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BOUNDARY VELOCITY SUBOPTIMAL CONTROL OF INCOMPRESSIBLE FLOW IN CYLINDRICALLY PERFORATED DOMAIN

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Abstract. In this paper we study an optimal boundary control problem for the 3D steady-state Navier-Stokes equation in a cylindrically perforated domain Ω_ε. The control is the boundary velocity field supported on the 'vertical' sides of thin cylinders. We minimize the vorticity of viscous flow through thick perforated domain. We show that an optimal solution to some limit problem in a non-perforated domain can be used as basis for the construction of suboptimal controls for the original control problem. It is worth noticing that the limit problem may take the form of either a variational calculation problem or an optimal control problem for Brinkman's law with another cost functional, depending on the cross-size of thin cylinders.

1. Introduction.

Optimal control problem for the Navier-Stokes equations has been the subject of extensive study in recent years. A complete and systematic mathematical and numerical analysis of optimal control problems of different types (e.g., having Dirichlet, Neumann, and distributed controls) for the steady-state Navier-Stokes system was given in [2, 16, 17, 18, 19, 21, 34]. Dirichlet controls, i.e., boundary velocity controls or boundary mass flux control, are common in applications [31]. However, as is shown in [20], even though the admissible controls are smooth, the optimality systems for optimal Dirichlet controls problems involve a boundary Laplacian or a boundary biharmonic equation. This circumstance makes the numerical resolution of the optimality systems, and hence the numerical calculation of an optimal control for such systems very complicated. So, many efforts are made for the development of penalty, approximation and relaxation methods for solving optimal Dirichlet control problems (see [2, 4, 21]).

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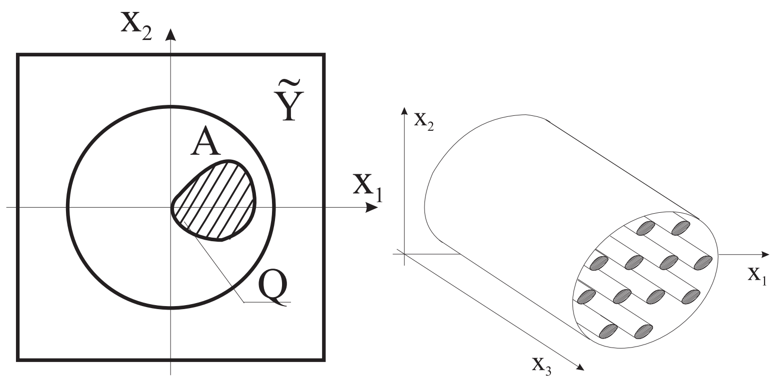

(a) The cell of perforation (b) The domain Ω_ε

FIGURE 1. An example of the cylindrically perforated domain

These problems are especially complicated in perforated domains and in domains with quickly oscillating boundaries (see [28]). So, our work deal with the development of approximate methods for the solution of optimal Dirichlet control problems for Navier-Stokes equations. The approach we propose gives the possibility to replace the original optimal control problem by some limit problem defined in a more simple domain. We show that an optimal control for the limit problem can be taken as a suboptimal control to the original one.

We turn now to a more detailed description of the main object of our study. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a bounded connected open domain with a simple connected smooth boundary $\partial\widetilde{\Omega}$, and let ε be a small positive parameter. To define a cylindrically perforated flow domain Ω_ε in $\mathbb{R}^3$, we introduce the following sets: $\widetilde{Y} = [-1/2, 1/2)^2$; Q is a compact subset of $\widetilde{Y}$ such that $0 \in \partial Q$,

$$\Theta_\varepsilon = \{\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2 : \varepsilon(r_\varepsilon Q + \mathbf{k}) \subset\subset \widetilde{\Omega}\}, \tag{1}$$

$$\widetilde{T}_\varepsilon^{\mathbf{k}} = \varepsilon(r_\varepsilon Q + \mathbf{k}), \mathbf{k} \in \Theta_\varepsilon; \quad \widetilde{T}_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \widetilde{T}_\varepsilon^{\mathbf{k}} = \bigcup_{j=1}^{J_\varepsilon} \widetilde{T}_j^{\mathbf{k}}, \tag{2}$$

$$T_\varepsilon = \widetilde{T}_\varepsilon \times [0, \ell]; \quad \widetilde{\Omega}_\varepsilon = \widetilde{\Omega} \setminus \widetilde{T}_\varepsilon; \quad \Omega = \widetilde{\Omega} \times (0, \ell), \tag{3}$$

$$\Gamma^1 = \widetilde{\Omega} \times \{0\}; \quad \Gamma^2 = \widetilde{\Omega} \times \{\ell\}; \quad \Gamma^3 = \partial\widetilde{\Omega} \times [0, \ell], \tag{4}$$

where the parameter r_ε denotes the cross-size of the thin cylinders $T_\varepsilon^{\mathbf{k}} = \widetilde{T}_\varepsilon^{\mathbf{k}} \times [0, \ell]$ and satisfies the conditions: $0 < r_\varepsilon \leq 1$ and $\lim_{\varepsilon\to 0} r_\varepsilon = 0$.

Then, the domain Ω_ε is defined by removing the cylinders $T_\varepsilon^{\mathbf{k}}$ from Ω, that is (see Fig. 1),

$$\Omega_\varepsilon = \Omega \setminus \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} T_\varepsilon^{\mathbf{k}}\right] = \Omega \setminus \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon(r_\varepsilon Q + \mathbf{k}) \times [0, \ell]\right]. \tag{5}$$

We use the following decomposition for the boundary of this domain : $\partial\Omega_\varepsilon = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2 \cup \Gamma^3 \cup \partial T_\varepsilon$, where

$$\Gamma_\varepsilon^1 = \widetilde{\Omega}_\varepsilon \times \{0\}, \quad \Gamma_\varepsilon^2 = \widetilde{\Omega}_\varepsilon \times \{\ell\}. \tag{6}$$

To characterize the different possible cross-sizes of thin cylinders which can be considered ('critical', smaller and larger cylinders), we define a ratio σ_ε between the current size of the cross-sections and the critical one:

$$\sigma_\varepsilon = \varepsilon^2 \left(\log 1/r_\varepsilon\right). \tag{7}$$

If the limit of σ_ε, as ε tends to zero, is positive and finite then the cross-size of the cylinders is called critical. If $\lim_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$, the cross-size of cylinders is smaller and if $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$, the cross-size is larger (see Cioranescu & Murat [9] and Allaire [3]).

The optimal control problem we consider is to minimize the vorticity of viscous, incompressible flow by choosing an appropriate boundary velocity on the 'vertical' sides of thin cylinders

$$T_\varepsilon^{\mathbf{k}} = \left\{(x_1, x_2, x_3) : (x_1, x_2) \in \varepsilon(r_\varepsilon Q + \mathbf{k}), 0 \le x_3 \le \ell\right\}, \quad \forall\, \mathbf{k} \in \Theta_\varepsilon.$$

Precisely, we study the following optimal control problem: find a boundary velocity field $\overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, ..., \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ and a corresponding velocity-pressure pair $(\mathbf{y}_\varepsilon, p_\varepsilon)$ such that the functional

$$\mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) = \lambda \int_{\Omega_\varepsilon} |\nabla\, \mathbf{y}_\varepsilon|^2\, dx + \frac{\beta\varepsilon}{r_\varepsilon} \sum_{j=1}^{J_\varepsilon} \int\limits_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}\right|^2 d\mathcal{H}^2 \tag{8}$$

is minimized subject to the steady-state Navier-Stokes equations

$$-\nu\triangle\mathbf{y}_\varepsilon + (\mathbf{y}_\varepsilon\cdot\nabla)\mathbf{y}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}_\varepsilon \quad \text{in } \ \Omega_\varepsilon, \tag{9}$$

$$\operatorname{div}\mathbf{y}_\varepsilon = 0 \quad \text{in } \ \Omega_\varepsilon, \tag{10}$$

$$\mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1,\ \mathbf{y}_\varepsilon|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2,\ \mathbf{y}_\varepsilon|_{\Gamma^3} = 0, \tag{11}$$

$$\mathbf{y}_\varepsilon\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall\, j = 1, \dots, J_\varepsilon. \tag{12}$$

Here, ν denotes the constant viscosity; $\mathbf{y}_\varepsilon$ and p_ε denote the velocity field and the pressure field, respectively; $\mathbf{f}_\varepsilon$ is a prescribed forcing term; $\mathbf{y}_\varepsilon^1$ and $\mathbf{y}_\varepsilon^2$ are given boundary velocities on the lower and upper boundaries Γ_ε^1 and Γ_ε^2, respectively, and $\overline{\alpha}_\varepsilon$ is the boundary velocity — the control field. Because of the divergence-free condition on $\mathbf{y}_\varepsilon$, the vector-valued functions $\mathbf{y}_\varepsilon^1$, $\mathbf{y}_\varepsilon^2$, and $\alpha_{\mathbf{k}_j}$ must necessarily satisfy the following relations:

$$\int_{\Gamma_\varepsilon^1} \mathbf{y}_\varepsilon^1\cdot\mathbf{n}\, d\mathcal{H}^2 = 0,\ \int_{\Gamma_\varepsilon^2} \mathbf{y}_\varepsilon^2\cdot\mathbf{n}\, d\mathcal{H}^2 = 0,\ \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \alpha_{\mathbf{k}_j}\cdot\mathbf{n}\, d\mathcal{H}^2 = 0,\ \forall\, j = 1, \dots, J_\varepsilon. \tag{13}$$

Through this paper we assume that there are functions $\mathbf{y^*}_\varepsilon \in \mathbf{H}^2(\Omega)\cap\mathbf{H}^1_{sol}(\Omega_\varepsilon)$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and $\mathbf{y}^* \in \mathbf{H}^2(\Omega)$ such that

$$\mathbf{y}_\varepsilon^*|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1,\ \mathbf{y}_\varepsilon^*|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2,\ \mathbf{y}_\varepsilon^*|_{\Gamma^3} = 0,\ \text{and } \mathbf{y}_\varepsilon^* \rightharpoonup \mathbf{y}^* \ \text{ in } \ \mathbf{H}^2(\Omega), \tag{14}$$

$$\mathbf{f}_\varepsilon \rightharpoonup \mathbf{f} \ \text{ in } \ \mathbf{L}^2(\Omega). \tag{15}$$

We also say that a boundary velocity field $\overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, ..., \alpha_{\mathbf{k}_{J_\varepsilon}}\right)$ is admissible if there exists a function $\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)$ (the so-called prototype of the boundary control $\overline{\alpha}_\varepsilon$) such that $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma$ (for a given value $\gamma > 0$) and

$$\mathbf{u}|_{\Gamma_\varepsilon^1} = \mathbf{y}_\varepsilon^1,\ \mathbf{u}|_{\Gamma_\varepsilon^2} = \mathbf{y}_\varepsilon^2,\ \mathbf{u}|_{\Gamma^3} = 0,\ \mathbf{u}\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} = \alpha_{\mathbf{k}_j}, \quad \forall j = 1, \dots, J_\varepsilon. \tag{16}$$

The constants λ and β in the functional (8) are two positive parameters that adjust the relative weights of the two terms in the functional. Such choice of the cost functional is motivated by the fact that irrotational flows have no local flow recirculations. On the other hand, for both physical and mathematical reasons, the size of the control should be constrained. So, following the representation

$$\begin{aligned} |\operatorname{curl}\mathbf{y}_\varepsilon|^2 &= \operatorname{curl}\mathbf{y}_\varepsilon : \operatorname{curl}\mathbf{y}_\varepsilon \\ &= |\nabla \mathbf{y}_\varepsilon|^2 + \left|{}^t(\nabla \mathbf{y}_\varepsilon)\right|^2 - \nabla \mathbf{y}_\varepsilon : {}^t(\nabla \mathbf{y}_\varepsilon) - {}^t(\nabla \mathbf{y}_\varepsilon) : \nabla \mathbf{y}_\varepsilon\,, \end{aligned} \tag{17}$$

we hope that minimizing the functional (8) will lead to reduction in flow recirculations.

Our main result is: the boundary velocity field

$$\overline{\alpha}_\varepsilon^{sub} = \left(\alpha_{\mathbf{k}_1}^{sub}, \alpha_{\mathbf{k}_2}^{sub}, ..., \alpha_{\mathbf{k}_{J_\varepsilon}}^{sub}\right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}$$

can be taken as the suboptimal control for the problem (8)–(12), where $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ is some linear bounded operator (see (109)–(110)), and $\mathbf{u}^0$ is a solution to one of the following problems:

1. in the case when $C_0 = \lim_{\varepsilon\to 0} \varepsilon^2 (\log 1/r_\varepsilon) = +\infty$, the functional

$$\int_\Omega |\mathbf{u}(x)|^2\, dx \tag{18}$$

is minimized subject to the constraints

$$\left\{ \mathbf{u}(x) \in \mathbf{H}^2(\Omega) \;\middle|\; \begin{array}{l} \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma, \quad \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u}|_{\Gamma^1} = \mathbf{y}^*|_{\Gamma^1}\,,\ \mathbf{u}|_{\Gamma^2} = \mathbf{y}^*|_{\Gamma^2}\,,\ \mathbf{u}|_{\Gamma^3} = 0, \\ \mathbf{y}^*\cdot\mathbf{n} = 0 \text{ on } \Gamma^1\cup\Gamma^2, \end{array} \right\} \tag{19}$$

2. when $0 < C_0 < +\infty$, the cost functional

$$\mathcal{J}_0(\mathbf{u},\mathbf{y}) = \lambda\int_\Omega |\nabla\mathbf{y}|^2\,dx + \frac{2\pi\lambda}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2\,dx + \beta|\partial Q|_H \int_\Omega |\mathbf{u}|^2\,dx \tag{20}$$

is minimized subject to the boundary value problem for Brinkman-type law

$$-\nu\triangle\mathbf{y} + \frac{2\pi\nu}{C_0}(\mathbf{y}-\mathbf{u}) + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} \quad \text{in } \Omega; \tag{21}$$

$$\operatorname{div}\mathbf{y} = 0 \quad \text{in } \Omega, \quad \mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega}; \tag{22}$$

$$\mathbf{u}|_{\Gamma^1} = \mathbf{y}^*|_{\Gamma^1}\,,\ \mathbf{u}|_{\Gamma^2} = \mathbf{y}^*|_{\Gamma^2}\,,\ \mathbf{u}|_{\Gamma^3} = 0; \tag{23}$$

$$p \in L^2_0(\Omega),\ \mathbf{u}\in\mathbf{H}^2(\Omega),\ \mathbf{y}-\mathbf{u}\in\mathbf{H}^1_{0,sol}(\Omega),\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma. \tag{24}$$

The plan of the paper is as follows. In Section 2 we give the main notations and preliminaries. In Section 3, we introduce the class of admissible Dirichlet controls and discuss the solvability and regularity of the corresponding solutions to the boundary value problem for Navier-Stokes equation. In Section 4, we deal with the sufficient conditions for the solvability of the original optimal control problem. In Section 5, we reformulate the original problem as some constrained optimization problem in the variable space. We also introduce and discuss the main Hypotheses on the perforation type that will be used later for the identification of the limit problem. Section 6 has a technical character and some concepts of the convergence in the variable spaces are introduced. In Section 7 we give the definition of the suboptimal controls and of the variational limit for the original optimal control problem and we show that the limit problem presents 'good variational properties'. In Sections 8 and 9 we demonstrate that the limit optimization problem can be recovered in an explicit form. Moreover, we focus on two cases: when cross-size of the thin cylinders is smaller, and when the cylinders have so-called critical size. In the first case we show that the limit problem has the structure of some calculus variation problem, whereas the second case leads us to the optimal control problem for the Brinkman's law. The limit cost functional has also a different structure from the original one. Finally, in Section 10 we describe the structure of the suboptimal controls and discuss their properties.

2. **Preliminaries and notation.** Throughout the paper we suppose that the boundaries $\partial\widetilde{\Omega}$ is of class C^∞, so, $\Omega = \widetilde{\Omega} \times (0, l)$ is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; Q is a compact subset of $\widetilde{Y}$ with Lipschitz boundary ∂Q, $\operatorname{int} Q$ is a strongly connected set, $Q \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_1 \geq 0\}$, and its boundary ∂Q contains the origin; $A = B(\mathbf{0}, r_0)$ is an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \subset\subset \widetilde{Y}$ and $Q \subset\subset A$ (see Fig. 1); C or C_i (where i is any subscript) denotes a constant independent of ε. For any subset $E \subset \mathbb{R}^n$ we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}^n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of the manifold ∂E on $\mathbb{R}^n$. We will use the standard notations for the Lebesgue function space $L^p(\Omega)$ and the Sobolev spaces $H^m(\Omega)$ of square-integrable functions for real smoothness indices m. For the definition of fractional ordered Sobolev spaces $H^l(\partial\Omega)$ (l non-integer) see [1]. Let $L^2_0(\Omega) = \left\{q \in L^2(\Omega) \ : \ \int_\Omega q\, dx = 0\right\}$ with the norm $\|p\|^2_{L^2_0(\Omega)} = \int_\Omega \left(p(x) - \int_\Omega p(x)\, dx\right)^2 dx$, and $L^2(\Omega, d\mu)$ be the Banach space of squared integrable functions in Ω with respect to the measure μ. If the measure μ is the Lebesgue one, we abbreviate the notation using $L^2(\Omega)$.

