nabla \varphi_\varepsilon dx + \int\limits_{\Omega_\varepsilon} \mathbf{y}_\varepsilon \cdot \nabla \theta_\varepsilon \varphi_\varepsilon \, dx = \int\limits_{\Omega_\varepsilon} \overrightarrow{\mathbf{e}_3} \cdot \mathbf{y}_\varepsilon \varphi_\varepsilon \, dx \\ \qquad\qquad \forall \varphi_\varepsilon \in H^1_0 \left(\Omega_\varepsilon\right). \end{cases} \tag{6.18}$$

The scheme of the proof is rather standard and is based on the energy method introduced by Tartar [35] and adapted later on by Allaire [3] for the Navier–Stokes equations.

Theorem 6.2. *Let $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions to $\mathbb{P}_\varepsilon$-problems such that*

$$(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \overset{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p, \theta). \tag{6.19}$$

Assume that the cross-size $\varepsilon r_\varepsilon$ of the cylinders $\mathrm{T}^{\mathbf{k}}_\varepsilon$ satisfies condition (6.12). Then $\mathbf{u} \in \mathbf{U}$, and a triplet $(\mathbf{y}, p, \theta) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ is a solution of the variational problem

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \quad \theta - b \in H^1_0(\Omega), \tag{6.20}$$

$$\begin{aligned} \int\limits_{\Omega} (\nabla \mathbf{y} \, : \, \nabla \mathbf{v}) \, dx + \frac{2\pi}{C_0} \int\limits_{\Omega} (\mathbf{y} - \mathbf{u}) \mathbf{v} \, dx + \int\limits_{\Omega} (\mathbf{y} \cdot \nabla) \mathbf{y} \cdot \mathbf{v} \, dx \\ + \int\limits_{\Omega} \nabla p \cdot \mathbf{v} \, dx = \int\limits_{\Omega} \theta \overrightarrow{\mathbf{e}_3} \cdot \mathbf{v} \, dx, \quad \forall \, \mathbf{v} \in \mathbf{H}^1_0(\Omega), \end{aligned} \tag{6.21}$$

$$\int\limits_{\Omega} q \, \operatorname{div} \, \mathbf{y} \, dx = 0, \quad \forall \, q \in L^2_0(\Omega),$$

$$\begin{aligned} \int\limits_{\Omega} \nabla \theta \cdot \nabla \varphi \, dx + \frac{\pi}{2C_0} \int\limits_{\Omega} (\theta - b) \, \varphi \, dx + \int\limits_{\Omega} \mathbf{y} \cdot \nabla \theta \varphi \, dx \\ = \int\limits_{\Omega} \mathbf{y} \cdot \overrightarrow{\mathbf{e}_3} \varphi \, dx, \quad \forall \, \varphi \in H^1_0 \left(\Omega\right). \end{aligned} \tag{6.22}$$

Remark 6.1. The corresponding limit boundary-value problem to (6.20)–(6.22) can be formally described as follows:

$$-\triangle \mathbf{y} + \frac{2\pi}{C_0}(\mathbf{y} - \mathbf{u}) + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \theta \overrightarrow{\mathbf{e}_3} \text{ in } \Omega; \tag{6.23}$$

$$\operatorname{div} \mathbf{y} = 0 \quad \text{in} \quad \Omega; \tag{6.24}$$

$$\mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega}; \tag{6.25}$$

$$-\Delta\theta + \frac{2\pi}{C_0}(\theta - b) + \mathbf{y} \cdot \nabla\theta = \mathbf{y} \cdot \overrightarrow{\mathbf{e}_3}; \tag{6.26}$$

$$\theta|_{\partial\Omega} = b|_{\partial\Omega}. \tag{6.27}$$

As follows from (6.23)–(6.27), in the case where the thin cylinders are of a critical size ($0 < C_0 < +\infty$), two additional terms $\frac{2\pi}{C_0}(\mathbf{y} - \mathbf{u})$ and $\frac{2\pi}{C_0}(\theta - b)$ appear in the limit equations. These relations correspond to the so-called Brinkman-type law of motion of a viscous incompressible fluid with heat supply (see [6]).

Proof. Let $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions, and let $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon, \breve{\theta}^b_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ be a sequence of their prototypes. Here, we suppose that $\{\breve{\mathbf{y}}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbf{H}^1(\Omega)$, $\{\breve{\theta}^b_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega)$, and each of $\breve{p}_\varepsilon$ is defined as $P_\varepsilon(p_\varepsilon) \in L^2_0(\Omega)$ (see Remark 3.4). The sequence $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon, \breve{\theta}^b_\varepsilon) \in \widehat{\Xi}_\varepsilon\}_{\varepsilon>0}$ is bounded in the variable space $\mathbb{X}_\varepsilon$. Hence, by (6.19), it converges to

$$(\mathbf{u}, \mathbf{y}, p, \theta) \in \left[\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega)\right] \times \mathbf{H}^1_{sol}(\Omega) \times L^2(\Omega) \times H^1(\Omega),$$

in the sense of Definition 4.3.