The (harmonic) capacity of a set E with respect to Ω ($\operatorname{cap}(E, \Omega)$) is defined as the infimum of $\int_\Omega |Dy|^2\, dx$ over the set of all functions $y \in H^1_0(\Omega)$ such that $y \geq 1$ a.e. in a neighbourhood of E. Let $\mathcal{M}_b(\Omega)$ be the space of bounded Borel measures on Ω. Let $\mathcal{M}^+_0(\Omega)$ be the cone of all nonnegative Borel measures μ on Ω such that $\mu(B) = 0$ for every set $B \subseteq \Omega$ with $\operatorname{cap}(B, \Omega) = 0$, and $\mu(B) = \inf\{\mu(U) \ : \ U \text{ quasi open}, B \subseteq U\}$ for every Borel set $B \subseteq \Omega$. The space $\mathcal{D}'(\Omega)$ of distribution in Ω is the dual of the space $C^\infty_0(\Omega)$. For $m \geq 0$, we introduce the subspaces $H^m_0(\Omega)$ of the Sobolev spaces $H^m(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $H^m(\Omega)$ and the dual spaces $H^{-m}(\Omega) = (H^m_0(\Omega))^*$. The duality pairing between a Sobolev space $H^s(\Omega)$ ($s > 0$) and its dual space is denoted by $\langle \cdot, \cdot \rangle_{H^{-s}(\Omega); H^s(\Omega)}$. The trace spaces $H^l(\partial\Omega)$ are the restriction to the boundary of $H^{l+1/2}(\Omega)$ (see [31]). The vector-valued counterparts of these spaces are denoted by boldface symbols, e.g. $\mathbf{L}^r(\Omega)$, $\mathbf{H}^m(\Omega)$, $\mathbf{H}^l(\partial\Omega)$, $\mathbf{H}^m_0(\Omega)$, and $\mathbf{C}^\infty_0(\Omega)$. If $\mathbf{u}$ is a vector-valued function from $\mathbb{R}^N$ to $\mathbb{R}^N$, then

1. The gradient of $\mathbf{u}$ is a $N \times N$ tensor: $\nabla \mathbf{u} = (\partial u_i / \partial x_j)_{1 \leq i,j \leq N}$;
2. The inner product of two $N \times N$ tensors $A = (a_{ij})$ and $B = (b_{ij})$ is denoted by: $A : B = \operatorname{tr}({}^t AB) = \sum_{1 \leq i,j \leq N} a_{ij} b_{ij}$.

We will also use the spaces of solenoidal vector fields

$$
\begin{aligned}
\mathbf{C}^\infty_{0,sol}(\Omega) &= \left\{\mathbf{y} \in \mathbf{C}^\infty_0(\Omega) \ : \ \operatorname{div} \mathbf{y} = \textstyle\sum_{i=1}^3 \frac{\partial y_i}{\partial x_i} = 0 \ \text{ in } \ \Omega\right\}, \\
\mathbf{H}^m_{sol}(\Omega) &= \left\{\mathbf{y} \in \mathbf{H}^m(\Omega) \ : \ \nabla \cdot \mathbf{y} = 0, \ \textstyle\int_{\partial\Omega} \mathbf{y} \cdot \mathbf{n}\, d\mathcal{H}^2 = 0\right\}, \\
\mathbf{H}^m_{0,sol}(\Omega) &= \text{the closure of } \ \mathbf{C}^\infty_{0,sol}(\Omega) \ \text{ in } \ \mathbf{H}^m(\Omega)\text{-norm},
\end{aligned}
$$

where, when $m = 0$, $\int_{\partial\Omega} \mathbf{y} \cdot \mathbf{n}\, d\mathcal{H}^2$ is the $\left(H^{-1/2}(\partial\Omega); H^{1/2}(\partial\Omega)\right)$ duality pairing between the function $(\mathbf{y} \cdot \mathbf{n}) \in H^{-1/2}(\partial\Omega)$ and the constant scalar function $1 \in$

$H^{1/2}(\partial\Omega)$. The norm on $\mathbf{H}^m_{sol}(\Omega)$ and $\mathbf{H}^m_{0,sol}(\Omega)$ is chosen to be that of $\mathbf{H}^m(\Omega)$. We define the divergence operator as follows

$$\langle \operatorname{div} \mathbf{y}, \varphi\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} = -\int_\Omega \mathbf{y}\cdot\nabla\varphi\,dx, \quad \forall\,\varphi\in H^1_0(\Omega). \tag{25}$$

Let $\mathbf{H}(\operatorname{div},\Omega) = \{\mathbf{y}\in\mathbf{L}^2(\Omega)\ :\ \operatorname{div}\mathbf{y}\in L^2(\Omega)\}$. We will need the following lemma on integration by parts for functions in the space $\mathbf{H}(\operatorname{div},\Omega)$ (for the proof see [31]).

Lemma 2.1. *Let* $\mathbf{w}\in\mathbf{H}(\operatorname{div},\Omega)$. *Then* $(\mathbf{w}\cdot\mathbf{n})|_{\partial\Omega}\in H^{-1/2}(\partial\Omega)$ *and*

$$\langle \mathbf{w}\cdot\mathbf{n}, v\rangle_{H^{-1/2}(\partial\Omega);H^{1/2}(\partial\Omega)} = \int_\Omega v\operatorname{div}\mathbf{w}\,dx + \int_\Omega \mathbf{w}\cdot\nabla v\,dx \quad \forall\,v\in H^1(\Omega). \tag{26}$$

To the end of this section, we define the standard bilinear, trilinear forms associated with the Navier-Stokes equations:

$$\begin{aligned}
a_\varepsilon(\mathbf{y},\mathbf{v}) &= \int_{\Omega_\varepsilon}(\nabla\mathbf{y}):(\nabla\mathbf{v})\,dx \quad \forall\,\mathbf{y},\mathbf{v}\in\mathbf{H}^1(\Omega_\varepsilon),\\
b_\varepsilon(\mathbf{y},q) &= -\int_{\Omega_\varepsilon} q\operatorname{div}\mathbf{y}\,dx \quad \forall\,\mathbf{y}\in\mathbf{H}^1(\Omega_\varepsilon),\forall\,q\in L^2(\Omega_\varepsilon),\\
c_\varepsilon(\mathbf{y},\mathbf{v},\mathbf{w}) &= \int_{\Omega_\varepsilon}(\mathbf{y}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx \quad \mathbf{y},\mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega_\varepsilon).
\end{aligned}$$

3. **Admissible controls and regularity solutions to the boundary value problem for Navier-Stokes equations.** We devote this section to the study of boundary value problem (9)-(12). For this we give the definition of a solution for the Navier-Stokes equations with inhomogeneous Dirichlet boundary conditions. Throughout, we assume that $\mathbf{f}_\varepsilon\in\mathbf{L}^2(\Omega)$ and $\sup_{\varepsilon>0}\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)} < +\infty$. We also assume that there are functions ${\mathbf{y}^*}_\varepsilon\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_{sol}(\Omega_\varepsilon)$ and $\mathbf{y}^*\in\mathbf{H}^2(\Omega)$ satisfying (14).

Definition 3.1. A pair $(\mathbf{y}_\varepsilon,p_\varepsilon)\in\mathbf{H}^1(\Omega_\varepsilon)\times L^2_0(\Omega_\varepsilon)$ is said to be a solution of the Navier-Stokes equations (9)-(12) iff

$$\nu a_\varepsilon(\mathbf{y}_\varepsilon,\mathbf{v}) + c_\varepsilon(\mathbf{y}_\varepsilon,\mathbf{y}_\varepsilon,\mathbf{v}) + b_\varepsilon(\mathbf{v},p_\varepsilon) = \int_{\Omega_\varepsilon}\mathbf{f}_\varepsilon\cdot\mathbf{v}\,dx, \quad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega_\varepsilon), \tag{27}$$

$$b_\varepsilon(\mathbf{y}_\varepsilon,q) = 0, \quad \forall\,q\in L^2_0(\Omega_\varepsilon), \tag{28}$$

and

$$\mathbf{y}_\varepsilon|_{\Gamma^1_\varepsilon} = \mathbf{y}^1_\varepsilon,\ \ \mathbf{y}_\varepsilon|_{\Gamma^2_\varepsilon} = \mathbf{y}^2_\varepsilon,\ \ \mathbf{y}_\varepsilon|_{\Gamma^3_\varepsilon} = 0,\ \ \mathbf{y}_\varepsilon|_{\partial T^{\mathbf{k}_j}_\varepsilon} = \alpha_{\mathbf{k}_j}, \quad \forall j=1,\dots,J_\varepsilon. \tag{29}$$

A proof of the existence of a solution in the sense of Definition 3.1 can be found in [31].

Let $\gamma > 0$ be a given value. We say that a boundary velocity field $\overline{\alpha}_\varepsilon = (\alpha_{\mathbf{k}_1},\dots,\alpha_{\mathbf{k}_{J_\varepsilon}})$ is admissible if there exists a function $\mathbf{u}\in\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)$ (the so-called prototype of the boundary control $\overline{\alpha}_\varepsilon$) such that $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le\gamma$ and conditions (16) are satisfied. Such choice of admissible controls is motivated by the fact that the trace space $\mathbf{H}^{1/2}(\partial\Omega_\varepsilon)$ is the restriction to the boundary of $\mathbf{H}^1(\Omega_\varepsilon)$. Hence, in view of the initial supposition, for a fixed $\overline{\alpha}_\varepsilon$ there exists a function $\mathbf{u}\in\mathbf{H}^1(\Omega)$ satisfying conditions (16). Then, due to (13) and the Stokes formula (26), we have $\int_{\Omega_\varepsilon}\operatorname{div}\mathbf{u}\,dx = 0$.

We denote by $\mathbf{U}_\varepsilon$ the set of all admissible controls for a fixed ε, i.e.

$$\mathbf{U}_\varepsilon=\left\{\overline{\alpha}_\varepsilon=\left(\alpha_{\mathbf{k}_1},\alpha_{\mathbf{k}_2},...,\alpha_{\mathbf{k}_{J_\varepsilon}}\right)\;\middle|\;\begin{array}{l}\alpha_{\mathbf{k}_j}=\mathbf{u}|_{\partial T_\varepsilon^{\mathbf{k}_j}},\ \forall j=1,\dots,J_\varepsilon\\ \mathbf{u}\in\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega),\\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le\gamma,\\ \mathbf{u}|_{\Gamma^1_\varepsilon}=\mathbf{y}^1_\varepsilon,\ \ \mathbf{u}|_{\Gamma^2_\varepsilon}=\mathbf{y}^2_\varepsilon,\ \ \mathbf{u}|_{\Gamma^3_\varepsilon}=0.\end{array}\right\} \tag{30}$$

Definition 3.2. We say that a triplet $(\overline{\alpha}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)$ is admissible to the optimal control problem (8)–(12), if $\overline{\alpha}_\varepsilon\in\mathbf{U}_\varepsilon$ and the pair $(\mathbf{y}_\varepsilon,p_\varepsilon)$ is a corresponding solution of the variational problem (27)–(29).

To establish the regularity properties of the admissible solutions we will use the following result (see [31]):

Theorem 3.3. *Let $(\mathbf{y},p)\in\mathbf{W}^{2,\alpha}(\Omega)\times W^{1,\alpha}(\Omega)$ $(1<\alpha<\infty)$ be a solution of the following Stokes problem:*

$$-\nu\triangle\mathbf{y}+\nabla p=\mathbf{f}\quad in\ \ \Omega,\quad \operatorname{div}\mathbf{y}=0\quad in\ \ \Omega,\quad \mathbf{y}=\varphi\quad on\ \ \partial\Omega.$$

Assume that $\mathbf{f}\in\mathbf{W}^{m,\alpha}(\Omega)$, $\varphi\in\mathbf{W}^{m+2-1/\alpha,\alpha}(\partial\Omega)$, and $\Omega\in C^{m+2}$ for some integer $m\ge1$. Then $\mathbf{y}\in\mathbf{W}^{m+2,\alpha}(\Omega)$, $p\in W^{m+1,\alpha}(\Omega)$.

We are now able to prove the main result of this section.

Theorem 3.4. *Let $\overline{\alpha}_\varepsilon$ be an admissible control ($\overline{\alpha}_\varepsilon\in\mathbf{U}_\varepsilon$), and let $\mathbf{u}_\varepsilon\in\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)$ be its prototype. Then there exists a corresponding velocity-pressure pair $(\mathbf{y}_\varepsilon,p_\varepsilon)\in\mathbf{H}^2_{sol}(\Omega_\varepsilon)\times\left[H^1(\Omega_\varepsilon)\cap L^2_0(\Omega_\varepsilon)\right]$ satisfying the boundary value problem (9)–(12) in the following variational sense:*

$$\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon\in\mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \tag{31}$$

$$\nu a_\varepsilon(\mathbf{y}_\varepsilon,\mathbf{v})+c_\varepsilon(\mathbf{y}_\varepsilon,\mathbf{y}_\varepsilon,\mathbf{v})=\int_{\Omega_\varepsilon}\mathbf{f}_\varepsilon\cdot\mathbf{v}\,dx,\quad\forall\,\mathbf{v}\in\mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \tag{32}$$

$$\nabla p_\varepsilon=\nu\triangle\mathbf{y}_\varepsilon-(\mathbf{y}_\varepsilon\cdot\nabla)\mathbf{y}_\varepsilon+\mathbf{f}_\varepsilon\quad in\ \ \mathcal{D}'(\Omega_\varepsilon). \tag{33}$$

Proof. The proof follows standard techniques dealing with the existence of a solution for the Navier-Stokes equations with inhomogeneous Dirichlet conditions and the regularity theory for such equations (see [16] and [31]). First, we set $\widehat{\mathbf{y}}_\varepsilon=\mathbf{y}_\varepsilon-\mathbf{u}_\varepsilon$. In view of the definition of control prototypes (see (30)) and conditions (10)–(12), we have:

$$\nabla\cdot\widehat{\mathbf{y}}_\varepsilon=0\ \text{ in }\ \Omega_\varepsilon,\quad\int_{\partial\Omega_\varepsilon}\widehat{\mathbf{y}}_\varepsilon\cdot\mathbf{n}\,d\mathcal{H}^2=0. \tag{34}$$

On the other hand, the function $\widehat{\mathbf{y}}_\varepsilon$ must satisfy the equation

$$-\nu\triangle\widehat{\mathbf{y}}_\varepsilon+(\widehat{\mathbf{y}}_\varepsilon\cdot\nabla)\widehat{\mathbf{y}}_\varepsilon+(\widehat{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon+(\mathbf{u}_\varepsilon\cdot\nabla)\widehat{\mathbf{y}}_\varepsilon+\nabla p_\varepsilon=\widehat{\mathbf{f}}_\varepsilon, \tag{35}$$

where $\widehat{\mathbf{f}}_\varepsilon=\mathbf{f}_\varepsilon+\nu\triangle\mathbf{u}_\varepsilon+(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon$. Using the regularity of $\mathbf{u}_\varepsilon$ and Sobolev Embedding Theorem, we obtain that $\widehat{\mathbf{f}}_\varepsilon\in\mathbf{L}^2(\Omega_\varepsilon)$. Hence, due to the well-known Temam's result (see [31]), there exists a pair $(\widehat{\mathbf{y}}_\varepsilon,p_\varepsilon)\in\mathbf{H}^1_{0,sol}(\Omega_\varepsilon)\times L^2_0(\Omega_\varepsilon)$ satisfying the problem (34)–(35) in the variational sense, i.e.

$$\nu a_\varepsilon(\widehat{\mathbf{y}}_\varepsilon,\mathbf{v})+c_\varepsilon(\widehat{\mathbf{y}}_\varepsilon,\widehat{\mathbf{y}}_\varepsilon,\mathbf{v})+c_\varepsilon(\widehat{\mathbf{y}}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v})+c_\varepsilon(\mathbf{u}_\varepsilon,\widehat{\mathbf{y}}_\varepsilon,\mathbf{v})$$
$$=\int_{\Omega_\varepsilon}\widehat{\mathbf{f}}_\varepsilon\cdot\mathbf{v}\,dx\quad\forall\,\mathbf{v}\in\mathbf{H}^1_{0,sol}(\Omega_\varepsilon).$$

It follows that $\widehat{\mathbf{y}}_\varepsilon+\mathbf{u}_\varepsilon=\mathbf{y}_\varepsilon\in\mathbf{H}^1(\Omega_\varepsilon)$. We wish to show that $(\mathbf{y}_\varepsilon,p_\varepsilon)\in\mathbf{H}^2(\Omega_\varepsilon)\times\left[H^1(\Omega_\varepsilon)\cap L^2_0(\Omega_\varepsilon)\right]$ and relation (32) holds true. Since the last statement is obvious, we prove the previous one. Indeed, using the embedding $\mathbf{H}^1(\Omega_\varepsilon) \hookrightarrow \mathbf{L}^6(\Omega_\varepsilon)$ and $\mathbf{H}^2(\Omega_\varepsilon) \hookrightarrow \mathbf{C}(\overline{\Omega}_\varepsilon)$, we have $\widehat{\mathbf{y}}_\varepsilon \in L^6(\Omega_\varepsilon)$. Therefore,

$$(\widehat{\mathbf{y}}_\varepsilon \cdot \nabla)\widehat{\mathbf{y}}_\varepsilon \in \mathbf{L}^{3/2}(\Omega_\varepsilon),\ (\widehat{\mathbf{y}}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \in \mathbf{L}^{3/2}(\Omega_\varepsilon),\quad \text{and}\quad (\mathbf{u}_\varepsilon \cdot \nabla)\widehat{\mathbf{y}}_\varepsilon \in \mathbf{L}^2(\Omega_\varepsilon).$$