Let $\left\{(\mathbf{w}^\varepsilon_k, q^\varepsilon_k) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)\right\}_{\varepsilon>0}$ be a sequence of functions defined by hypotheses (H1)–(H4). Let $\varphi \in C^\infty_0(\Omega)$ and $\mathbf{f}_\varepsilon = \theta_\varepsilon \overrightarrow{\mathbf{e}_3}$ be fixed functions. It is clear that $\varphi \mathbf{w}^\varepsilon_k \in \mathbf{H}^1_{0,sol}(\Omega)$ and $\varphi q^\varepsilon_k \in L^2_0(\Omega)$ for any $\varepsilon > 0$. Now, we consider the variational problem (6.18) with the test functions $\mathbf{v} = \varphi \mathbf{w}^\varepsilon_k \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$, $q = \varphi q^\varepsilon_k \in L^2_0(\Omega_\varepsilon)$. As a result, we obtain

$$\int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla(\varphi \mathbf{w}^\varepsilon_k)\, dx$$
$$+ \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon : \nabla(\varphi \mathbf{w}^\varepsilon_k)\, dx + \int_\Omega \chi_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla)\breve{\mathbf{y}}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx$$
$$+ \int_\Omega \chi_\varepsilon ((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx$$

$$+\int\limits_{\Omega}\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx-\int\limits_{\Omega}\chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx$$
$$-\int\limits_{\Omega_\varepsilon}p_\varepsilon\ \mathrm{div}\,(\varphi\mathbf{w}_k^\varepsilon)\,dx=\int\limits_{\Omega}\chi_\varepsilon\mathbf{f}_\varepsilon\cdot(\varphi\mathbf{w}_{\mathbf{k}}^\varepsilon)\,dx\,;\quad(6.28)$$

$$\int\limits_{\Omega_\varepsilon}\varphi\,q_k^\varepsilon\ \mathrm{div}\ \mathbf{y}_\varepsilon\,dx=\int\limits_{\Omega_\varepsilon}\varphi\,q_k^\varepsilon\ \mathrm{div}\,(\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon)\,dx=0.\qquad(6.29)$$

Expanding (6.28) and using the fact that $\mathbf{w}_k^\varepsilon$ is divergence-free, we obtain

$$\int\limits_{\Omega_\varepsilon}\varphi\nabla(\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon)\,:\,\nabla\mathbf{w}_k^\varepsilon\,dx+\int\limits_{\Omega_\varepsilon}\nabla(\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon)\,:\,\mathbf{w}_k^\varepsilon\nabla\varphi\,dx$$
$$+\int\limits_{\Omega_\varepsilon}\varphi(\nabla\mathbf{u}_\varepsilon\,:\,\nabla\mathbf{w}_k^\varepsilon)\,dx+\int\limits_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon\,:\,\mathbf{w}_k^\varepsilon\nabla\varphi\,dx$$
$$+\int\limits_{\Omega}\chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx$$
$$+\int\limits_{\Omega}\chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx+\int\limits_{\Omega}\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx$$
$$-\int\limits_{\Omega}\chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\,dx-\int\limits_{\Omega_\varepsilon}p_\varepsilon\mathbf{w}_k^\varepsilon\cdot\nabla\varphi\,dx$$
$$=\int\limits_{\Omega}\chi_\varepsilon\mathbf{f}_\varepsilon\cdot\varphi\mathbf{w}_{\mathbf{k}}^\varepsilon\,dx.\quad(6.30)$$

After the integration of (6.29) by parts, we get

$$\langle\nabla q_k^\varepsilon,\varphi\,\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}+\int\limits_{\Omega}\chi_\varepsilon q_k^\varepsilon\,\nabla\varphi\cdot(\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\,dx=0.\ (6.31)$$

Then, adding two last equations and using the fact that

$$\int\limits_{\Omega_\varepsilon}\varphi\nabla(\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon)\,:\,\nabla\mathbf{w}_k^\varepsilon\,dx=-\left\langle\Delta\mathbf{w}_k^\varepsilon,\varphi\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}$$
$$-\int\limits_{\Omega}\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon-\mathbf{u}_\varepsilon)\nabla\varphi:\nabla\mathbf{w}_k^\varepsilon\,dx,\quad(6.32)$$

we obtain

$$\langle \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon, \varphi\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} + \int_\Omega \chi_\varepsilon \nabla \mathbf{u}_\varepsilon : \mathbf{w}_k^\varepsilon \nabla\varphi \, dx$$
$$+ \int_\Omega \chi_\varepsilon q_k^\varepsilon \nabla\varphi \cdot (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx - \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\nabla\varphi : \nabla \mathbf{w}_k^\varepsilon \, dx$$
$$+ \int_\Omega \chi_\varepsilon \nabla(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) : \mathbf{w}_k^\varepsilon \nabla\varphi\, dx + \int_\Omega \chi_\varepsilon \varphi(\nabla \mathbf{u}_\varepsilon : \nabla \mathbf{w}_k^\varepsilon)\, dx$$
$$+ \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\varphi \mathbf{w}_k^\varepsilon)\, dx + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon \cdot (\varphi\mathbf{w}_k^\varepsilon)\, dx$$
$$+ \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx$$
$$- \int_\Omega \chi_\varepsilon \breve{p}_\varepsilon \mathbf{w}_k^\varepsilon \cdot \nabla\varphi\, dx = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon \cdot \varphi \mathbf{w}_\mathbf{k}^\varepsilon \, dx. \quad (6.33)$$

Now, we can pass to the limit in (6.33) as $\varepsilon \to 0$. To do so, we recall the following facts: $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$; $\nabla \mathbf{w}_j^\varepsilon$ converges pointwise and weakly in $\mathbf{L}^2(\Omega)$ to zero; $\chi_\varepsilon \to 1$ in $L^2(\Omega)$; $q_k^\varepsilon \rightharpoonup 0$ in $L_0^2(\Omega)$; $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon) \rightharpoonup p$ in $L_0^2(\Omega)$; the sequence $\{\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\}$ satisfies the conditions of hypothesis (H4); the nonlinear term $((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)$ converges strongly to $((\mathbf{y}-\mathbf{u})\cdot\nabla)(\mathbf{y}-\mathbf{u})$ in $\mathbf{H}^{-1}(\Omega)$; condition (6.19) implies $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $\mathbf{H}^1(\Omega)$. As a result, we obtain

$$\langle \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon, \varphi\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}$$
$$\to \langle \mu_k, \varphi(\mathbf{y}-\mathbf{u})\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \overset{\text{by Lemma 6.5}}{=} \frac{2\pi}{C_0}\int_\Omega \varphi\, \overrightarrow{\mathbf{e}_k}\cdot(\mathbf{y}-\mathbf{u})\, dx.$$

Hence, using Rellich's theorem and (6.33), we get

$$\frac{2\pi}{C_0}\int_\Omega \varphi\, \overrightarrow{\mathbf{e}_k}\cdot(\mathbf{y}-\mathbf{u})\, dx + \int_\Omega \nabla\mathbf{y} : \overrightarrow{\mathbf{e}_k}\nabla\varphi\, dx$$
$$+ \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{y}\cdot(\varphi\overrightarrow{\mathbf{e}_k})\, dx - \int_\Omega p\,\overrightarrow{\mathbf{e}_k}\cdot\nabla\varphi\, dx = \int_\Omega \mathbf{f}\cdot\varphi\overrightarrow{\mathbf{e}_k}\, dx,$$
$$\forall\, \varphi \in C_0^\infty(\Omega) \ \text{ for every } \ k = 1,2,3, \quad (6.34)$$

where $\mathbf{f} = \theta\overrightarrow{\mathbf{e}_3}$, $\theta = \theta^* + b$, $\breve{\theta}_\varepsilon^b - b \rightharpoonup \theta^*$ in $H_0^1(\Omega)$.

Integrating the term $\int_\Omega p\,\overrightarrow{\mathbf{e}_k}\cdot\nabla\varphi\, dx$ by parts and regrouping (6.34) and (4.6), we deduce that the limit tuple $(\mathbf{u}, \mathbf{y}, p, \theta)$ must satisfy the relations

$$\int_{\Omega} \nabla \mathbf{y} : \nabla \Phi \, dx + \frac{2\pi}{C_0} \int_{\Omega} (\mathbf{y} - \mathbf{u}) \cdot \Phi \, dx + \int_{\Omega} (\mathbf{y} \cdot \nabla)\mathbf{y} \cdot \Phi \, dx$$

$$+ \int_{\Omega} \nabla p \cdot \Phi \, dx = \int_{\Omega} \theta \overrightarrow{\mathbf{e}_3} \cdot \Phi \, dx, \quad \forall \, \Phi \in \mathbf{C}_0^{\infty}(\Omega); \quad (6.35)$$

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \quad \mathbf{u} \in \mathbf{U}. \quad (6.36)$$

We now consider the last relation in (6.18). Putting $\varphi_\varepsilon = \omega_\varepsilon \varphi$, we obtain

$$\int_{\Omega_\varepsilon} \nabla \theta_\varepsilon \cdot \nabla [\omega_\varepsilon \varphi] \, dx + \int_{\Omega_\varepsilon} \mathbf{y}_\varepsilon \cdot \nabla \theta_\varepsilon [\omega_\varepsilon \varphi] \, dx = \int_{\Omega_\varepsilon} \overrightarrow{\mathbf{e}_3} \cdot \mathbf{y}_\varepsilon [\omega_\varepsilon \varphi] \, dx. \quad (6.37)$$

We note that

$$\int_{\Omega_\varepsilon} \nabla \theta_\varepsilon \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx = \int_{\Omega} \nabla \breve{\theta}^*_\varepsilon \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx + \int_{\Omega} \nabla b \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx,$$

where $\breve{\theta}^*_\varepsilon = \breve{\theta}^b_\varepsilon - b$, $\breve{\theta}^*_\varepsilon \rightharpoonup \theta^*$ in $H^1_0(\Omega)$. Using the facts that

$$\int_{\Omega} \nabla b \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx \to \int_{\Omega} \nabla b \cdot \nabla \varphi \, dx,$$

$$\int_{\Omega} \nabla \theta^*_\varepsilon \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx = \int_{\Omega} \nabla \theta^*_\varepsilon \cdot \omega_\varepsilon \nabla \varphi \, dx + \int_{\Omega} \nabla \theta^*_\varepsilon \cdot \varphi \nabla \omega_\varepsilon \, dx,$$

$$\int_{\Omega} \nabla \theta^*_\varepsilon \cdot \omega_\varepsilon \nabla \varphi \, dx \to \int_{\Omega} \nabla \theta^* \cdot \nabla \varphi \, dx,$$

$$\int_{\Omega} \nabla \theta^*_\varepsilon \cdot \varphi \nabla \omega_\varepsilon \, dx = \langle -\Delta \omega_\varepsilon, \varphi \theta^*_\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} - \int_{\Omega} \theta^*_\varepsilon \nabla \varphi \cdot \nabla \omega_\varepsilon \, dx,$$