Then, Theorem 3.3 leads us to the conclusion that $\widehat{\mathbf{y}}_\varepsilon \in \mathbf{W}^{2,3/2}(\Omega_\varepsilon)$. Using the imbedding results for Sobolev spaces, we get $\widehat{\mathbf{y}}_\varepsilon \in \mathbf{L}^\alpha(\Omega_\varepsilon)$ for every $\alpha \in [1,+\infty)$. Hence, $(\widehat{\mathbf{y}}_\varepsilon \cdot \nabla)\widehat{\mathbf{y}}_\varepsilon \in \mathbf{W}^{-1,\alpha}(\Omega_\varepsilon)$. From Theorem 3.3 we get $\widehat{\mathbf{y}}_\varepsilon \in \mathbf{W}^{1,\alpha}(\Omega_\varepsilon)$ and $p_\varepsilon \in L^\alpha(\Omega_\varepsilon)$. Hence, $(\widehat{\mathbf{y}}_\varepsilon \cdot \nabla)\widehat{\mathbf{y}}_\varepsilon \in \mathbf{L}^\alpha(\Omega_\varepsilon)$. Repeating this procedure ones more, we obtain: $\widehat{\mathbf{y}}_\varepsilon \in \mathbf{W}^{2,2}(\Omega_\varepsilon)$ and $p_\varepsilon \in W^{1,2}(\Omega_\varepsilon)$. So, taking into account the regularity of the function $\mathbf{u}_\varepsilon$, we come to the required result. □

At the end of this section, we cite the following result (its proof can be found in Duvaut & Lions [14], Fursikov [16], and Temam [31]) that will be useful in the sequel:

Proposition 1. *Let $\mathbf{u}_\varepsilon \in \mathbf{U}_\varepsilon$ be a prototype of some admissible control. Assume $\mathbf{y}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ satisfies conditions (31)–(32). Then there exists a continuous positive function $B : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (independent of ε) such that the following estimate holds true:*

$$\|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)} \le B\left(\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)}, \|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(\Omega)}, \|\mathbf{u}_\varepsilon\|^2_{\mathbf{H}^1(\Omega)}\right). \tag{36}$$

4. On solvability of the optimal boundary control problem.

The optimal Dirichlet control problem we consider can be precisely stated as follows: seek a triplet $(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \mathbf{U}_\varepsilon \times \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ such that the cost functional (8) is minimized subject to the relations (27)–(29). We denote this problem by $(\mathbb{P}_\varepsilon)$. We also define the set of admissible solutions for $(\mathbb{P}_\varepsilon)$ by

$$\Xi_\varepsilon = \left\{ (\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \;\middle|\; \begin{array}{l} \overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1}, \alpha_{\mathbf{k}_2}, ..., \alpha_{\mathbf{k}_{\mathrm{J}_\varepsilon}}\right) \in \mathbf{U}_\varepsilon, \\ \mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon), \\ (\mathbf{y}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon), \\ (\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \text{ satisfies (27)–(29).} \end{array} \right\}. \tag{37}$$

Now we are able to prove the existence of a solution to $(\mathbb{P}_\varepsilon)$.

Theorem 4.1. *The optimal control problem $(\mathbb{P}_\varepsilon)$ has a solution iff this problem is regular, that is, $\Xi_\varepsilon \neq \emptyset$ for every fixed $\varepsilon > 0$.*

Proof. It is clear that we need to prove only sufficient conditions of this statement. Let us fix $\varepsilon > 0$. Since $\mathbf{U}_\varepsilon \neq \emptyset$, it obviously follows from Theorem 3.4 that the set Ξ_ε is non-empty as well. Hence we may choose a minimizing sequence $\{(\overline{\alpha}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}, p_{\varepsilon,m}) \in \Xi_\varepsilon\}$ such that

$$\lim_{m\to\infty} \mathcal{J}_\varepsilon(\overline{\alpha}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}) = \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon).$$

The boundedness of $\{\mathcal{J}_\varepsilon(\overline{\alpha}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m})\}$ and implicit control constraints imply the boundedness of control prototypes $\left\{\|\mathbf{u}_{\varepsilon,m}\|_{\mathbf{H}^2(\Omega)}\right\}$ and $\left\{\|\nabla\, \mathbf{y}_{\varepsilon,m}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\right\}$. Then using (36), we see that the set $\left\{\|\mathbf{y}_{\varepsilon,m}\|_{\mathbf{H}^1(\Omega_\varepsilon)}\right\}$ is also bounded independent of m. Hence we may extract subsequences (still denoted by $\mathbf{u}_{\varepsilon,m}$ and $\mathbf{y}_{\varepsilon,m}$, respectively) such that $\mathbf{u}_{\varepsilon,m} \rightharpoonup \mathbf{u}_\varepsilon$ in $\mathbf{H}^2(\Omega)$, and $\mathbf{y}_{\varepsilon,m} \rightharpoonup \mathbf{y}_\varepsilon$ in $\mathbf{H}^1(\Omega)$ for some $(\mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega)$, as $m \to \infty$. Since $\mathbf{u}_{\varepsilon,m} \in \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ for every $m \in \mathbb{N}$, $\|\mathbf{u}_{\varepsilon,m}\|_{\mathbf{H}^2(\Omega)} \le \gamma$, and $\mathbf{u}_{\varepsilon,m} \to \mathbf{u}_\varepsilon$ in $\mathbf{H}^1(\Omega)$, it follows that $\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega_\varepsilon)$, $\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^2(\Omega)} \le \gamma$, that is, $\mathbf{u}_\varepsilon$ is an admissible control prototype ($\mathbf{u}_\varepsilon \in \mathbf{U}_\varepsilon$). On the other hand, since the trace of $\mathbf{H}^2(\Omega)$ equals $\mathbf{H}^{3/2}(\partial T_\varepsilon)$ and the space $\mathbf{H}^{3/2}(\partial T_\varepsilon)$ is compactly imbedded into $\mathbf{C}(\partial T_\varepsilon)$, we have:

$$\mathbf{u}_{\varepsilon,m} \rightharpoonup \mathbf{u}_\varepsilon \ \text{ in } \ \mathbf{H}^2(\Omega) \text{ implies } \ \mathbf{u}_{\varepsilon,m}\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} \equiv \alpha_{\mathbf{k}_j,m} \to \alpha_{\mathbf{k}_j} \equiv \mathbf{u}_\varepsilon\big|_{\partial T_\varepsilon^{\mathbf{k}_j}} \ \text{ in } \ \mathbf{C}(\Omega)$$

as $m \to \infty$ for all $j = 1, \dots, J_\varepsilon$.

Moreover, due to the imbedding results for Sobolev spaces, we also get the strong convergence $\mathbf{y}_{\varepsilon,m} \to \mathbf{y}_\varepsilon$ in $\mathbf{L}^4(\Omega_\varepsilon)$ as $m \to \infty$. Then, using standard techniques in proving the existence of a solution to the steady-state Navier-Stokes equations, we may pass to the limit in the relation

$$\mathbf{y}_{\varepsilon,m} - \mathbf{u}_{\varepsilon,m} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon),$$
$$\nu a_\varepsilon(\mathbf{y}_{\varepsilon,m}, \mathbf{v}) + c_\varepsilon(\mathbf{y}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}, \mathbf{v}) = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$$

as $m \to \infty$ to conclude that $\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$ and $\mathbf{y}_\varepsilon$ satisfies equation (32). As a result, applying Theorem 3.4, we get that $\mathbf{y}_\varepsilon \in \mathbf{H}^2_{sol}(\Omega_\varepsilon)$ and there exists the corresponding pressure $p_\varepsilon \in H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon)$ such that the relations (27)–(28) hold true, i.e., the triplet $(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon, p_\varepsilon)$ is admissible to the problem $(\mathbb{P}_\varepsilon)$.

Finally, using the compact imbedding results and the sequential weak lower semi-continuity of the cost functional $\mathcal{J}_\varepsilon(\cdot,\cdot)$ with respect to the product of the weak topologies of $\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega_\varepsilon)$, we obtain

$$\mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon) \le \liminf_{m\to\infty} \mathcal{J}_\varepsilon(\overline{\alpha}_{\varepsilon,m}, \mathbf{y}_{\varepsilon,m}) = \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon).$$

Hence, $(\mathbf{u}_\varepsilon|_{\partial T_\varepsilon}, \mathbf{y}_\varepsilon)$ is an optimal triplet for the problem $(\mathbb{P}_\varepsilon)$. □

Since the boundary value problem (9)–(12) may have a non-unique solution under a fixed boundary control, in what follows we define the binary relation $\langle L; \Xi_\varepsilon \rangle$ on each of the sets Ξ_ε by the rule: $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\, L\, (\overline{\widehat{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon)$ if and only if $\overline{\alpha}_\varepsilon = \overline{\widehat{\alpha}}_\varepsilon$. It is easily seen that $\langle L; \Xi_\varepsilon \rangle$ is an equivalence relation. So, hereinafter we will not distinguish the triplets belonging to the same class of equivalence.

5. **Reformulation of the problem $(\mathbb{P}_\varepsilon)$.** We begin this section with the description of the geometry of the perforated domain Ω_ε. We describe the class of admissible solutions to the optimal control problem (8)–(12) in terms of singular periodic Borel measures on $\mathbb{R}^3$, using the approach of Zhikov, Bouchitté and Fragala (see [5, 33]).

Let us denote by Q^r the homothetic contractions of the set Q at r^{-1} times. In what follows it is assumed that $0 < r \le 1$. Let η_0^r be the normalized periodic Borel measure on $\mathbb{R}^2$ with the periodicity cell $\widetilde{Y}$ such that η_0^r is concentrated and uniformly distributed on the set ∂Q^r, and is proportional to the one-dimensional Hausdorff measure. It is clear that in this case $\eta_0^r(\widetilde{Y} \setminus \partial Q^r) = 0$.

Now we consider the following measure $d\eta^r = d\eta_0^r \times dx_3$ in $Y = [-1/2, 1/2)^2 \times [0, 1)$. It is easy to see that this measure concentrated on the set $\partial Q^r \times [0, 1)$, and for any smooth function g we have

$$\int_Y g\, d\eta^r = \int_0^1 \int_{\widetilde{Y}} g\, dx_3\, d\eta_0^r \;=\; \left[\mathcal{H}^2(\partial Q^r \times [0,1))\right]^{-1} \int_{\partial Q^r \times [0,1)} g\, d\mathcal{H}^2.$$

However, as follows from the properties of the Hausdorff measure, we have $\mathcal{H}^2(\partial Q^r \times [0, 1)) = \mathcal{H}^1(\partial Q^r) = r\mathcal{H}^1(\partial Q)$ (see [15]). Using the notation $|\partial Q|_H = \mathcal{H}^1(\partial Q)$, the previous relation can be rewritten in the following form

$$r\int_Y g\,d\eta^r = r\int_0^1\int_{\widetilde{Y}} g\,dx_3\,d\eta_0^r \;=\; |\partial Q|_H^{-1}\int_{\partial Q\times(0,1)} g\,d\mathcal{H}^2. \tag{38}$$

Thus, $|\partial Q|_H$ is the one-dimensional Hausdorff measure of the set ∂Q.

We introduce the scaling measure η_ε^r by setting $\eta_\varepsilon^r(B) = \varepsilon^3\eta^r(\varepsilon^{-1}B)$ for every Borel set $B\subset\mathbb{R}^3$. The parameters r and ε are related by the rule $r(\varepsilon) = r_\varepsilon$, where $0 < r_\varepsilon \le 1$ and $\lim_{\varepsilon\to 0} r_\varepsilon = 0$. Then $\int_{\varepsilon Y} d\eta_\varepsilon^r = \varepsilon^3\int_Y d\eta^r = \varepsilon^3$. It means that the measure η_ε^r weakly converges to the Lebesgue measure: $d\eta_\varepsilon^r \rightharpoonup dx$, that is, for every $\varphi\in C_0^\infty(\mathbb{R}^3)$ we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\varphi\,d\eta_\varepsilon^r = \int_{\mathbb{R}^3}\varphi\,dx. \tag{39}$$

By analogy it can be shown that the scaling measure $\eta_{0,\varepsilon}^r$ on $\mathbb{R}^2$ (by definition, $\eta_{0,\varepsilon}^r(B) = \varepsilon^2\eta_0^r(\varepsilon^{-1}B)$ for every Borel set $B\subset\mathbb{R}^2$) weakly converges to the two-dimensional Lebesgue measure.

Remark 1. It is easy to see that the scaling measure η_ε^r belong to the class $\mathcal{M}_0^+(\Omega)$. Note that if $\eta\in\mathcal{M}_0^+(\Omega)$, then the functions of $H^1(\Omega)$ are defined η-almost everywhere and are η-measurable in Ω. Hence the space $\mathbf{H}^1(\Omega)\cap\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ is well defined (see [12]).

We note that, in view of relation (38), the term

$$\frac{\beta\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon}\int_{\partial T_\varepsilon^{\mathbf{k}_j}}\left|\alpha_{\mathbf{k}_j}\right|^2 d\mathcal{H}^2$$

can be rewritten in the equivalent form

$$\begin{aligned}\frac{\beta\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon}\int_{\partial T_\varepsilon^{\mathbf{k}_j}}\left|\alpha_{\mathbf{k}_j}\right|^2 d\mathcal{H}^2 &= \beta\varepsilon^2|\partial Q|_H\sum_{j=1}^{J_\varepsilon}\int_0^1\int_{\varepsilon(\widetilde{Y}+\mathbf{k}_j)}|\mathbf{u}_\varepsilon|^2\,d\eta_0^{r(\varepsilon)}(x/\varepsilon)\,dx_3\\ &=\beta\varepsilon^3|\partial Q|_H\sum_{j=1}^{J_\varepsilon}\int_{\varepsilon(Y+\mathbf{k}_j)}|\mathbf{u}_\varepsilon|^2\,d\eta^{r(\varepsilon)}(x/\varepsilon) = \beta|\partial Q|_H\int_\Omega|\mathbf{u}_\varepsilon|^2\,d\eta_\varepsilon^r\;,\end{aligned} \tag{40}$$

where $\mathbf{u}_\varepsilon$ is a prototype of the control function $\overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1},\alpha_{\mathbf{k}_2},...,\alpha_{\mathbf{k}_{J_\varepsilon}}\right)$. As a result, the original cost functional (8) takes the form

$$\mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon,\mathbf{y}_\varepsilon) = \lambda\int_\Omega\chi_\varepsilon|\nabla\,\check{\mathbf{y}}_\varepsilon|^2\,dx + \beta|\partial Q|_H\int_\Omega|\mathbf{u}_\varepsilon|^2\,d\eta_\varepsilon^{r(\varepsilon)}. \tag{41}$$

Here χ_ε is the characteristic function of the perforated domain Ω_ε, and $\check{\mathbf{y}}_\varepsilon\in\mathbf{H}^1(\Omega)$ is 'some' extension of the function $\mathbf{y}_\varepsilon$ on the whole Ω.

Remark 2. Note that any admissible control $\overline{\alpha}_\varepsilon = \left(\alpha_{\mathbf{k}_1},\alpha_{\mathbf{k}_2},...,\alpha_{\mathbf{k}_{J_\varepsilon}}\right)\in\mathbf{U}_\varepsilon$ can be obviously interpreted as an element of the space $\mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})$. Indeed, let $\mathbf{u}_\varepsilon\in\mathbf{H}^1_{sol}(\Omega_\varepsilon)\cap\mathbf{H}^2(\Omega)$ be a prototype of $\overline{\alpha}_\varepsilon$. Then, using the imbedding result $\mathbf{H}^2(\Omega)\hookrightarrow\mathbf{C}(\Omega)$, we get that $\mathbf{u}_\varepsilon\in\mathbf{C}(\Omega)$. Hence $\mathbf{u}_\varepsilon$ is a $\eta_\varepsilon^r(\varepsilon)$-measurable function such that

$$\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2(\Omega,d\eta_\varepsilon^{r(\varepsilon)})} = \int_\Omega|\mathbf{u}_\varepsilon|^2\,d\eta_\varepsilon^r \le \|\mathbf{u}_\varepsilon\|^2_{\mathbf{C}(\Omega)}\eta_\varepsilon^r(\varepsilon)(\Omega) < +\infty. \tag{42}$$

So, $\mathbf{u}_\varepsilon\in\mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})$ and we obtain the required result.

Definition 5.1. Let $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$ be any admissible solution to the problem $(\mathbb{P}_\varepsilon)$. Then we say that a triplet $(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \mathbb{X}_\varepsilon$ is a prototype to $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)$, if

$$\mathbb{X}_\varepsilon = \left[\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right] \times \left[\mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^1(\Omega))\right] \times L^2_0(\Omega),$$

$\mathbf{u}_\varepsilon$ is a control prototype, and $(\breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon)$ are some extensions of the functions $(\mathbf{y}_\varepsilon, p_\varepsilon)$ on the whole Ω.