$$\int_{\Omega} \theta^*_\varepsilon \nabla \varphi \cdot \nabla \omega_\varepsilon \, dx \to \int_{\Omega} \theta^* \nabla \varphi \cdot \nabla 1 \, dx = 0,$$

and, by hypothesis $(H8)$,

$$\langle -\Delta \omega_\varepsilon, \varphi \theta^*_\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \to \langle \mu, \varphi \theta^* \rangle_{H^{-1}(\Omega), H^1_0(\Omega)},$$

we finally have

$$\int_{\Omega_\varepsilon} \nabla \theta_\varepsilon \cdot \nabla \left[\omega_\varepsilon \varphi\right] dx \to \int_{\Omega} \nabla \theta \cdot \nabla \varphi \, dx + \langle \mu, \varphi \theta^* \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}, \quad (6.38)$$

where $\theta = \theta^* + b$. Hence,

$$\int_{\Omega} \breve{\mathbf{y}}_\varepsilon \cdot \nabla\theta_\varepsilon \left[\omega_\varepsilon\varphi\right] dx = \int_{\Omega} \breve{\mathbf{y}}_\varepsilon \cdot \nabla\theta^*_\varepsilon \left[\omega_\varepsilon\varphi\right] dx + \int_{\Omega} \breve{\mathbf{y}}_\varepsilon \cdot \nabla b \left[\omega_\varepsilon\varphi\right] dx \\ \to \int_{\Omega} \mathbf{y} \cdot \nabla\theta^*\varphi\, dx + \int_{\Omega} \mathbf{y} \cdot \nabla b\varphi\, dx,$$

$$\int_{\Omega_\varepsilon} \mathbf{y}_\varepsilon \cdot \overrightarrow{\mathbf{e}_3} \left[\omega_\varepsilon\varphi\right] dx = \int_{\Omega} \breve{\mathbf{y}}_\varepsilon \cdot \overrightarrow{\mathbf{e}_3} \left[\omega_\varepsilon\varphi\right] dx \to \int_{\Omega} \mathbf{y} \cdot \overrightarrow{\mathbf{e}_3}\varphi\, dx.$$

As a result, we conclude

$$\int_{\Omega} \nabla\theta \cdot \nabla\varphi\, dx + \langle \mu, \varphi\theta^* \rangle_{H^{-1}(\Omega), H^1_o(\Omega)} + \int_{\Omega} \mathbf{y} \cdot \nabla\theta\varphi\, dx \\ = \int_{\Omega} \mathbf{y} \cdot \overrightarrow{\mathbf{e}_3}\varphi\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega), \tag{6.39}$$

where $\theta^b_\varepsilon \rightharpoonup \theta = \theta^* + b$ in $H^1(\Omega)$. The proof is completed. □

7. Variational limit of the cost functional $\mathcal{I}_\varepsilon$

In this section, we study the asymptotic behavior of the cost functional $\mathcal{I}_\varepsilon : \Xi_\varepsilon \to \mathbb{R}$ (2.10) as $\varepsilon \to 0$, following the approach of [26]. Note that the functional $\mathcal{I}_\varepsilon$ is well defined on the set of admissible solutions Ξ_ε, and it does not have a sense outside of this set. In view of this, we use the concept of variational convergence of functionals in the variable spaces, because the limit functional may take another form in this case than the Γ-limit of $\{\mathcal{I}_\varepsilon\}_{\varepsilon>0}$ [27].

Definition 7.1. *We say that a functional $\mathcal{I}_o : \Xi_o \to \overline{\mathbb{R}}$ is the variational limit of the sequence $\{\mathcal{I}_\varepsilon : \Xi_\varepsilon \to \mathbb{R}\}_{\varepsilon>0}$ with respect to the* w*-convergence if:*

(*i*) *Ξ_o is a topological* w*-limit of $\{\Xi_\varepsilon\}_{\varepsilon>0}$ in the Kuratowski sense;*

(*ii*) *for any tuple $(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_o$ there exist a constant $\varepsilon_o > 0$ and a* w*-convergent sequence $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)\}_{\varepsilon>0}$ such that $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon$ $\forall\varepsilon \in (0, \varepsilon_o)$, and*

$$\mathcal{I}_o(\mathbf{u}, \mathbf{y}, p, \theta) \geq \limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon\left(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon\right);$$

(*iii*) *for any subsequence* $\{\Xi_{\varepsilon_i}\}_{i\in\mathbb{N}}$ *and any sequence* $\{(\overline{\alpha}_i, \mathbf{y}_i, p_i, \theta_i) \in \Xi_{\varepsilon_i}\}_{i\in\mathbb{N}}$ w*-convergent to some tuple* $(\mathbf{u}, \mathbf{y}, p, \theta)$*, we have*

$$\mathcal{I}_o(\mathbf{u}, \mathbf{y}, p, \theta) \le \liminf_{i\to\infty} \mathcal{I}_{\varepsilon_i}\left(\overline{\alpha}_i, \mathbf{y}_i, p_i, \theta_i\right).$$

Note that, in view of (3.3), the functional $\mathcal{I}_\varepsilon\left(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon\right)$ can be represented in the form

$$\mathcal{I}_\varepsilon\left(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon\right) = \int\limits_{\Omega_\varepsilon} \left|\mathbf{y}_\varepsilon - \mathbf{z}_\varepsilon^\partial\right|^2 dx + \beta|\partial Q|_H \int\limits_{\Omega} |\mathbf{u}_\varepsilon|^2 \, d\eta_\varepsilon^{r_\varepsilon}. \tag{7.1}$$

We begin with the following result concerning the identification the limit cost functional $\mathcal{I}_o : \Xi_o \to \overline{\mathbb{R}}$.