Remark 3. Let us recall that the perforated domain Ω_ε, considered here, satisfies the so-called "condition of strong connectedness" (see [26]). It means that there exist a family $\{\mathbf{P}_\varepsilon\}_{\varepsilon>0}$ of extension operators $\mathbf{P}_\varepsilon : \mathbf{H}^1(\Omega_\varepsilon) \to \mathbf{H}^1(\Omega)$ and a constant C independent of ε and $\mathbf{y}_\varepsilon$, such that $\|(\mathbf{P}_\varepsilon \mathbf{y}_\varepsilon)\|_{\mathbf{H}^1(\Omega)} \le C\, \|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}$ for every $\mathbf{y}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)$. So, we can assume that $\breve{\mathbf{y}}_\varepsilon := \mathbf{P}_\varepsilon \mathbf{y}_\varepsilon$ for some extension operator with the above properties. The main problem is to find an a priori estimate for the pressure p_ε, which yields the existence of the corresponding prototype $\breve{p}_\varepsilon \in L^2_0(\Omega)$ with a uniformly bounded norm in $L^2_0(\Omega)$. In fact, it is not obvious how to construct such extension of the pressure p_ε. For that purpose, following Allaire [3], we introduce an abstract framework of hypotheses on the cylindrical holes.

To begin with, we denote by $\{\mathbf{e}_k\}_{k=1,2,3}$ the canonical basis in $\mathbb{R}^3$, and by $\widetilde{\ }$ the extension by zero onto the cylindrical holes.

Hypotheses (H1)-(H5). Let us assume that there exist functions $(\mathbf{w}^\varepsilon_k, q^\varepsilon_k, \mu_k)$ $(1 \le k \le 3)$ and a linear map $R_\varepsilon \in \mathcal{L}\left(\mathbf{H}^1_0(\Omega); \mathbf{H}^1_0(\Omega_\varepsilon)\right)$ such that:

(H1) $\mathbf{w}^\varepsilon_k \in \mathbf{H}^1(\Omega)$, $q^\varepsilon_k \in L^2_0(\Omega)$, $\mu_k \in \mathbf{W}^{-1,\infty}(\Omega)$;
(H2) $\nabla \cdot \mathbf{w}^\varepsilon_k = 0$ in Ω and $\mathbf{w}^\varepsilon_k = 0$ in T_ε;
(H3) $\mathbf{w}^\varepsilon_k \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, and $q^\varepsilon_k \rightharpoonup 0$ in $L^2_0(\Omega)$;
(H4) $\forall\, \mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega)$ and $\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ in $\mathbf{H}^1(\Omega)$, $\mathbf{v}_\varepsilon = 0$ on the cylinders T_ε, it follows that

$$\lim_{\varepsilon\to 0} \langle \nabla q^\varepsilon_k - \nu \Delta \mathbf{w}^\varepsilon_k, \varphi\, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = \langle \mu_k, \varphi\, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}\,, \quad \forall\, \varphi \in C^\infty_0(\Omega);$$

(H5) If $\mathbf{y} \in \mathbf{H}^1_0(\Omega_\varepsilon)$, then $R_\varepsilon\left(\chi_\varepsilon \mathbf{y}\right) = \mathbf{y}$ in Ω_ε; if $\nabla \cdot \mathbf{y} = 0$ in Ω, then $\nabla \cdot (R_\varepsilon\, \mathbf{y}) = 0$ in Ω_ε;

$$\|R_\varepsilon\, \mathbf{y}\|_{\mathbf{H}^1_0(\Omega_\varepsilon)} \le C \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} \ \text{ and } \ C \ \text{ does not depend on } \ \varepsilon.$$

Remark 4. In this case, following Tartar's [30] idea, a linear continuous extension operator $P_\varepsilon \in \mathcal{L}(L^2_0(\Omega_\varepsilon); L^2_0(\Omega))$ can be constructed as

$$\langle \nabla \left[P_\varepsilon q_\varepsilon\right], \mathbf{w} \rangle_{H^{-1}(\Omega); H^1_0(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon \mathbf{w} \rangle_{H^{-1}(\Omega_\varepsilon); H^1_0(\Omega_\varepsilon)}\,, \quad \forall\, \mathbf{w} \in \mathbf{H}^1_0(\Omega). \tag{43}$$

Then it can be easily proved that the following properties hold true (see [3] Theorem 1.1.8):

(i) $P_\varepsilon q_\varepsilon = q_\varepsilon$ in $L^2_0(\Omega_\varepsilon)$;
(ii) $\|P_\varepsilon q_\varepsilon\|_{L^2_0(\Omega)} \le C \|q_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}$;
(iii) $\|\nabla \left[P_\varepsilon q_\varepsilon\right]\|_{\mathbf{H}^{-1}(\Omega)} \le C \|\nabla q_\varepsilon\|_{\mathbf{H}^{-1}(\Omega_\varepsilon)}$,

where the constant C is independent of q_ε and ε.

6. **Convergence in the variable space $\mathbb{X}_\varepsilon$.** The characteristic feature of the optimal control problem $(\widehat{\mathbb{P}}_\varepsilon)$ is the fact that, for every fixed value of ε, each of admissible triplets $(\mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \widehat{\Xi}_\varepsilon$ belong to the corresponding functional space $\mathbb{X}_\varepsilon$. Since our main goal in the next sections is to study the asymptotic behaviour of the $\widehat{\mathbb{P}}_\varepsilon$-problem as ε tends to zero, we recall the main types of convergence in variable spaces

(see [33]). We cite them with respect to the family of the periodic Borel measure η^r_ε. Let $\left\{\mathbf{u}^r_\varepsilon \in \mathbf{L}^2(\Omega, d\eta^r_\varepsilon)\right\}$ be a bounded sequence, i.e. $\limsup_{\varepsilon\to 0}\int_\Omega |\mathbf{u}^r_\varepsilon|^2\, d\eta^r_\varepsilon < +\infty$.

1. The weak convergence $\mathbf{u}^r_\varepsilon \rightharpoonup \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$ means that $\mathbf{u}\in\mathbf{L}^2(\Omega)$ and $\lim_{\varepsilon\to 0}\int_\Omega \mathbf{u}^r_\varepsilon\cdot\varphi\, d\eta^r_\varepsilon = \int_\Omega \mathbf{u}\cdot\varphi\, dx$ for any $\varphi\in C^\infty_0(\Omega)$;
2. The strong convergence $\mathbf{u}^r_\varepsilon \to \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$ means that $\mathbf{u}\in\mathbf{L}^2(\Omega)$ and $\lim_{\varepsilon\to 0}\int_\Omega \mathbf{u}^r_\varepsilon\cdot\mathbf{w}^r_\varepsilon\, d\eta^r_\varepsilon = \int_\Omega \mathbf{u}\cdot\mathbf{w}\, dx$ if $\mathbf{w}^r_\varepsilon \rightharpoonup \mathbf{w}$ in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$.

The following properties of the convergence in variable spaces hold (see [33]):

(a) *Compactness criterium:* if a sequence is bounded in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$, then this sequence is compact in the sense of the weak convergence;

(b) *Property of lower semicontinuity:* if $\mathbf{u}^r_\varepsilon \rightharpoonup \mathbf{u}$ in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$, then

$$\liminf_{\varepsilon\to 0}\int_\Omega |\mathbf{u}^r_\varepsilon|^2\, d\eta^r_\varepsilon \ge \int_\Omega |\mathbf{u}|^2\, dx. \tag{44}$$

We observe that in fact the characteristic function χ_ε of the perforated domain Ω_ε is a two-parametric and ε-periodic one, i.e.

$$\chi_\varepsilon(x) = \chi^r_\varepsilon(x) = \chi^r\left(\frac{x}{\varepsilon}\right), \quad \chi^r(x) = \begin{cases} 1, & x\in Y\setminus[Q^r\times[0,1)],\\ 0, & x\in Q^r\times[0,1) = rQ\times[0,1). \end{cases}$$

It means that the Radon measure $\chi_\varepsilon(x)dx$ can be viewed as a scaling measure $d\eta^r_\varepsilon$ such that $\eta^r_\varepsilon(B) = \varepsilon^3\eta^r(\varepsilon^{-1}B)$ for every Borel set $B\subset\mathbb{R}^3$, where η^r is the Y-periodic Borel measure in $\mathbb{R}^3$ concentrated on $Y\setminus[Q^r\times[0,1)]$, and proportional there to the Lebesgue measure $\mathcal{L}^3$, i.e. $\int_Y d\eta^r = |\widetilde{Y}\setminus Q^r|_{\mathcal{L}^2} = |\widetilde{Y}|_{\mathcal{L}^2} - r^2\,|Q|_{\mathcal{L}^2} = 1 - r^2\,|Q|_{\mathcal{L}^2}$. Therefore $d\eta^r \rightharpoonup |\widetilde{Y}|_{\mathcal{L}^2}\, dx = dx$ as $r\to 0$, and

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\chi_\varepsilon\, dx = \lim_{\varepsilon\to 0}\int_\Omega \varphi\chi^{r(\varepsilon)}\left(\frac{x}{\varepsilon}\right)\, dx = |Y|\int_\Omega \varphi\, dx = \int_\Omega \varphi\, dx \tag{45}$$

for every $\varphi\in C^\infty_0(\Omega)$.

Taking the definition of strong convergence in variable space $L^2(\Omega, \chi_\varepsilon\, dx)$ and relation (45) into account, we get the following obvious result.

Lemma 6.1. *The characteristic function χ_ε converges strongly to 1 both in $L^2(\Omega)$ and in the variable space $L^2(\Omega, \chi_\varepsilon\, dx)$ as $\varepsilon\to 0$.*

To introduce the convergence formalism for the sequences $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$, we begin with the following concepts:

Definition 6.2. We say that a sequence of controls $\{\overline{\alpha}_\varepsilon\in\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_a-converges to a function $\mathbf{a}_0$ if some sequence of its prototypes $\left\{\mathbf{u}_\varepsilon\in\mathbf{H}^2(\Omega)\cap\mathbf{L}^2(\Omega, d\eta^{r(\varepsilon)}_\varepsilon)\right\}_{\varepsilon>0}$ converges to $\mathbf{a}_0$ weakly in $\mathbf{H}^2(\Omega)$.

Definition 6.3. We say that a sequence $\{\overline{\alpha}_\varepsilon\in\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ w_b-converges to a function $\mathbf{b}_0\in\mathbf{L}^2(\Omega)$ if some sequence of its prototypes $\left\{\mathbf{u}_\varepsilon\in\mathbf{H}^2(\Omega)\cap\mathbf{L}^2(\Omega, d\eta^{r(\varepsilon)}_\varepsilon)\right\}_{\varepsilon>0}$ converges to $\mathbf{b}_0$ weakly in $\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)$, i.e.

$$\limsup_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^2(\Omega, d\eta^r_\varepsilon)} < +\infty \quad\text{and}\quad \mathbf{u}^r_\varepsilon \rightharpoonup \mathbf{b}_0 \quad\text{in}\quad \mathbf{L}^2(\Omega, d\eta^r_\varepsilon). \tag{46}$$

In order to relate w_a- and w_b-limits, and to check the correctness of these definitions we prove the following result:

Lemma 6.4. *Any sequence of admissible controls $\{\overline{\alpha}_\varepsilon\in\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ contains a subsequence for which the w_a- and w_b-limits coincide almost everywhere in Ω.*

Proof. Let $\left\{\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$ be a sequence of some control prototypes. Since this sequence is bounded in $\mathbf{H}^2(\Omega)$, we may suppose that there is an element $\mathbf{a}_0 \in \mathbf{H}^2(\Omega)$ and a subsequence of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index) such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{a}_0$ as $\varepsilon \to 0$.

By Sobolev Imbedding Theorem we have that $\mathbf{a}_0 \in \mathbf{C}(\Omega)$ and $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ in $\mathbf{C}(\Omega)$. Then

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon|^2\, d\eta_\varepsilon^r \le\; & 2\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{a}_0|^2 d\eta_\varepsilon^r + 2\limsup_{\varepsilon\to 0}\int_\Omega |\mathbf{a}_0|^2 d\eta_\varepsilon^r \\ \le & 2\limsup_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|^2_{\mathbf{C}(\Omega)}\, \eta_\varepsilon^r\left(\overline{\Omega}\right) + 2\,\|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)}\, \eta_\varepsilon^r\left(\overline{\Omega}\right) = 2\,\|\mathbf{a}_0\|^2_{\mathbf{C}(\Omega)}\, \eta_\varepsilon^r\left(\overline{\Omega}\right).\end{aligned}$$

Hence the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, and, by the compactness criterium of the weak convergence in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$, we may suppose that there is an element $\mathbf{b}_0 \in \mathbf{L}^2(\Omega)$ such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{b}_0$ in $\mathbf{L}^2(\Omega, d\eta_\varepsilon^r)$ (passing to a subsequence, if it is necessary). On the other hand, for any function $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega \varphi\,(\mathbf{a}_0 - \mathbf{b}_0)\; dx \le & \left|\int_\Omega \varphi \mathbf{a}_0\, dx - \int_\Omega \varphi \mathbf{a}_0\, d\eta_\varepsilon^r\right| + \left|\int_\Omega \varphi\,(\mathbf{a}_0 - \mathbf{u}_\varepsilon)\; d\eta_\varepsilon^r\right| \\ & + \left|\int_\Omega \varphi \mathbf{u}_\varepsilon\, d\eta_\varepsilon^r - \int_\Omega \varphi\, \mathbf{b}_0\, dx\right| \le \left|\int_\Omega \varphi \mathbf{a}_0\, dx - \int_\Omega \varphi \mathbf{a}_0\, d\eta_\varepsilon^r\right| \\ & + \|\mathbf{u}_\varepsilon - \mathbf{a}_0\|_{\mathbf{C}(\Omega)} \int_\Omega |\varphi|\, d\eta_\varepsilon^r + \left|\int_\Omega \varphi \mathbf{u}_\varepsilon\, d\eta_\varepsilon^r - \int_\Omega \varphi\, \mathbf{b}_0\, dx\right| \\ = & I_1 + I_2 + I_3.\end{aligned} \tag{47}$$

Owing to the weak convergence $\eta_\varepsilon^r \rightharpoonup dx$ and to the fact that $(\varphi\, \mathbf{a}_0) \in \mathbf{C}_0(\Omega)$, we obtain $I_1 \to 0$ as $\varepsilon \to 0$. By analogy, we also have that $I_2 \to 0$ as $\varepsilon \to 0$. Then, taking into account (46), inequality (47) leads us to the following conclusion: $\int_\Omega \varphi\,(\mathbf{a}_0 - \mathbf{b}_0)\; dx = 0, \forall\, \varphi \in C_0^\infty(\Omega)$, that is, $\mathbf{a}_0 = \mathbf{b}_0$ almost everywhere in Ω. The proof is complete. ☐

As a consequence, the following statements are readily true:

Lemma 6.5. *Let $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible controls. Then the weak limits in $\mathbf{H}^2(\Omega)$ of any weakly convergent sequences of prototypes*

$$\left\{\mathbf{u}_\varepsilon^{(1)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0} \quad \text{and} \quad \left\{\mathbf{u}_\varepsilon^{(2)} \in \mathbf{H}^2(\Omega) \cap \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})\right\}_{\varepsilon>0}$$

are the same.

Lemma 6.6. *Any sequence of admissible controls $\{\overline{\alpha}_\varepsilon \in \mathbf{U}_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the w_a-convergence. Moreover, its w_a-limit $\mathbf{u}_0$ belongs to the set*

$$\mathbf{U} = \left\{\mathbf{u} \in \mathbf{H}^2(\Omega) \;:\; \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma\right\}. \tag{48}$$

Using the above results, we are now able to introduce the convergence concept for the admissible triplets. As follows from Remark 3, for any uniformly bounded sequence of functions $\left\{\mathbf{y}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ there are extension operators $\mathbf{P}_\varepsilon : \mathbf{H}^1(\Omega_\varepsilon) \to \mathbf{H}^1(\Omega)$ and a constant C independent of ε, such that $\|\breve{\mathbf{y}}_\varepsilon = (\mathbf{P}_\varepsilon \mathbf{y}_\varepsilon)\|_{\mathbf{H}^1(\Omega)} \le C\,\|\mathbf{y}_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}$ for every ε. Let us suppose that there are two different bounded sequences of prototypes $\{\breve{\mathbf{y}}_\varepsilon^{(1)} = \mathbf{P}_\varepsilon^{(1)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0}$ and $\{\breve{\mathbf{y}}_\varepsilon^{(2)} = \mathbf{P}_\varepsilon^{(2)}(\mathbf{y}_\varepsilon)\}_{\varepsilon>0}$ such that

$\breve{\mathbf{y}}_\varepsilon^{(1)} \longrightarrow \mathbf{y}_1^*$ and $\breve{\mathbf{y}}_\varepsilon^{(2)} \longrightarrow \mathbf{y}_2^*$ weakly in $\mathbf{H}^1(\Omega)$. Then, using Lemma 6.1 and passing to the limit in the integral identity

$$\int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(1)} \cdot \varphi \, dx = \int_\Omega \chi_\varepsilon \breve{\mathbf{y}}_\varepsilon^{(2)} \cdot \varphi \, dx, \quad \forall \varphi \in \mathbf{H}^1(\Omega)$$

as ε tends to zero, we obtain $\int_\Omega \mathbf{y}_1^* \cdot \varphi \, dx = \int_\Omega \mathbf{y}_2^* \cdot \varphi \, dx$, $\forall \varphi \in \mathbf{H}^1(\Omega)$. Hence $\mathbf{y}_1^* = \mathbf{y}_2^*$. In view of this we give the following notion:

Definition 6.7. We say that a bounded sequence $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a triplet $(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ in the variable space $\mathbb{X}_\varepsilon$ as ε tends to zero (in symbols, $(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \stackrel{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p)$), if some bounded sequence of its prototypes $\left\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ converges to $(\mathbf{u}, \mathbf{y}, p)$ in the following sense: (i) $\mathbf{u}_\varepsilon \stackrel{w_a}{\longrightarrow} \mathbf{u}$ in $\mathbf{H}^2(\Omega)$; (ii) $\breve{p}_\varepsilon \rightharpoonup p$ in $L_0^2(\Omega)$; (iii) $\breve{\mathbf{y}}_\varepsilon \rightharpoonup \mathbf{y}$ in $\mathbf{H}^1(\Omega)$.