Theorem 7.1. *Assume* $\mathbf{z}_\varepsilon^\partial \to \mathbf{z}_\partial$ *strongly in* $\mathbb{L}^2(\Omega)$*. Then the variational limit of* $\{\mathcal{I}_\varepsilon : \Xi_\varepsilon \to \mathbb{R}\}_{\varepsilon>0}$ *takes the form*

$$\mathcal{I}_o(\mathbf{u}, \mathbf{y}, p, \theta) = \int\limits_{\Omega} |\mathbf{y} - \mathbf{z}_\partial|^2 \, dx + \beta|\partial Q|_H \int\limits_{\Omega} |\mathbf{u}|^2 \, dx. \tag{7.2}$$

Proof. To begin with, we check condition (*iii*) of Definition 7.1. Let

$$\Xi_{\varepsilon_i} \ni (\overline{\alpha}_i, \mathbf{y}_i, p_i, \theta_i) \overset{\mathrm{w}}{\to} (\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_o.$$

Then, by the Sobolev embedding theorem, we have

$$\mathcal{L}_{\mathbf{u}_i}(\mathbf{y}_i) = \left\{ \begin{array}{l} \mathbf{y}_i, \ \text{at } x \in \Omega_{\varepsilon_i} \\ \mathbf{u}_i, \ \text{at } x \in \Omega \setminus \Omega_{\varepsilon_i} \end{array} \right\} \to \mathbf{y}$$

strongly in $\mathbb{L}^2(\Omega)$ and weakly in $\mathbb{H}^2(\Omega)$. Moreover, the inequality

$$\liminf_{i\to\infty} \int\limits_{\Omega} |\mathbf{u}_i|^2 \, d\eta_{\varepsilon_i}^r \ge \int\limits_{\Omega} |\mathbf{u}|^2 \, dx$$

holds true. Hence,

$$\begin{aligned}
&\liminf_{i\to\infty} \mathcal{I}_{\varepsilon_i}\left(\overline{\alpha}_i, \mathbf{y}_i, p_i, \theta_i\right) \\
&\quad = \liminf_{i\to\infty} \left\{ \int\limits_{\Omega} \chi_{\varepsilon_i} \left|\mathcal{L}_{\mathbf{u}_{\varepsilon_i}}\left(\mathbf{y}_{\varepsilon_i}\right) - \mathbf{z}_{\varepsilon_i}^\partial\right|^2 dx + \beta\,|\partial Q|_H \int\limits_{\Omega} |\mathbf{u}_{\varepsilon_i}|^2 \, d\eta_{\varepsilon_i}^r \right\} \\
&\qquad \ge \lim_{i\to\infty} \int\limits_{\Omega} \chi_{\varepsilon_i} \left|\mathcal{L}_{\mathbf{u}_{\varepsilon_i}}\left(\mathbf{y}_{\varepsilon_i}\right) - \mathbf{z}_{\varepsilon_i}^\partial\right|^2 dx + \beta\,|\partial Q|_H \int\limits_{\Omega} |\mathbf{u}|^2 \, dx \\
&\qquad\qquad = \int\limits_{\Omega} |\mathbf{y} - \mathbf{z}_\partial|^2 \, dx + \beta\,|\partial Q|_H \int\limits_{\Omega} |\mathbf{u}|^2 \, dx.
\end{aligned}$$

Here, χ_ε denotes the characteristic function of Ω_ε. Thus, property (iii) is proved. In order to verify item (ii), we fix an arbitrary tuple $(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_o$. Then there exists a sequence of admissible tuples $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \Xi_\varepsilon\}$ such that

$$(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) \stackrel{\mathrm{w}}{\to} (\mathbf{u}, \mathbf{y}).$$

Hence, $\overline{\alpha}_\varepsilon \stackrel{\mathrm{w}_a}{\to} \mathbf{u}$. Using the compact embedding $\mathbb{H}^2(\Omega) \subset \mathbb{C}(\Omega)$, we have $\mathbf{u}_\varepsilon \to \mathbf{u}$ with respect to the norm of $\mathbb{C}(\Omega)$. From this, we conclude

$$\int_\Omega |\mathbf{u}_\varepsilon|^2 \, d\eta_\varepsilon^{r_\varepsilon} \to \int_\Omega |\mathbf{u}|^2 \, dx \quad \text{as } \varepsilon \to 0.$$

Hence,

$$\limsup_{\varepsilon \to 0} \left[\int_{\Omega_\varepsilon} \left| \mathbf{y}_\varepsilon - \mathbf{z}_\varepsilon^\partial \right|^2 dx + \beta \left| \partial C \right|_H \int_\Omega \left| \mathbf{u}_\varepsilon \right|^2 d\eta_\varepsilon^r \right] = \mathcal{I}_o \left(\mathbf{u}, \mathbf{y}, p, \theta \right),$$

and this concludes the proof. □

Thus, if the domain Ω is perforated by thin cylinders with the cross-section of critical size $0 < C_0 < +\infty$ (see condition (6.12)), then, for the family of optimal control problems (2.8)–(2.10), there exists a limit variational problem (5.4) which can be represented in the form of the following optimal control problem:

$$\text{Minimize } \mathcal{I}_o(\mathbf{u}, \mathbf{y}, p, \theta) = \int_\Omega |\mathbf{y} - \mathbf{z}_\partial|^2 \, dx + \beta |\partial Q|_H \int_\Omega |\mathbf{u}|^2 \, dx \tag{7.3}$$

subject to the constraints

$$(\mathbf{u}, \mathbf{y}, p, \theta) \in \Xi_o, \tag{7.4}$$

where the set of admissible solutions Ξ_o is defined as

$$\Xi_0 = \left\{ (\mathbf{u}, \mathbf{y}, p, \theta) \;\middle|\; \begin{array}{l} p \in L_0^2(\Omega), \ \mathbf{u} \in \mathbf{H}^2(\Omega), \ \mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq \gamma, \quad \mathbf{u}|_{\Gamma^3} = 0, \\ -\triangle \mathbf{y} + \frac{2\pi}{C_0}(\mathbf{y} - \mathbf{u}) + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \theta \overrightarrow{\mathbf{e}_3} \ \text{ in } \Omega; \\ \operatorname{div} \mathbf{y} = 0 \ \text{ in } \Omega; \\ \mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} \\ -\Delta\theta + \frac{2\pi}{C_0}(\theta - b) + \mathbf{y} \cdot \nabla\theta = \mathbf{y} \cdot \overrightarrow{\mathbf{e}_3} \ \text{ in } \Omega; \\ \theta|_{\partial\Omega} = b|_{\partial\Omega}. \end{array} \right\} \tag{7.5}$$

Problem (7.3)–(7.5) is called a limit or homogenized optimal control problem.

8. Suboptimal controls and their approximation properties

In this section, we deal with the the construction of suboptimal controls to the original optimal boundary control problem (2.1)–(2.10) in the case where the cross-size $\varepsilon r_\varepsilon$ of thin cylinders $\mathrm{T}_\varepsilon^{\mathbf{k}}$ satisfies condition (6.12). To begin with, we note that, by hypothesis (H5), there exists an operator $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ with property

$$\text{if } \ \mathbf{y} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon), \ \text{ then } \ \Lambda_\varepsilon(\breve{\mathbf{y}}) = \mathbf{y} \ \text{ in } \ \Omega_\varepsilon, \tag{8.1}$$

$$\|\Lambda_\varepsilon(\mathbf{u})\|_{\mathbf{H}^1(\Omega_\varepsilon)} \le C\|\mathbf{u}\|^1_{\mathbf{H}}(\Omega), \quad \text{where a constant } C > 0 \text{ is independent of } \varepsilon. \tag{8.2}$$

The main result can be formulated as follows:

Theorem 8.1. *Let* $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ *be a linear continuous map such that properties* (8.1)–(8.2) *hold true, and let* $(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0) \in \Xi_0$ *be an optimal solution to the limit problem* (5.4). *Then the function*

$$\overline{\alpha}_\varepsilon^{sub} = \left(\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, \ldots, \alpha^{sub}_{\mathbf{k}_{\mathrm{J}_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon} \tag{8.3}$$

is an asymptotically suboptimal control for the original problem $(\mathbb{P}_\varepsilon)$ *in the sense of Definition* 5.1.

Proof. As follows from [12, Theorem 9.3], we have the following result: let $(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0)$ be an optimal solution to the limit problem (5.4). Then

$$\Lambda_\varepsilon(\mathbf{u}^0) \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)}), \quad \forall\, \varepsilon > 0, \tag{8.4}$$

$$\begin{gathered}\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon} \xrightarrow{w_a} \mathbf{u}^0 \ \text{ at } \ \varepsilon \to 0,\\ \text{and } \ \lim_{\varepsilon\to 0}\int_\Omega \left|\Lambda_\varepsilon(\mathbf{u}^0)\right|^2\, d\eta_\varepsilon^{r(\varepsilon)} = \int_\Omega \left|\mathbf{u}^0\right|^2\, dx.\end{gathered} \tag{8.5}$$

Let us consider a sequence $\{(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$, where

$$(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon) = \left(\widehat{\mathbf{y}}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}), \widehat{p}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}, \widehat{\theta}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon})\right)$$

is a weak solution to the boundary-value problem (2.1)–(2.6). Then, by analogy with [12] (see Theorem 9.3), it can be proved that this sequence is compact with respect to the w-convergence. Passing to a subsequence (if necessary), we obtain

$$(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon) = \left(\widehat{\mathbf{y}}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}), \widehat{p}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}, \widehat{\theta}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon})\right) \xrightarrow{\varepsilon\to 0} (\mathbf{u}^0, \widehat{\mathbf{y}}^0, \widehat{p}^0, \widehat{\theta}^0),$$

where $(\mathbf{u}^0, \widehat{\mathbf{y}}^0, \widehat{p}^0, \widehat{\theta}^0)$ and $(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0)$ belong to the same class of equivalence. Therefore, $\mathcal{I}_0(\mathbf{u}^0, \mathbf{y}^0) = \mathcal{I}_0(\mathbf{u}^0, \widehat{\mathbf{y}}^0)$.