As a consequence, we have the following result:

Theorem 6.8. *Let $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible triplets for the $\mathbb{P}_\varepsilon$-problems. Then there exist a subsequence $\{(\overline{\alpha}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'})\}_{\varepsilon'>0}$ and a triplet $(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ such that : $\mathbf{u} \in \mathbf{U}$, $(\overline{\alpha}_{\varepsilon'}, \mathbf{y}_{\varepsilon'}, p_{\varepsilon'}) \stackrel{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p)$, and*

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega). \tag{49}$$

Proof. We set $\breve{\mathbf{y}}_\varepsilon = \mathbf{u}_\varepsilon$ in $\Omega \setminus \Omega_\varepsilon$. Thanks to Lemma 6.6, Proposition 1, and Remark 3, the sequence $\{\breve{\mathbf{y}}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathbf{H}^1(\Omega)$. Let

$$\left\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0} \tag{50}$$

be a sequence of some prototypes, where $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon) \in L_0^2(\Omega)$ (see Remark 4). Then the sequence of pressures $\left\{\breve{p} \in L_0^2(\Omega)\right\}_{\varepsilon>0}$ is also uniformly bounded. So, the sequence (50) is relatively compact with respect to the weak convergence in the space $\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$. Let $(\mathbf{u}, \mathbf{y}, p) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ be its weak limit.

We aim to show that $\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega)$. Since $\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon) \cap \mathbf{H}^1_0(\Omega)$ and

$$\operatorname{div} : \mathbf{H}^1_0(\Omega_\varepsilon) \mapsto L^2(\Omega_\varepsilon)/\mathbb{R} = \left\{ g \in L^2(\Omega_\varepsilon) \ : \ \int_{\Omega_\varepsilon} g(x) \, dx = 0 \right\},$$

it follows that $\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon \in \mathbf{H}(\operatorname{div}, \Omega_\varepsilon)$ for every ε. Therefore

$$0 = \int_{\Omega_\varepsilon} \varphi \operatorname{div}(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \, dx = -\int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3) \tag{51}$$

$\forall \varepsilon > 0$ (see Lemma 2.1). However, we have

$$\int_\Omega \varphi \, \chi_\varepsilon \operatorname{div}(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \, dx = -\int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3).$$

Hence, $\operatorname{div}[\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)] = \chi_\varepsilon \operatorname{div}(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)$, and we get $\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}(\operatorname{div}, \Omega)$. Then, due to Lemma 6.1, we can pass to the limit in (51). As a result, we have

$$0 = \lim_{\varepsilon \to 0} \int_\Omega \chi_\varepsilon (\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dx = \int_\Omega (\mathbf{y} - \mathbf{u}) \nabla \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3).$$

Thus $(\mathbf{y} - \mathbf{u}) \in H^1_{0,div}(\Omega)$, and this concludes the proof. ∎

Corollary 1. *The supposition of Theorem 6.8 and condition (14) imply the inclusions*

$$\mathbf{y} \in \mathbf{H}^1_{sol}(\Omega), \quad \mathbf{u} \in \mathbf{U} = \left\{ \mathbf{u} \in \mathbf{H}^2_{sol}(\Omega) \ : \ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le \gamma \right\}. \tag{52}$$

Proof. Since $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{sol}(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega)$ for every ε, it follows that $\int_{\Omega_\varepsilon} v \operatorname{div} \mathbf{u}_\varepsilon \, dx = 0$ for every $v \in H^1(\Omega)$. On the other hand, $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $\mathbf{H}^2(\Omega)$ as $\varepsilon \to 0$. Hence, taking into account (25), we get

$$0 = \lim_{\varepsilon \to 0} \int_\Omega \chi_\varepsilon \, \mathbf{u}_\varepsilon \cdot \nabla v \, dx = \int_\Omega \mathbf{u} \cdot \nabla v \, dx = \langle \operatorname{div} \mathbf{u}, v \rangle_{H^{-1}(\Omega); H^1_0(\Omega)}, \quad \forall \, v \in H^1_0(\Omega).$$

Since $\operatorname{div} \mathbf{u} \in L^2(\Omega)$, this yields $\operatorname{div} \mathbf{u} = 0$. As a result, we have $\mathbf{u} \in \mathbf{H}^1_{sol}(\Omega)$. The inclusion $\mathbf{y} \in \mathbf{H}^1_{sol}(\Omega)$ immediately follows from the previous one and (49). □

7. **Definition of suboptimal controls.** The main question we are going to consider in this section concerns the approximation of the optimal solutions to the original problem $(\mathbb{P}_\varepsilon)$ for ε small enough. We focus our attention on the possibility to define the so-called suboptimal solutions which have to guarantee the closeness of the corresponding value of the cost functional $\mathcal{J}_\varepsilon(\mathbf{u}^{sub}_\varepsilon, \mathbf{y}^{sub}_\varepsilon)$ to its minimum if ε is small enough. To do so, we introduce the following concept:

Definition 7.1. We say that a sequence of functions

$$\left\{ \overline{\alpha}^{sub}_\varepsilon = \left(\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, \dots, \alpha^{sub}_{\mathbf{k}_{\mathrm{J}_\varepsilon}} \right) \right\}_{\varepsilon > 0}$$

is an asymptotically suboptimal controls for the problem $(\mathbb{P}_\varepsilon)$ if

$$\alpha^{sub}_{\mathbf{k}_j} \in \mathbf{H}^{1/2}(\partial T^{\mathbf{k}_j}_\varepsilon), \quad \int_{\partial T^{\mathbf{k}_j}_\varepsilon} \mathbf{n} \cdot \alpha^{sub}_{\mathbf{k}_j} \, d\mathcal{H}^2 = 0, \quad \forall j = 1, \dots, \mathrm{J}_\varepsilon, \tag{53}$$

and for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that

$$\left| \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) - \mathcal{J}_\varepsilon(\overline{\alpha}^{sub}_\varepsilon, \mathbf{y}^{sub}_\varepsilon) \right| < \delta, \quad \forall \, \varepsilon < \varepsilon_0, \tag{54}$$

where $\mathbf{y}^{sub}_\varepsilon = \mathbf{y}_\varepsilon(\overline{\alpha}^{sub}_\varepsilon)$ denotes the corresponding solution of the boundary value problem (9)–(12).

To construct such controls, we use an approach coming from the variation convergence of constrained minimization problems (see [7], [13], [23], [24], [25], [28]). In view of this, we study the asymptotic behaviour of the problem $(\mathbb{P}_\varepsilon)$ as $\varepsilon \to 0$. We represent the $\mathbb{P}_\varepsilon$-problem for various values of ε, in the form of the following sequence:

$$\left\{ \left\langle \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) \right\rangle ; \varepsilon > 0 \right\}. \tag{55}$$

Then the definition of an appropriate limit problem to the family (55), can be reduced to the analysis of the limit properties of this sequence as $\varepsilon \to 0$. To get this limit in the form of some constrained minimization problem, we introduce the following definition:

Definition 7.2. We say that a minimization problem

$$\left\langle \inf_{(\mathbf{u}, \mathbf{y}, p) \in \Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \right\rangle \tag{56}$$

is the variational w-limit of the sequence (55), if the following conditions are satisfied:

(1) if a sequence $\{(\overline{\alpha}_k, \mathbf{y}_k, p_k)\}_{k\in\mathbb{N}}$ w-converges to a triplet $(\mathbf{u}, \mathbf{y}, p)$ as $k \to \infty$, and there exists a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $(\overline{\alpha}_k, \mathbf{y}_k, p_k) \in \Xi_{\varepsilon_k}$ for all k, then

$$(\mathbf{u}, \mathbf{y}, p) \in \Xi_0; \quad \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \le \liminf_{k\to\infty} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}_k, \mathbf{y}_k); \tag{57}$$

(2) for every class of equivalence Ξ_0/L there exist a triplet $(\mathbf{u}, \mathbf{y}, p) \in \Xi_0/L$ and a realizing sequence $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that

$$(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \xrightarrow{\ w\ } (\mathbf{u}, \mathbf{y}, p), \text{ and } \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \ge \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\alpha_\varepsilon, \mathbf{y}_\varepsilon). \tag{58}$$

The following theorem deals with the main property of the variational w-limit problems.

Theorem 7.3. *Assume that (56) is a weak variational w-limit of the sequence (55), and this problem has a solution. Let $\left\{(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal triplets for the $\mathbb{P}_\varepsilon$-problems. Then there exists a triplet $(\mathbf{u}^0, \mathbf{y}^0, p^0) \in \Xi_0$ such that*

$$(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \xrightarrow{\ w\ } (\mathbf{u}^0, \mathbf{y}^0, p^0), \tag{59}$$

$$\inf_{(\mathbf{u},\mathbf{y},p)\in\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) = \mathcal{J}_0\left(\mathbf{u}^0, \mathbf{y}^0\right) = \lim_{\varepsilon\to 0} \inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon). \tag{60}$$

Proof. First, observe that in view of Theorem 6.8, the w-compactness property holds true for the sequence of optimal solutions $\left\{(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$. So, we may suppose that there exist a subsequence $\left\{(\overline{\alpha}^0_k, \mathbf{y}^0_k, p^0_k)\right\}_{k\in\mathbb{N}}$ of the sequence of optimal solutions and a triplet $(\mathbf{u}^*, \mathbf{y}^*, p^*)$, such that $(\overline{\alpha}^0_k, \mathbf{y}^0_k, p^0_k) \xrightarrow{\ w\ } (\mathbf{u}^*, \mathbf{y}^*, p^*)$ as $\varepsilon_k \to 0$. Hence, property (59) leads us to the conclusion that $(\mathbf{u}^*, \mathbf{y}^*, p^*) \in \Xi_0$, and

$$\begin{aligned}\liminf_{k\to\infty} \min_{(\overline{\alpha},\mathbf{y})\in\widehat{\Xi}_{\varepsilon_k}} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}, \mathbf{y}) &= \liminf_{k\to\infty} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}^0_k, \mathbf{y}^0_k, p^0_k)\\ &\ge \mathcal{J}_0(\mathbf{u}^*, \mathbf{y}^*) \ge \min_{(\mathbf{u},\mathbf{y},p)\in\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) = \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0),\end{aligned} \tag{61}$$

where $(\mathbf{u}^0, \mathbf{y}^0)$ is an optimal pair to (56). Then, by property (ii) of Definition 7.2 there exists a realizing sequence $\left\{(\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ such that $(\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon) \xrightarrow{\ w\ } (\mathbf{u}^0, \mathbf{y}^0, p^0)$, and

$$\mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0) \ge \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon).$$

Using this fact, we have

$$\begin{aligned}\min_{(\mathbf{u},\mathbf{y},p)\in\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) &= \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0)\\ &\ge \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon) \ge \limsup_{\varepsilon\to 0} \min_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon)\\ &\ge \limsup_{k\to\infty} \min_{(\overline{\alpha},\mathbf{y},p)\in\Xi_{\varepsilon_k}} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}, \mathbf{y}) = \limsup_{k\to\infty} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}^0_{\varepsilon_k}, \mathbf{y}^0_{\varepsilon_k}).\end{aligned} \tag{62}$$

From this and (61) we deduce that

$$\liminf_{k\to\infty} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}^0_{\varepsilon_k}, \mathbf{y}^0_{\varepsilon_k}) \ge \limsup_{k\to\infty} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}^0_{\varepsilon_k}, \mathbf{y}^0_{\varepsilon_k}).$$

Then, combining (61) and (62), we get

$$\mathcal{J}_0(\mathbf{u}^*, \mathbf{y}^*) = \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0) = \min_{(\mathbf{u},\mathbf{y},p)\in\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) = \lim_{k\to\infty} \min_{(\overline{\alpha},\mathbf{y},p)\in\Xi_{\varepsilon_k}} \mathcal{J}_{\varepsilon_k}(\overline{\alpha}, \mathbf{y}). \tag{63}$$

Using these relations and the fact that the problem (56) has solutions, we may suppose that $(\mathbf{u}^*, \mathbf{y}^*) = (\mathbf{u}^0, \mathbf{y}^0)$. Since equality (63) holds for the w-limits of all subsequences of $\left\{(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0, p_\varepsilon^0)\right\}_{\varepsilon>0}$, it follows that these limits coincide and, therefore, $(\mathbf{u}^0, \mathbf{y}^0, p^0)$ is the w-limit of the whole sequence $\left\{(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0, p_\varepsilon^0)\right\}_{\varepsilon>0}$. Then, using the same argument for the sequence of minimizers as for the subsequence $\left\{(\overline{\alpha}_{\varepsilon_k}^0, \mathbf{y}_{\varepsilon_k}^0, p_{\varepsilon_k}^0)\right\}_{k\in\mathbb{N}}$, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \min_{(\overline{\alpha},\mathbf{y},p)\in\,\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}, \mathbf{y}) &= \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0) \ge \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0) = \min_{(\mathbf{u},\mathbf{y},p)\in\,\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) \\ &\ge \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\widehat{\overline{\alpha}}_\varepsilon, \widehat{\mathbf{y}}_\varepsilon) \ge \limsup_{\varepsilon\to 0} \min_{(\overline{\alpha},\mathbf{y},p)\in\,\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}, \mathbf{y}) \\ &= \limsup_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon^0, \mathbf{y}_\varepsilon^0), \quad \forall\, \delta > 0\end{aligned}$$

and this concludes the proof. □

In the last section we show that any solution of the w-limit problem (56) can be taken as prototype for the construction of the suboptimal controls in the sense of Definition 7.1.

8. **Convergence Theorem.** The main question of this section is the study of asymptotic behaviour of the boundary value problem (9)–(12) as ε tends to zero. To begin with, we give some technical lemmas. We suppose that Hypotheses (H1)-(H5) are fulfilled.

Lemma 8.1. $\mu_{ij} \in \mathcal{M}_b(\Omega), \quad \forall\, i, j \ : (1 \le i, j \le 3)$.

Proof. We will prove that for every compact set $B \subset \Omega$ of zero capacity, we have $\mu_{ij}(K) = 0$. By standard properties of Radon measures, it follows that $\mu_{ij}(D) = 0$ for any Borel set $D \subset \Omega$ of zero capacity.

Let K be a compact subset of Ω. Then for any $k \in \mathbb{N}$ there exists $\varphi_k \in C_0^\infty(\Omega)$ such that $\varphi_k \ge \chi_K$, $0 \le \varphi_k \le 1$, $\|\varphi_k\|_{H_0^1(\Omega)} \le \frac{1}{k}$. In view of Hypothesis (H4), we have

$$\lim_{\varepsilon\to 0} \langle \nabla q_i^\varepsilon - \nu \Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = 0, \tag{64}$$
$$\forall\, \mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega) \ : \ \mathbf{v}_\varepsilon \rightharpoonup 0 \ \text{ in } \ \mathbf{H}^1(\Omega) \ \text{ and } \ \mathbf{v}_\varepsilon = 0 \ \text{ on } \ T_\varepsilon.$$

Applying this to the sequence $\left\{\mathbf{v}_{\varepsilon,k} = \varphi_k \mathbf{w}_j^\varepsilon\right\}$, we obtain that for any $\delta > 0$

$$\left|\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}\right| + \left|\nu \left\langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \right\rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}\right| \le \delta$$

$\forall\, \varepsilon < \varepsilon_0(\delta)$, $k > k_0(\delta)$. Since $\mathbf{w}_j^\varepsilon$ is divergence-free (see Hypothesis (H2)), it follows that

$$\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = -\int_\Omega q_i^\varepsilon \mathbf{w}_j^\varepsilon \cdot \nabla \varphi_k \, dx.$$

Then, taking into account the following obvious estimates

$$\left|\langle \nabla q_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}\right| \le \|q_i^\varepsilon\|_{L_0^2(\Omega)} \|\|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\mathrm{div},\Omega)} \|\varphi_k\|_{H_0^1(\Omega)} \le \frac{C}{k},$$

$$\langle -\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = \int_\Omega \varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \, dx + \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, dx,$$

and $\left|\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla \varphi_k \, dx\right| \le \|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\nabla \varphi_k\|_{\mathbf{L}^2(\Omega)} \le \frac{C}{k}$, we obtain

$$\int_\Omega \left|\varphi_k \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| \, dx \le 2\delta, \quad \forall\, \varepsilon < \varepsilon_1(\delta), \ k > k_1(\delta). \tag{65}$$

Due to the Hypothesis (H3), each of the sequences $\left\{\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right\}$ $(1 \le i, j \le 3)$ is bounded in $L^1(\Omega)$. So, extracting, if necessary, a subsequence, we can suppose the existence of a symmetric matrix $\mathbf{M} = \{\mu_{ij}\}_{1\le i,j\le 3}$ of bounded Radon measures μ_{ij} such that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon$ converges to μ_{ij} in the weak-* sense of the space $\mathcal{M}_b(\Omega)$. Then, passing to the limit in inequality (65) as $\varepsilon \to 0$, we get $\int_\Omega \varphi_k \, d\mu_{ij} \le 2\delta, \quad \forall\ k > k_1(\delta)$. Since $\varphi_k \ge \chi_K$, this yields $\mu_{ij}(K) \le 2\delta$, $\forall\ \delta > 0$, and we obtain the required result. ☐

Lemma 8.2. *For any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and each $i, j\ : (1 \le i, j \le 3)$, we have*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \, dx = \int_\Omega \varphi \, d\mu_{ij}. \tag{66}$$

Proof. Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. By analogy with Casado-Diaz [8], we consider a sequence of functions $\{\varphi_k \in C_0^\infty(\Omega)\}$ satisfying conditions

$$\sup_{k\in\mathbb{N}} \|\varphi_k\|_{L^\infty(\Omega)} < +\infty, \quad \varphi_k \to \varphi \ \text{ in } \ H_0^1(\Omega) \ \text{ and } \ \mu_{ij} - \text{a.e. in } \ \Omega.$$

(the existence of such sequence has been proved in [32]). From Lebesgue's dominated convergence theorem, we have $\varphi \in \varphi_k$ and φ_k converges strongly to φ in φ_k. Then

$$\begin{aligned}\left|\int_\Omega \varphi \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| \, dx - \int_\Omega \varphi \, d\mu_{ij}\right| \le & \int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) |\varphi - \varphi_k| \, dx \\ & + \left|\int_\Omega \varphi_k \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \, dx - \int_\Omega \varphi_k \, d\mu_{ij}\right| \\ & + \int_\Omega |\varphi_k - \varphi| \, d\mu_{ij}.\end{aligned}$$

Passing to the limit in this relation for a fixed k and taking into account the weak-* convergence of $\left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)$ to μ_{ij} in $\mathcal{M}_b(\Omega)$, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \left|\int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \, dx - \int_\Omega \varphi \, d\mu_{ij}\right| \\ &\le \limsup_{\varepsilon\to 0} \int_\Omega \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| |\varphi - \varphi_k| \, dx + \limsup_{\varepsilon\to 0} \int_\Omega |\varphi_k - \varphi| \, d\mu_{ij}.\end{aligned}$$

Passing, now, to the limit as $k \to \infty$, we find

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \left|\int_\Omega \varphi \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \, dx - \int_\Omega \varphi \, d\mu_{ij}\right| \\ &\le \lim_{k\to\infty} \limsup_{\varepsilon\to 0} \int_\Omega \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| |\varphi - \varphi_k| \, dx.\end{aligned}$$

Let us show that the limit in the right hand side is zero. We apply the property (64) to the sequence $\left\{\mathbf{v}_{\varepsilon,k} = \pm|\varphi_k - \varphi|\mathbf{w}_j^\varepsilon\right\}$. One gets

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \left[\pm \int_\Omega |\varphi_k - \varphi| \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \, dx \pm \int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx\right] = 0.$$

Since

$$\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx \le 2\|\mathbf{w}_i^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}_j^\varepsilon\|_{\mathbf{H}^1(\Omega)} \|\nabla|\varphi_k - \varphi|\|_{\mathbf{L}^2(\Omega)}$$

and φ_k tends to φ strongly in $H_0^1(\Omega)$, it immediately follows that

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \left|\int_\Omega \nabla \mathbf{w}_i^\varepsilon : \mathbf{w}_j^\varepsilon \nabla|\varphi_k - \varphi| \, dx\right| = 0.$$

Thus $\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega \left|\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right| |\varphi - \varphi_k|\, dx = 0$ and this concludes the proof. □

Lemma 8.3. *If a sequence* $\left\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\right\}$ *and* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ *are such that* $\mathbf{v}_\varepsilon = 0$ *on* T_ε *and* $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ *in* $\mathbf{H}_0^1(\Omega)$*, then* $\mathbf{v} \in \mathbf{L}^1(\Omega, d\mu_i)$ *for each* $i\ :\ (1 \le i \le 3)$.

Proof. For every $k > 0$ we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = k$ if $s \ge k$, $T_k(s) = s$ if $-k \le s \le k$, and $T_k(s) = -k$ if $s \le -k$. We denote the vector-valued counterpart of this function with the bold symbol $\mathbf{T}_k : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\left\{\mathbf{v}_\varepsilon \in \mathbf{H}_0^1(\Omega)\right\}$ and $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ be above given functions. We wish to show the fulfilment of the following relation

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\left|\langle \nabla q_i^\varepsilon - \nu\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i\right| = 0, \tag{67}$$

which implies that $\int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i$ is bounded independently of k, and hence, by the Beppo Levi's monotone convergence theorem, $\mathbf{v} \in \mathbf{L}^1(\Omega, d\mu_i)$.

For every $\varepsilon > 0$ and $k \in \mathbb{R}$, we define the functions $\mathbf{v}_{\varepsilon,k}$ by the rule $\mathbf{v}_\varepsilon = \mathbf{T}_k(\mathbf{v}_\varepsilon) + \mathbf{v}_{\varepsilon,k}$ and note that

$$\begin{aligned}
&\left|\langle \nabla q_i^\varepsilon - \nu\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_\varepsilon\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i\right| \\
\le &\left|\langle \nabla q_i^\varepsilon - \nu\Delta \mathbf{w}_i^\varepsilon, \mathbf{v}_{\varepsilon,k}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)}\right| \\
+ &\left|\langle \nabla q_i^\varepsilon - \nu\Delta \mathbf{w}_i^\varepsilon, \mathbf{T}_k(\mathbf{v}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} - \int_\Omega \mathbf{T}_k(\mathbf{v})\, d\mu_i\right| = I_1 + I_2.
\end{aligned} \tag{68}$$

Since I_2 tends to zero as $\varepsilon \to 0$ for a fixed k by Hypothesis (H4) and Lemma 8.1, and I_1 tends to zero when $k \to \infty$ and $\varepsilon \to 0$ by property (64), this concludes the proof. □

Our next step concerns the structural identification of the matrix $\mathbf{M} = \{\mu_{ij}\}$, where

$$\mu_{ij} \ \text{ is the limit of } \ \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon \ \text{ in the weak-* sense of the space } \ \mathcal{M}_b(\Omega). \tag{69}$$

Lemma 8.4. *Assume the functions* $\mathbf{w}_j^\varepsilon = \left(w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon\right)^t \in \mathbf{H}^1(\Omega)$ $(1 \le j \le 3)$ *are such that*

$$w_{j,k}^\varepsilon \in H_0^1(\Omega) \quad \text{if} \quad j \ne k. \tag{70}$$

Then $\mathbf{M} = \operatorname{diag}(\mu_{11}, \mu_{22}, \mu_{33})$.

Proof. According to Hypothesis (H3) and Rellich-Kondrachov's compactness theorem, we conclude that $\mathbf{w}_j^\varepsilon = \left(w_{j,1}^\varepsilon, w_{j,2}^\varepsilon, w_{j,3}^\varepsilon\right)^t$ converges to $\mathbf{e}_j$ strongly in $\mathbf{L}^2(\Omega)$. Besides, due to the initial assumptions (70), we also have $w_{j,k}^\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$, $w_{j,k}^\varepsilon \to 0$ in $L^2(\Omega)$, $\forall\, j \ne k$. Then, by Poincaré inequality in Ω, we can deduce

$$\limsup_{\varepsilon\to 0}\|\nabla w_{j,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \le C \limsup_{\varepsilon\to 0}\|w_{j,k}^\varepsilon\|_{L^2(\Omega)} = 0, \quad \forall\, j \ne k. \tag{71}$$

Let (i,j) be a pair of indices such that $i \ne j$ and $1 \le i, j \le 3$. Extracting, if necessary, a subsequence, we can assume the existence of bounded Radon measure μ_{ij} such that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon$ converges to μ_{ij} in the weak-* sense of $\mathcal{M}_b(\Omega)$. It means that

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right)\varphi\, dx = \int_\Omega \varphi\, d\mu_{ij}, \quad \forall\, \varphi \in C_0(\Omega).$$

Observing that $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon = \sum_{k=1}^3 \nabla w_{i,k}^\varepsilon \cdot \nabla w_{j,k}^\varepsilon$, we have the estimate

$$\int_\Omega \left(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_j^\varepsilon\right) \varphi \, dx \le \|\varphi\|_{C_0(\Omega)} \sum_{k=1}^3 \|\nabla w_{i,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \|\nabla w_{j,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)}. \tag{72}$$

Since each of the terms $\|\nabla w_{i,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \|\nabla w_{j,k}^\varepsilon\|_{\mathbf{L}^2(\Omega)}$ consists at least of one multiplier with non-coinciding indices, we can apply property (71). As a result, passing to the limit in (72) as $\varepsilon \to 0$, we obtain the required result: $\mu_{ij} = 0$. □

To check the Hypotheses (H1)–(H5) and obtain a precise description for the measures μ_{ij}, we partition the set $\widetilde{\Omega}$ into squares $\varepsilon \widetilde{Y}$ with edges ε and denote these squares with $\varepsilon \widetilde{Y}_j$. The corresponding cylindrical cells $\varepsilon \widetilde{Y}_j \times (0, \ell)$ are denoted by Z_j^ε. Following the ideas of G. Cioranescu & F. Murat [9] and G. Allaire [3], we introduce the functions $\mathbf{w}_k^\varepsilon \in \mathbf{H}^1(Z_j^\varepsilon)$ and $q_k^\varepsilon \in L^2(Z_j^\varepsilon)$ $(k = 1, 2, 3)$ with $\int_{Z_j^\varepsilon} q_k \, dx = 0$ as follows:

1. For each cell Z_j^ε that meets the boundary Γ_3

$$\{\mathbf{w}_k^\varepsilon = \mathbf{e}_k, \quad q_k^\varepsilon = 0\} \quad \text{in} \quad Z_j^\varepsilon \cap \Omega. \tag{73}$$

2. For each cell Z_j^ε entirely included in Ω (precisely, for $Z_{\mathbf{k}}^\varepsilon$ where $\mathbf{k} \in \Theta_\varepsilon$),

$$\begin{aligned} &\left\{ \begin{array}{c} \mathbf{w}_k^\varepsilon = \mathbf{e}_k \\ q_k^\varepsilon = 0 \end{array} \right\} && \text{in} \quad Z_{\mathbf{k}}^\varepsilon \setminus [\varepsilon\,(A + \mathbf{k}) \times (0, \ell)]\,, \\ &\left\{ \begin{array}{c} -\nu \Delta \mathbf{w}_k^\varepsilon + \nabla q_k^\varepsilon = 0 \\ \nabla \cdot \mathbf{w}_k^\varepsilon = 0 \end{array} \right\} && \text{in} \quad \varepsilon\,(A \setminus Q^{r_\varepsilon} + \mathbf{k}) \times (0, \ell), \\ &\left\{ \begin{array}{c} \mathbf{w}_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{array} \right\} && \text{in} \quad \varepsilon\,(Q^{r_\varepsilon} + \mathbf{k}) \times (0, \ell). \end{aligned} \tag{74}$$

It is now clear that the functions $\mathbf{w}_k^\varepsilon$ are independent of x_3, that is,

$$\mathbf{w}_k^\varepsilon(x_1, x_2, x_3) = \mathbf{w}_k^\varepsilon(x_1, x_2), \quad 1 \le k \le 3. \tag{75}$$

Following closely Allaire [3], a quite similar result can be proved.

Proposition 2. *Assume that the size of thin cylinders $T_\varepsilon^{\mathbf{k}}$ satisfies the following condition*

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = \lim_{\varepsilon \to 0} \varepsilon^2 \left(\log 1/r_\varepsilon\right) > 0, \tag{76}$$

and the functions $(\mathbf{w}_k, q_k^\varepsilon)$ $(k = 1, 2, 3)$ are defined as in (73)–(74). Then, there exists a symmetric positive matrix $\mathbf{M} = (\mu_1, \mu_2, \mu_3) \in L(\mathbb{R}^3; \mathbb{R}^3)$ such that Hypotheses (H1)–(H3) are satisfied. Moreover, in this case there exists a linear map R_ε satisfying (H5) such that the extension P_ε of the pressure p_ε turns out to be equal to $P_\varepsilon = p_\varepsilon$ in Ω_ε, and

$$P_\varepsilon = \frac{1}{\ell |A^\varepsilon \setminus Q^{\varepsilon r_\varepsilon}|} \int_{(A^\varepsilon \setminus Q^{\varepsilon r_\varepsilon} + \varepsilon \mathbf{k}) \times (0, \ell)} p_\varepsilon \, dx \text{ in each cylinder } T_\varepsilon^{\mathbf{k}}.$$

Now, it is clear that, in view of Lemma 8.4 and relations (73)–(74), the matrix $\mathbf{M} = \{\mu_{ij}\}_{1 \le i,j \le 3}$, which appears in Proposition 2, has the diagonal structure $\mathbf{M} = \operatorname{diag}(\mu_{11}, \mu_{22}, \mu_{33})$. In fact, the measure $\mu_{ii} \in \mathcal{M}_0^+$ that appeared as the weak limit of $\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_i^\varepsilon$ in the space $\mathcal{M}_b(\Omega)$, can be recovered in an explicit form. For this, we give the following result:

Lemma 8.5. *Let Q be a compact subset of $\widetilde{Y}$ with Lipschitz boundary ∂Q, $\operatorname{int} Q$ be a strongly connected set, $Q \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 \geq 0\}$, and its boundary ∂Q contains the origin. Let $A = B(\mathbf{0}, r_0)$ be an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \subset\subset \widetilde{Y}$ and $Q \subset\subset A$ (see Fig. 1). Then, under condition (76) for a sequence $\left\{\mathbf{w}_i^\varepsilon \in \mathbf{H}^1(\Omega)\right\}$ $(1 \leq i \leq 3)$ defined by (73)–(74), we have $(\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_i^\varepsilon) \rightharpoonup \mu_{ii}^*$ weakly-* in $\mathcal{M}_b(\Omega)$, where*

$$\mu_{ii}^* = 2\pi \lim_{\varepsilon \to 0} \sigma_\varepsilon^{-1}, \quad \forall\, i : 1 \leq i \leq 3. \tag{77}$$

Proof. The proof follows standard techniques (see [15]) and, in some aspects, it is similar to the one given in [11]. To begin with, we use the notation $|\nabla \mathbf{w}_i^\varepsilon|^2 = (\nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_i^\varepsilon)$ and partition the cylindrical domain Ω into cubes εY with edges ε. We denote these cubes by εY_j. It is clear that $|\varepsilon Y_j| = \varepsilon |\varepsilon \widetilde{Y}_j| = \varepsilon^3$.