Let $\left\{(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0, p_\varepsilon^0, \theta_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be an optimal solution to $(\mathbb{P}_\varepsilon)$. Then

$$\begin{aligned}
&\left| \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon)\in \Xi_\varepsilon} \mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon, \theta_\varepsilon) - \mathcal{I}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon)\right| \\
&\quad = \left|\mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0, p_\varepsilon^0, \theta_\varepsilon^0) - \mathcal{I}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon, \widehat{\theta}_\varepsilon)\right| \\
&\quad \le \left|\mathcal{I}_\varepsilon(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0, p_\varepsilon^0, \theta_\varepsilon^0) - \mathcal{I}_0(\mathbf{u}^0, \mathbf{y}^0, p^0, \theta^0)\right| \\
&\quad + \left| \int_\Omega |\widehat{\mathbf{y}}^0 - \mathbf{z}_\partial|^2\,dx - \int_{\Omega_\varepsilon} |\widehat{\mathbf{y}}_\varepsilon - \mathbf{z}_\varepsilon^\partial|^2\,dx \right| \\
&\quad + \beta\left| |\partial Q|_H \int_\Omega \left|\mathbf{u}^0\right|^2\,dx - \frac{\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon} \int_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}} \left|\,\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\widetilde{\partial}\mathrm{T}_\varepsilon^{\mathbf{k}_j}}\right|^2 d\mathcal{H}^2 \right| \\
&\quad = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

To conclude the proof, we note that, for a given $\delta > 0$, one can always find: (1) $\varepsilon_1 > 0$ such that $\mathcal{I}_1 < \delta/3$ for all $\varepsilon < \varepsilon_1$ by Theorem 7.1 and the well-known properties of variational convergence; (2) $\varepsilon_2 > 0$ such that $\mathcal{I}_2 < \delta/3$ for all $\varepsilon < \varepsilon_2$ by the proof of Theorem 7.1; (3) $\varepsilon_3 > 0$ such that $\mathcal{I}_3 < \delta/3$ for all $\varepsilon < \varepsilon_3$ by property (8.5). Thus, taking $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and using Definition 5.1, we come to the required conclusion. □

9. Conclusion

The results obtained above show that, in the case where the domain Ω is perforated by thin cylinders with critical cross-size, the limit optimal control problem (7.3)–(7.5), which is a basis for the construction of suboptimal controls, essentially differs from the original problem. First, the control object is a steady-state motion of the viscous incompressible flow in a homogeneous cylindrical domain, and this process is described by a Brinkman-type law in thermodynamics (since the additional terms $\frac{2\pi}{C_0}(\mathbf{y} - \mathbf{u})$ and $\frac{2\pi}{C_0}(\theta - b)$ appear in the equations of state (6.20) and (6.26), respectively). Second, the control is realized through the boundary Dirichlet conditions on the upper and lower bases of the cylindrical domain. Here, as admissible controls, we have the functions of the class $\mathbf{H}^2(\Omega)$, for which the following restriction is satisfied:

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma, \quad \mathbf{u}\big|_{\Gamma_3} = 0.$$

As a result, the set of admissible controls in the limit problem takes another form. Third, despite the fact that the control is both boundary and distributed over the whole domain, the limit cost functional (7.3) contains the square of the weighted norm of a control in the space $\mathbf{L}^2(\Omega)$. Therefore, from the viewpoint of control theory, the limit problem (7.3)–(7.5) essentially differs from the initial optimal control problem in the perforated domain. At the same time, it has the following feature: this problem is considered in the domain with a much simpler geometry. This fact allows the effective use of existing numerical algorithms for its analysis and the realization of suboptimal controls.

References

[1] R. Adams, *Sobolev Spaces*, New York: Academic Press, 1975.

[2] F. Abergel, R. Temam, *On some control problems in fluids mechanics* // Theoret. Comput. Fluid Dynamics, **1** (1990), 303–325.

[3] G. Allaire, *Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. I. Abstract frameworks, a volume distribution of holes* // Arch. Ration. Mech. Anal., **13** (1991), 209–259.

[4] I. Babuška, *The finite element method with penalty* // Math Comp., **27** (1973), 221–228.

[5] B. Birnir, N. Svanstedt, *Existence theory and strong attractors for the Rayleigh–Bénard problem with a large aspect ratio* // Discrete and Continuous Dynamical Systems, **10** (2004), N 1-2, 55–74.

[6] H. C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles* // Appl. Sci. Res., **A1** (1947), 27–34.

[7] G. Bouchitté, I. Fragala, *Homogenization of thin structures by two-scale method with respect to measures* // SIAM J. Math. Anal., **32** (2001), 1198–1226.

[8] G. Buttazzo, G. Dal Maso, *Γ-convergence and optimal control problems* // J. Optim. Theory Appl., **38** (1982), 385–407.

[9] J. Casado-Díaz, *Existence of a sequence satisfying Cioranescu–Murat conditions in homogenization of Dirichlet problems in perforated domains* // Rend. di Matem. Roma, (1996), Ser. VII, 387–413.

[10] D. Cioranescu, F. Murat, *Un terme étrage venu d'ailleurs* // Nonlinear Partial Differential Equations and their applications. Collége de France Seminar, Vol. **II**, 58–138; Vol. **III**, 157–178, Research Notes in Mathematics, Pitman, London, 1981.

[11] C. D'Apice, U. De Maio, P. I. Kogut, *Suboptimal boundary controls for elliptic equation in critically perforated domain* // Annales de l'institut Henri Poincaré: (C) Analyse non Linéare, **25** (2008), Iss. 6, 1073–1101.

[12] C. D'Apice, U. De Maio, P. I. Kogut, *Boundary velocity suboptimal control of incompressible flow in cylindrically perforated domain* // Discrete and Continuous Dynamical Systems, Series B, **11** (2009), N 2, 283–314.

[13] A. Corbo Esposito, C. D'Apice, A. Gaudiello, *A homogenization problem in a perforated domain with both Dirichlet and Neumann conditions on the boundary of the holes* // Asymp. Anal, **31** (2002), 297–316.

[14] G. Dal Maso, F. Murat, *Asymptotic behaviour and correctors for Dirichlet problem in perforated domains with homogeneous monotone operators* // Ann. Scoula Norm. Sup. Pisa Cl. Sci., (1997), N 4, 239–290.

[15] Z. Denkowski, S. Mortola, *Asymptotic behavior of optimal solutions to control problems for systems described by differential inclusions corresponding to partial differential equations* // J. Optimiz. Theory Appl., **78** (1993), 365–391.

[16] G. Duvaut, J.-L. Lions, *Les Inéquations en Méchanique et en Physique*, Paris: Dunod, 1971.

[17] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[18] A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and applications*, Providence, RI: AMS, 2000.

[19] A. V. Fursikov, M. D. Gunzburger, L. S. Hou, *Boundary value problems and optimal boundary control for the Navier–Stokes system* // SIAM J. Control Optim. **36** (1998), 852–894.

[20] M. D. Gunzburger, L. S. Hou, T. Svobodny, *Boundary velocity control of incompressible flow with an application to viscous drag reduction* // SIAM J. Control Optim., **30** (1992), 167–181.

[21] V. I. Ivanenko, V. S. Mel'nik, *Variational Methods in Optimal Control Problems for Distributed Systems*, Kyiv: Naukova Dumka, 1989 (in Russian).

[22] L. S. Hou, T. Svobodny, *Optimization problems for the Navier-Stokes equations with regular boundary controls* // J. Math. Anal. Appl. **177** (1993), 342–367.

[23] L. S. Hou, S. S. Ravindran, *A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier–Stokes equations* // SIAM J. Control Optim. **36** (1998), 1795–1814.

[24] A. V. Kapustyan, J. Valero, *Global continuous solutions, uniqueness and attractors for the 3D Navier–Stokes systems* // Preprint I-2005-24, Universidad Miguel Hernández, Centro de Investigación Operativa, 2005, 26 p.

[25] P. I. Kogut, *S-convergence of constrained minimization problems and its variational properties* // Probl. Upravl. i Inform., (1997), N 4, 64–79 (in Russian).

[26] P. I. Kogut, G. Leugering, *S-convergence of optimal control problems in Banach spaces* //Math. Nachr., **233–234** (2002), 141–169.

[27] P. I. Kogut, G. Leugering, *On S-homogenization of an optimal control problem with control and state constraints* // J. Analysis and its Applications, **20** (2001), 395–429.

[28] P. I. Kogut, G. Leugering, *On the homogenization of optimal control problems on periodic graphs* // Lecture Notes in Pure and Applied Mathematics, **252** (2007), 55–74.

[29] P. I. Kogut, G. Leugering, *Asymptotic Analysis of State Constrained Semilinear Optimal Control Problems* // Journal of Optimization Theory and Applications (JOTA), **135** (2007), N 2, 301–321.

[30] A. A. Kovalevskii, *G-convergence and homogenization of nonlinear elliptic operators in divergence form with variable domain* // Russian Acad. Sci. Izv. Math., **44** (1995), N 3, 431–460.

[31] V. A. Marchenko, E. Ya. Khruslov, *Boundary Value Problems in Domains with Fine-Grained Boundaries.* Kyiv: Naukova Dumka, 1974 (in Russian).

[32] F. F. Reuss, *Notice sur un nouvel effect de l'électricité galvanique*, Mémoire Soc. Sup. Imp. de Moscou, 1809.

[33] J. Saint Jean Paulin, H. Zoubairi, *Optimal control and "strange term" for the Stokes problem in perforated domains* //Portugaliac Mathematica, **275** (2002), N 2, 161–178.

[34] E. Sanches-Palencia, *Non Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Berlin: Springer, 1980.

[35] L. Tartar, *Convergence of the homogenization process*, Appendix of [34].

[36] R. Temam, *Navier–Stokes Equations, Theory and Numerical Methods*, Amsterdam: North-Holland, 1979.

[37] W. P. Ziemmer, *Weakly Differentiable Functions*, New York: Springer, 1989.

[38] V. V. Zhikov, *On an extension of the method of two-scale convergence and its applications* // Sbornik Math., **191** (2000), N 7, 973–1014.

[39] M. Z. Zgurovski, V. S. Mel'nik, *Nonlinear Analysis and Control Problems for Systems with Distributed Parameters*, Kyiv: Naukova Dumka, 1999 (in Russian).

Contact information

Peter I. Kogut
Department of Differential Equations,
Dnipropetrovsk National University
Kozakova str., 18/14,
49050 Dnipropetrovsk,
Ukraine
E-Mail: p.kogut@i.ua

Vladimir V. Gotsulenko
Department of Technical Cybernetics,
Dnipropetrovsk Technical University,
Lazarjan str., 2,
49010 Dnipropetrovsk,
Ukraine
E-Mail: gosul@ukr.net