Let us observe that $|\nabla \mathbf{w}_i^\varepsilon|^2 = \sum_{k=1}^3 \left|\nabla w_{i,k}^\varepsilon\right|^2$, where $w_{i,i}^\varepsilon \rightharpoonup 1$ in $H^1(\Omega)$, and $w_{i,k}^\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$ for all $k \neq i$. Then

$$\begin{aligned}\int_{\varepsilon Y_j} \varphi \left|\nabla \mathbf{w}_i^\varepsilon\right|^2 dx &= \int_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 dx + \sum_{k \neq i} \int_{\varepsilon Y_j} \varphi \left|\nabla w_{ik}^\varepsilon\right|^2 dx \\ &= \int_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 dx + S_j(\varepsilon)\end{aligned}$$

for any $\varphi \in C_0(\Omega)$. Hence, using (75), we have the following obvious relation:

$$\begin{aligned}\varphi(x_j^\varepsilon)\varepsilon \int_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx = \varphi(x_j^\varepsilon) \int_{\varepsilon Y_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx &\leq \int_{\varepsilon Y_j} \varphi \left|\nabla w_{ii}^\varepsilon\right|^2 dx \\ \leq \varphi(y_j^\varepsilon) \int_{\varepsilon Y_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx &= \varphi(y_j^\varepsilon)\varepsilon \int_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx,\end{aligned}$$

where $x_j^\varepsilon, y_j^\varepsilon \in \varepsilon Y_j$. Combining this relation with the previous one, we obtain

$$\varepsilon \int_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx + S_j(\varepsilon) \leq \int_{\varepsilon Y_j} \left|\nabla \mathbf{w}_i^\varepsilon\right|^2 \varphi\, dx \leq \varphi(y_j^\varepsilon)\varepsilon \int_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx + S_j(\varepsilon). \tag{78}$$

From the definition of the capacity and its properties, it readily follows that

$$\int_{\varepsilon \widetilde{Y}_j} \left|\nabla w_{ii}^\varepsilon\right|^2 dx = \operatorname{cap}\left(Q^{\varepsilon r(\varepsilon)}, A^\varepsilon\right) = \operatorname{cap}\left(r_\varepsilon Q, A\right) = \operatorname{cap}\left(Q, r_\varepsilon^{-1} A\right). \tag{79}$$

Since $0 \in \partial Q$ and using the arguments of the paper [11] (see Lemma 3.3), we can conclude that

$$\operatorname{cap}\left(Q, r_\varepsilon^{-1} A\right) = \frac{2\pi}{\log(1/r_\varepsilon)} \left(1 + c_\varepsilon\right) = 2\pi \varepsilon^2 \sigma_\varepsilon^{-1} \left(1 + c_\varepsilon\right), \quad \text{where} \quad \lim_{\varepsilon \to 0} c_\varepsilon = 0. \tag{80}$$

Then, summing up inequalities (78) for all admissible indices j and taking into account relations (79)–(80), we obtain

$$\begin{aligned}&2\pi \sigma_\varepsilon^{-1} \left(1 + c_\varepsilon\right) \sum_j \varepsilon^3 \varphi(x_j^\varepsilon) + \sum_j S_j(\varepsilon) \\ \leq &\sum_j \int_{\varepsilon Y_j} \left|\nabla \mathbf{w}_i^\varepsilon\right|^2 \varphi\, dx \\ \leq &2\pi \sigma_\varepsilon^{-1} \left(1 + c_\varepsilon\right) \sum_j \varepsilon^3 \varphi(y_j^\varepsilon) + \sum_j S_j(\varepsilon).\end{aligned} \tag{81}$$

Observing that

$$-\|\varphi\|_{C_0(\Omega)} \lim_{\varepsilon\to 0} \int_\Omega |\nabla w^\varepsilon_{ik}|^2\, dx \le \lim_{\varepsilon\to 0} \sum_j \int_{\varepsilon Y_j} \varphi\, |\nabla w^\varepsilon_{ik}|^2\, dx$$
$$\le \|\varphi\|_{C_0(\Omega)} \lim_{\varepsilon\to 0} \int_\Omega |\nabla w^\varepsilon_{ik}|^2\, dx,$$

and taking into account Lemma 8.4 and the construction of the Riemann sum for the integral $\int_\Omega \varphi\, dx$, we can pass to the limit in (81) as ε tends to zero. As a result, we obtain

$$2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \int_\Omega \varphi\, dx \le \lim_{\varepsilon\to 0} \int_\Omega |\nabla \mathbf{w}^\varepsilon_i|^2\, \varphi\, dx \le 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \int_\Omega \varphi\, dx, \quad \forall\, \varphi \in C_0(\Omega).$$

Hence, $\lim_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|^2 \varphi\, dx = 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1}$, and therefore Lemma 8.5 is proved. □

As a consequence of this result we have the following one.

Corollary 2. *Let $\mathbf{w}^\varepsilon_k \in \mathbf{H}^1(Z^\varepsilon_j)$ and $q^\varepsilon_k \in L^2(Z^\varepsilon_j)$ $(k = 1, 2, 3)$ be the functions defined by (73)–(74). Then*

$$\lim_{\varepsilon\to 0} \langle \nabla q^\varepsilon_i - \nu \Delta \mathbf{w}^\varepsilon_i, \varphi \mathbf{w}^\varepsilon_k \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = 2\pi \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \delta_{ik} \int_\Omega \varphi\, dx \tag{82}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $1 \le k, i \le 3$, where $\delta_{ik} = 0$ if $i \ne k$, and $\delta_{ii} = 1$.

Proof. Integrating the expression $\langle \nabla q^\varepsilon_i - \nu \Delta \mathbf{w}^\varepsilon_i, \varphi \mathbf{w}^\varepsilon_k \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}$ by parts and using the property (H2), we can reduce it to the form

$$\begin{aligned} \langle \nabla q^\varepsilon_i - \nu \Delta \mathbf{w}^\varepsilon_i, \varphi \mathbf{w}^\varepsilon_k \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} &= -\int_\Omega q^\varepsilon_i (\mathbf{w}^\varepsilon_k \cdot \nabla \varphi)\, dx \\ &+ \nu \int_\Omega \nabla \mathbf{w}^\varepsilon_i : \mathbf{w}^\varepsilon_k \nabla \varphi\, dx + \nu \int_\Omega (\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_k)\, \varphi\, dx. \end{aligned} \tag{83}$$

Then combining Rellich's theorem, the conditions $\mathbf{w}^\varepsilon_k \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$ and $q^\varepsilon_k \rightharpoonup 0$ in $L^2_0(\Omega)$, and Lemmas 8.4–8.5, we can pass to the limit in the above equation as $\varepsilon \to 0$. As a result, we conclude

$$\begin{aligned} &\lim_{\varepsilon\to 0} \langle \nabla q^\varepsilon_i - \nu \Delta \mathbf{w}^\varepsilon_i, \varphi \mathbf{w}^\varepsilon_k \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} \\ &\qquad = \nu \lim_{\varepsilon\to 0} \int_\Omega (\nabla \mathbf{w}^\varepsilon_i : \nabla \mathbf{w}^\varepsilon_k)\, \varphi\, dx = 2\pi\nu \lim_{\varepsilon\to 0} \sigma_\varepsilon^{-1} \delta_{ik} \int_\Omega \varphi\, dx. \end{aligned}$$

□

Now we are able to state and prove the main result of this section concerning the passage to the limit as $\varepsilon \to 0$ in the following integral identities

$$\begin{aligned} \nu \int_{\Omega_\varepsilon} (\nabla \mathbf{y}_\varepsilon : \nabla \mathbf{v})\, dx &+ \int_{\Omega_\varepsilon} (\mathbf{y}_\varepsilon \cdot \nabla) \mathbf{y}_\varepsilon \cdot \mathbf{v}\, dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{v}\, dx \\ &= \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v}\, dx, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega_\varepsilon), \\ \int_{\Omega_\varepsilon} q \operatorname{div} \mathbf{y}_\varepsilon\, dx &= 0, \quad \forall\, q \in L^2_0(\Omega_\varepsilon), \\ \mathbf{y}_\varepsilon|_{\partial\Omega_\varepsilon} &= \mathbf{u}_\varepsilon|_{\partial\Omega_\varepsilon}. \end{aligned} \tag{84}$$

The scheme of the proof is rather standard and is based on the energy method, introduced by Tartar [30], and adapted later by Allaire [3] for the Navier-Stokes equations.

Theorem 8.6. *Let* $\mathbf{f}_\varepsilon \rightharpoonup \mathbf{f}$ *in* $\mathbf{L}^2(\Omega)$, *and let* $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *by a sequence of admissible triplets for the* $\mathbb{P}_\varepsilon$*-problems such that*

$$(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \overset{w}{\longrightarrow} (\mathbf{u}, \mathbf{y}, p). \tag{85}$$

Then $\mathbf{u} \in \mathbf{U}$, *and the pair* $(\mathbf{y}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ *is a solution of the following variational problem*

$$\mathbf{y} - \mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \tag{86}$$

$$\begin{aligned} &\nu \textstyle\int_\Omega (\nabla \mathbf{y} : \nabla \mathbf{v})\, dx + \frac{2\pi\nu}{C_0} \int_\Omega (\mathbf{y} - \mathbf{u})\mathbf{v}\, dx + \int_\Omega (\mathbf{y} \cdot \nabla)\mathbf{y}_\varepsilon \cdot \mathbf{v}\, dx \\ &\quad + \textstyle\int_\Omega \nabla p \cdot \mathbf{v}\, dx = \int_\Omega f \cdot \mathbf{v}\, dx, \ \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \end{aligned} \tag{87}$$

$$\textstyle\int_\Omega q \operatorname{div} \mathbf{y}\, dx = 0, \quad \forall q \in L^2_0(\Omega).$$

Remark 5. The corresponding limit boundary problem to (86)–(87) can be formally described as follows

$$-\nu \triangle \mathbf{y} + \frac{2\pi\nu}{C_0}(\mathbf{y} - \mathbf{u}) + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} \quad \text{ in } \ \Omega; \tag{88}$$

$$\operatorname{div} \mathbf{y} = 0 \quad \text{ in } \ \Omega; \tag{89}$$

$$\mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega}. \tag{90}$$

These relations correspond to the so-called Brinkman-type law that was introduced in the late 1940's in [6] as a new set of equations, intermediate between the Darcy and Stokes equations. The Brinkman's law is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity. In our case the term $\frac{2\pi\nu}{C_0}(\mathbf{y} - \mathbf{u})$ takes the role of that one, and expresses the presence of the cylindrical holes of critical size ($0 < C_0 < +\infty$) and Dirichlet controls supported on their boundaries, which disappeared passing to the limit.

Proof. For a given sequence of admissible solutions, let $\left\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of their prototypes. Here, we suppose that $\{\breve{\mathbf{y}}_\varepsilon\}$ are uniformly bounded in $\mathbf{H}^1(\Omega)$, and each of $\breve{p}_\varepsilon$ is defined as $P_\varepsilon(p_\varepsilon) \in L^2_0(\Omega)$ (see Remark 4). Due to Theorem 6.8, we may assume that the sequence $\{(\mathbf{u}_\varepsilon, \breve{\mathbf{y}}_\varepsilon, \breve{p}_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $\mathbb{X}_\varepsilon$, and hence there exists a triplet

$$(\mathbf{u}, \mathbf{y}, p) \in \left[\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{sol}(\Omega)\right] \times \mathbf{H}^1_{sol}(\Omega) \times L^2(\Omega)$$

satisfying (85).

Let $\left\{(\mathbf{w}^\varepsilon_k, q^\varepsilon_k) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)\right\}_{\varepsilon>0}$ be a sequence defined by Hypotheses (H1)–(H4). Let $\varphi \in C^\infty_0(\Omega)$ be a fixed function. It is clear that $\varphi \mathbf{w}^\varepsilon_k \in H^1_{0,sol}(\Omega)$ and $\varphi q^\varepsilon_k \in L^2_0(\Omega)$ for every $\varepsilon > 0$. Now, we consider the variational problem (84) with the following test functions $\mathbf{v} = \varphi \mathbf{w}^\varepsilon_k \in H^1_{0,sol}(\Omega_\varepsilon)$, $q = \varphi q^\varepsilon_k \in L^2_0(\Omega_\varepsilon)$. As a result, we obtain

$$\begin{aligned} &\nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla(\varphi \mathbf{w}^\varepsilon_k)\, dx + \nu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon : \nabla(\varphi \mathbf{w}^\varepsilon_k)\, dx \\ &\quad + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon \cdot \nabla)\breve{\mathbf{y}}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx + \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot \nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx \\ &\quad + \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_k)\, dx \\ &\quad - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}(\varphi \mathbf{w}^\varepsilon_k)\, dx = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon \cdot (\varphi \mathbf{w}^\varepsilon_{\mathbf{k}})\, dx\,; \end{aligned} \tag{91}$$

$$\int_{\Omega_\varepsilon} \varphi\, q_k^\varepsilon \operatorname{div} \mathbf{y}_\varepsilon \, dx = \int_{\Omega_\varepsilon} \varphi\, q_k^\varepsilon \operatorname{div}(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\, dx = 0. \tag{92}$$

Expanding (91), and using the fact that $\mathbf{w}_k^\varepsilon$ is divergence-free, we have

$$\begin{aligned}
&\nu \int_{\Omega_\varepsilon} \varphi \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla \mathbf{w}_k^\varepsilon \, dx + \nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \mathbf{w}_k^\varepsilon \nabla\varphi \, dx \\
&+\nu \int_{\Omega_\varepsilon} \varphi(\nabla \mathbf{u}_\varepsilon : \nabla \mathbf{w}_k^\varepsilon)\, dx + \nu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon : \mathbf{w}_k^\varepsilon \nabla\varphi\, dx \\
&+ \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\varphi \mathbf{w}_k^\varepsilon)\, dx + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx \\
&+ \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx \\
&- \int_{\Omega_\varepsilon} p_\varepsilon \mathbf{w}_k^\varepsilon\cdot\nabla\varphi\, dx = \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon\cdot\varphi\mathbf{w}_\mathbf{k}^\varepsilon\, dx.
\end{aligned} \tag{93}$$

Since $\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}_0^1(\Omega)$, after integration of (92) by parts, we get

$$\langle \nabla q_k^\varepsilon, \varphi\,\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} + \int_\Omega \chi_\varepsilon q_k^\varepsilon \nabla\varphi\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx = 0 \tag{94}$$

Then adding two last equations and using the fact that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \varphi\nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla\mathbf{w}_k^\varepsilon\, dx \\
&= \int_\Omega \varphi\nabla(\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)) : \nabla\mathbf{w}_k^\varepsilon\, dx \\
&= -\left\langle \Delta\mathbf{w}_k^\varepsilon, \varphi\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\right\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} - \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\nabla\varphi : \nabla\mathbf{w}_k^\varepsilon\, dx,
\end{aligned} \tag{95}$$

we obtain

$$\begin{aligned}
&\langle \nabla q_k^\varepsilon - \nu\Delta\mathbf{w}_k^\varepsilon, \varphi\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} + \nu\int_\Omega \chi_\varepsilon\nabla\mathbf{u}_\varepsilon : \mathbf{w}_k^\varepsilon\nabla\varphi\, dx \\
&+ \int_\Omega \chi_\varepsilon q_k^\varepsilon \nabla\varphi\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx - \nu\int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\nabla\varphi : \nabla\mathbf{w}_k^\varepsilon\, dx \\
&+ \nu\int_\Omega \chi_\varepsilon\nabla(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon) : \mathbf{w}_k^\varepsilon\nabla\varphi\, dx + \nu\int_\Omega \chi_\varepsilon\varphi(\nabla\mathbf{u}_\varepsilon : \nabla\mathbf{w}_k^\varepsilon)\, dx \\
&+ \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx \\
&+ \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\varphi\mathbf{w}_k^\varepsilon)\, dx \\
&- \int_\Omega \chi_\varepsilon\breve{p}_\varepsilon\mathbf{w}_k^\varepsilon\cdot\nabla\varphi\, dx = \int_\Omega \chi_\varepsilon\mathbf{f}_\varepsilon\cdot\varphi\mathbf{w}_\mathbf{k}^\varepsilon\, dx.
\end{aligned} \tag{96}$$

Now we can pass to the limit in (96) as ε tends to zero. To do so, we recall the following facts: $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$; $\nabla\mathbf{w}_j^\varepsilon$ converges pointwise and weakly in $\mathbf{L}^2(\Omega)$ to zero; $\chi_\varepsilon \to 0$ in $L^2(\Omega)$; $q_k^\varepsilon \rightharpoonup 0$ in $L_0^2(\Omega)$; $\breve{p}_\varepsilon = P_\varepsilon(p_\varepsilon) \rightharpoonup p$ in $L_0^2(\Omega)$; the sequence $\{\chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\}$ fulfills the conditions of Hypothesis (H4); the nonlinear term $((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)$ converges strongly to $((\mathbf{y}-\mathbf{u})\cdot\nabla)(\mathbf{y}-\mathbf{u})$ in $\mathbf{H}^{-1}(\Omega)$;

condition (85) implies $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $\mathbf{H}^1(\Omega)$. As a result, we obtain

$$\langle \nabla q_k^\varepsilon - \nu \Delta \mathbf{w}_k^\varepsilon, \varphi \chi_\varepsilon (\check{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} \to \nu \langle \mu_k, \varphi(\mathbf{y}-\mathbf{u})\rangle_{\mathbf{H}^{-1}(\Omega);\mathbf{H}_0^1(\Omega)} \overset{\text{by Lemma 8.5}}{=} \frac{2\pi\nu}{C_0}\int_\Omega \varphi\, \mathbf{e}_k \cdot (\mathbf{y}-\mathbf{u})\, dx,$$

and hence, using Rellich's Theorem, from (96) we get

$$\begin{aligned} &\frac{2\pi\nu}{C_0}\int_\Omega \varphi\, \mathbf{e}_k \cdot (\mathbf{y}-\mathbf{u})\, dx + \nu \int_\Omega \nabla \mathbf{y} : \mathbf{e}_k \nabla \varphi\, dx + \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{y}\cdot(\varphi \mathbf{e}_k)\, dx \\ &\quad - \int_\Omega p\, \mathbf{e}_k \cdot \nabla\varphi\, dx = \int_\Omega \mathbf{f}\cdot \varphi \mathbf{e_k}\, dx, \quad \forall \varphi \in C_0^\infty(\Omega) \text{ and each } k = 1,2,3. \end{aligned} \tag{97}$$

Integrating the term $\int_\Omega p\, \mathbf{e}_k \cdot \nabla\varphi\, dx$ by parts and regrouping (97) and (49), we deduce that the limit triplet $(\mathbf{u}, \mathbf{y}, p)$ must satisfy the following relations:

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{y} : \nabla \Phi\, dx + \frac{2\pi\nu}{C_0}\int_\Omega (\mathbf{y}-\mathbf{u})\cdot \Phi\, dx + \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{y}\cdot\Phi\, dx \\ + \int_\Omega \nabla p \cdot \Phi\, dx = \int_\Omega \mathbf{f}\cdot\Phi\, dx, \quad \forall \Phi \in \mathbf{C}_0^\infty(\Omega); \end{aligned} \tag{98}$$

$$\mathbf{y}-\mathbf{u} \in \mathbf{H}^1_{0,sol}(\Omega), \quad \mathbf{u} \in \mathbf{U}. \tag{99}$$

The proof is complete. □

9. **Identification of the limit optimal control problem.** In this section we show that for the sequence of constrained minimization problems (55), there exists a weak variational limit with respect to the w-convergence, and it can be recovered in an explicit form. We begin with the following result:

Lemma 9.1. *Let $\{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of admissible solutions, assumed to be w-convergent to a triplet $(\mathbf{u}, \mathbf{y}, p) \in \left[\mathbf{H}^2(\Omega)\cap \mathbf{H}^1_{sol}(\Omega)\right] \times \mathbf{H}^1_{sol} \times L^2_0(\Omega)$. Then*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla \mathbf{y}_\varepsilon|^2\, dx = \int_\Omega |\nabla \mathbf{y}|^2\, dx + \frac{2\pi}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2\, dx. \tag{100}$$

Proof. We first observe that

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla \mathbf{y}_\varepsilon|^2\, dx = \int_\Omega \chi_\varepsilon |\nabla \check{\mathbf{y}}_\varepsilon - \nabla \mathbf{u}_\varepsilon|^2\, dx \\ + 2\int_\Omega \chi_\varepsilon (\nabla \check{\mathbf{y}}_\varepsilon : \nabla \mathbf{u}_\varepsilon)\, dx - \int_\Omega \chi_\varepsilon |\nabla \mathbf{u}_\varepsilon|^2\, dx. \end{aligned} \tag{101}$$

Then, taking into account the facts that $\nabla \mathbf{u}_\varepsilon \to \nabla \mathbf{u}$ in $\mathbf{L}^2(\Omega)$, $\check{\mathbf{y}}_\varepsilon \rightharpoonup \mathbf{y}$ in $\mathbf{H}^1(\Omega)$, and $\chi_\varepsilon \to 1$ in $L^2(\Omega)$, we have

$$\int_\Omega \chi_\varepsilon (\nabla \check{\mathbf{y}}_\varepsilon : \nabla \mathbf{u}_\varepsilon)\, dx \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \nabla \mathbf{y} : \nabla \mathbf{u}\, dx, \tag{102}$$

$$\int_\Omega \chi_\varepsilon |\nabla \mathbf{u}_\varepsilon|^2\, dx \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla \mathbf{u}|^2\, dx. \tag{103}$$

Since $(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) \in \mathbf{H}^1_{0,sol}(\Omega_\varepsilon)$ for every $\varepsilon > 0$, it follows that we can take $\mathbf{v} = \mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon$ as a test function in (84). Then the following equality is ensured:

$$\nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\, dx = -\nu \int_\Omega \chi_\varepsilon \nabla \mathbf{u}_\varepsilon : \nabla(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx$$
$$- \int_\Omega \chi_\varepsilon((\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla)(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx - \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\breve{\mathbf{y}}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx$$
$$- \int_\Omega \chi_\varepsilon(\breve{\mathbf{y}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx + \int_\Omega \chi_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx$$
$$+ \int_\Omega \chi_\varepsilon \breve{p}_\varepsilon \operatorname{div}(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx + \int_\Omega \chi_\varepsilon \mathbf{f}_\varepsilon\cdot(\breve{\mathbf{y}}_\varepsilon - \mathbf{u}_\varepsilon)\, dx\,.$$

By arguments of the previous theorem, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \nu \int_{\Omega_\varepsilon} \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon) : \nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)\, dx &= -\nu \int_\Omega \nabla\mathbf{u} : \nabla(\mathbf{y} - \mathbf{u})\, dx \\ &- \int_\Omega ((\mathbf{y}-\mathbf{u})\cdot\nabla)(\mathbf{y}-\mathbf{u})\cdot(\mathbf{y}-\mathbf{u})\, dx - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{y}\cdot(\mathbf{y}-\mathbf{u})\, dx \\ &- \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y}-\mathbf{u})\, dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y}-\mathbf{u})\, dx \\ &+ \int_\Omega p \operatorname{div}(\mathbf{y}-\mathbf{u})\, dx + \int_\Omega \mathbf{f}\cdot(\mathbf{y}-\mathbf{u})\, dx\,. \end{aligned} \tag{104}$$

We now consider the integral identity (97) with the test function $\mathbf{y} - \mathbf{u}$. By a rearrangement, we have

$$\begin{aligned}&\frac{2\pi\nu}{C_0}\int_\Omega \varphi\, |\mathbf{y}-\mathbf{u}|^2\, dx + \nu\int_\Omega |\nabla(\mathbf{y}-\mathbf{u})|^2\, dx + \nu\int_\Omega \nabla\mathbf{u} : \nabla(\mathbf{y}-\mathbf{u})\, dx \\ &+ \int_\Omega ((\mathbf{y}-\mathbf{u})\cdot\nabla)(\mathbf{y}-\mathbf{u})\cdot(\mathbf{y}-\mathbf{u})\, dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{y}\cdot(\mathbf{y}-\mathbf{u})\, dx \\ &+ \int_\Omega (\mathbf{y}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y}-\mathbf{u})\, dx - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{y}-\mathbf{u})\, dx \\ =& \int_\Omega p\operatorname{div}(\mathbf{y}-\mathbf{u})\, dx + \int_\Omega \mathbf{f}\cdot(\mathbf{y}-\mathbf{u})\, dx\,.\end{aligned} \tag{105}$$

The comparison of (104) with (105) leads to the following equality:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(\mathbf{y}_\varepsilon - \mathbf{u}_\varepsilon)|^2\, dx = \int_\Omega |\nabla(\mathbf{y}-\mathbf{u})|^2\, dx + \frac{2\pi}{C_0}\int_\Omega \varphi\, |\mathbf{y}-\mathbf{u}|^2\, dx$$

which, together with (101)–(103), concludes the proof. □

Remark 6. Using this approach, it can be proved a more general result: under the supposition of Lemma 9.1, the following relation is valid

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \nabla\mathbf{y}_\varepsilon : (\nabla\mathbf{y}_\varepsilon)^t\, dx = \int_\Omega \nabla\mathbf{y} : (\nabla\mathbf{y})^t\, dx + \frac{2\pi}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2\, dx. \tag{106}$$

We are now able to establish the identification result of the variational limit for the sequence of constrained minimization problems (55).

Theorem 9.2. *For the sequence* (55) *there exists a unique variational w-limit which can be represented in the form* (56), *where the cost functional* $\mathcal{J}_0$ *and the set of* *admissible solutions* Ξ_0 *are defined as follows:*

$$\mathcal{J}_0(\mathbf{u},\mathbf{y}) = \lambda \int_\Omega |\nabla \mathbf{y}|^2\, dx + \frac{2\pi\lambda}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2\, dx + \beta|\partial Q|_H \int_\Omega |\mathbf{u}|^2\, dx; \qquad (107)$$

$$\Xi_0 = \left\{ (\mathbf{u},\mathbf{y},p) \;\middle|\; \begin{array}{l} p\in L^2_0(\Omega),\ \ \mathbf{u}\in \mathbf{H}^2(\Omega),\ \ \mathbf{y}-\mathbf{u}\in \mathbf{H}^1_{0,sol}(\Omega),\\ \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le \gamma, \quad \mathbf{u}|_{\Gamma_3}=0,\\ -\nu\triangle \mathbf{y} + \frac{2\pi\nu}{C_0}(\mathbf{y}-\mathbf{u}) + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} \quad in\ \ \Omega;\\ \operatorname{div}\mathbf{y}=0 \quad in\ \ \Omega;\\ \mathbf{y}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} \end{array}\right\} \qquad (108)$$

Proof. The proof of this theorem is divided into two steps, each of them concerns with the verification of the corresponding item of Definition 7.2.

STEP 1: *Statement (1) of Definition 7.2 is valid.*

Let $\{(\overline{\alpha}_\varepsilon,\mathbf{y}_\varepsilon,p_\varepsilon)\}_{\varepsilon>0}$ be a bounded sequence of admissible triplets which is w-convergent to a triplet $(\mathbf{u},\mathbf{y},p)$. Let $\{\varepsilon_k\}$ be a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k\to 0$ as $k\to\infty$ and $(\overline{\alpha}_k,\mathbf{y}_k,p_k)\in \Xi_{\varepsilon_k}$ for all $k\in\mathbb{N}$. Then, due to Theorem 8.6, we have that the w-limit triplet $(\mathbf{u},\mathbf{y},p)$ satisfies relations (98)–(99), and moreover

$$\gamma \ge \liminf_{k\to\infty} \|\mathbf{u}_k\|_{\mathbf{H}^2(\Omega)} \ge \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}$$

by the lower semicontinuity of $\|\cdot\|_{H^2(\Omega)}$ with respect to the weak convergence in $\mathbf{H}^2(\Omega)$. So, inclusion (57) holds true. The fulfilment of inequality (57) immediately follows from Lemma 9.1 and the property of lower semicontinuity of the weak convergence in variable spaces (see (44))

$$\lim_{k\to\infty}\left[\beta|\partial Q|_H \int_\Omega |\mathbf{u}_k|^2\, d\eta^{r(\varepsilon_k)}_{\varepsilon_k}\right] \ge \beta|\partial Q|_H\int_\Omega |\mathbf{u}|^2\, dx.$$

STEP 2: *Statement (2) of Definition 7.2 holds true.*

Let $(\mathbf{u},\mathbf{y},p)\in\Xi_0$ be an admissible triplet for the minimization problem (56), (107), (108). In view of Hypothesis (H5) we may suppose that there exist operators $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega)\mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ such that

$$\text{if}\ \ \mathbf{y}\in \mathbf{H}^1_{sol}(\Omega_\varepsilon),\ \ \text{then}\ \ \Lambda_\varepsilon(\check{\mathbf{y}}) = \mathbf{y}\ \ \text{in}\ \ \Omega_\varepsilon, \qquad (109)$$

$$\|\Lambda_\varepsilon(\mathbf{u})\|_{\mathbf{H}^1(\Omega_\varepsilon)} \le C\|\mathbf{u}\|^1_{\mathbf{H}}(\Omega),\ \ \text{where}\ \ C>0\ \ \text{is independent of}\ \ \varepsilon. \qquad (110)$$

Hence, for a given $\mathbf{u}\in\mathbf{U}$ we have $\nabla\cdot\Lambda_\varepsilon(\mathbf{u})=0$, and $\int_{\partial\Omega_\varepsilon}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\, d\mathcal{H}^2=0$. As follows from the structure of the set of admissible controls $\mathbf{U}$,

$$\int_{\partial\Omega_\varepsilon}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\, d\mathcal{H}^2 = 0 \quad\Leftrightarrow\quad \int_{\partial T^{\mathbf{k}_j}_\varepsilon}\Lambda_\varepsilon(\mathbf{u})\cdot\mathbf{n}\, d\mathcal{H}^2 = 0,\quad \forall\, j=1,\dots,J_\varepsilon. \qquad (111)$$

Let us define the sequence

$$\left\{\overline{\alpha}^*_\varepsilon = \left(\alpha^*_{\mathbf{k}_1},\alpha^*_{\mathbf{k}_2},...,\alpha^*_{\mathbf{k}_{J_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u})|_{\partial T_\varepsilon}\right\}_{\varepsilon>0}. \qquad (112)$$

Here we suppose that the sequence $\{\mathbf{w}_\varepsilon = \Lambda_\varepsilon(\mathbf{u})\}$ can be extended on the whole domain Ω with uniformly bounded norms in $\mathbf{H}^1(\Omega)$ (see Remark 3). It is easy to see that each of the functions $\{\overline{\alpha}^*_\varepsilon\}$ satisfy the conditions (53), and $\Lambda_\varepsilon(\mathbf{u})\in\mathbf{L}^2(\Omega,d\eta^{r_\varepsilon}_\varepsilon)$ (see Remark 1). Moreover, in view of the estimates $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\le\gamma$ and (110), the sequence $\{\Lambda_\varepsilon(\mathbf{u})\}$ is uniformly bounded in the variable space $\mathbf{L}^2(\Omega,d\eta^{r_\varepsilon}_\varepsilon)$. Hence, we may suppose the existence of a function $\mathbf{u}^*\in L^2(\Omega)$ such that

$$\{\Lambda_\varepsilon(\mathbf{u})\}\to\mathbf{u}^*\ \ \text{weakly in}\ \ \mathbf{L}^2(\Omega,d\eta^{r_\varepsilon}_\varepsilon). \qquad (113)$$

On the other hand, thanks to Lemmas 6.4–6.6, we have

$$\mathbf{u} \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r_\varepsilon}), \quad \forall\, \varepsilon > 0, \tag{114}$$

$$\mathbf{u} \rightharpoonup \mathbf{u} \ \text{ in } \ \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r_\varepsilon}) \text{ as } \ \varepsilon \to 0, \quad \lim_{\varepsilon\to 0} \int_\Omega |\mathbf{u}|^2 \, d\eta_\varepsilon^{r_\varepsilon} = \int_\Omega |\mathbf{u}|^2 \, dx, \tag{115}$$

and hence, $\mathbf{u}|_{\partial T_\varepsilon} \xrightarrow{w_a} \mathbf{u}$. Let us show that $\mathbf{u} = \mathbf{u}^*$.

To do so, we note that $\mathbf{u} \in \mathbf{C}(\Omega)$ by the imbedding results for Sobolev spaces. Hence, for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\left| \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \mathbf{u} \cdot \mathbf{n} \, d\mathcal{H}^2 \right| < \delta, \qquad j = 1, \dots, J_\varepsilon \text{ and } \ \forall\, \varepsilon < \varepsilon_0. \tag{116}$$

Let us partition the set Ω into cubes εY with edges ε and denote these cubes with εY_i. Then, combining (111) with (116), we obtain that the following estimate holds true $\forall\, \varepsilon < \varepsilon_0$

$$\left| \int_{\partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i} (\mathbf{u} - \Lambda_\varepsilon(\mathbf{u})) \cdot \mathbf{n} \, d\mathcal{H}^2 \right| < \delta, \qquad \forall\, j = 1, \dots, J_\varepsilon\,, \ \forall\, i \ : \ \partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i \neq \emptyset.$$

Hence, for every fixed indices (i, j) there exist points $x^i \in \partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i$ such that

$$\left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \Lambda_\varepsilon(\mathbf{u}) \, d\eta_\varepsilon^{r_\varepsilon} \right| = \left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\partial T_\varepsilon^{\mathbf{k}_j} \cap \varepsilon Y_i} \Lambda_\varepsilon(\mathbf{u}) \, d\mathcal{H}^2 \right| < \delta. \tag{117}$$

We are able to estimate the difference $\left| \int_\Omega \varphi\, \mathbf{u}\, dx - \int_\Omega \varphi\, \mathbf{u}^*\, dx \right|$, where $\varphi \in C_0(\Omega)$. As a result, we have

$$\begin{aligned} &\left| \int_\Omega \varphi\, \mathbf{u}\, dx - \int_\Omega \varphi\, \mathbf{u}^*\, dx \right| \le \left| \int_\Omega \varphi\, \mathbf{u}\, dx - \int_\Omega \varphi\, \mathbf{u}\, d\eta_\varepsilon^{r_\varepsilon} \right| \\ &+ \left| \int_\Omega \varphi\, \mathbf{u}\, d\eta_\varepsilon^{r_\varepsilon} - \int_\Omega \varphi\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right| + \left| \int_\Omega \varphi\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} - \int_\Omega \varphi\, \mathbf{u}^*\, dx \right| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Taking into account the weak convergence of the sequences $\left\{\mathbf{u} \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r_\varepsilon})\right\}$ and $\left\{\Lambda_\varepsilon(\mathbf{u}) \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r_\varepsilon})\right\}$ (see (113), (115)), we observe that $I_1 \to 0$ and $I_3 \to 0$ as ε tends to zero. Now we show that $I_2 \to 0$. Since $\mathbf{u} \in \mathbf{C}(\Omega)$, it follows that there exists a constant C^* such that

$$\begin{aligned} I_2 &\le \sum_i \left| \int_{\varepsilon Y_i} \varphi\, \mathbf{u}\, d\eta_\varepsilon^{r_\varepsilon} - \int_{\varepsilon Y_i} \varphi\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right| \\ &\le C^* \varepsilon^3 \sum_i \left| \varphi(x^i)\, \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \varphi\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right| \\ &\le C^* \varepsilon^3 \|\varphi\|_{C(\Omega)} \sum_i \left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \frac{\varphi(x)}{\varphi(x^i)}\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right|. \end{aligned}$$

Let us suppose the converse, that is,

$$\lim_{\varepsilon\to 0} \varepsilon^3 \sum_i \left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \frac{\varphi(x)}{\varphi(x^i)}\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right| > 0,$$

then there exist a constant D^* and a value $\varepsilon^* > 0$ such that

$$\left| \mathbf{u}(x^i) - \frac{1}{\varepsilon^3} \int_{\varepsilon Y_i} \frac{\varphi(x)}{\varphi(x^i)}\, \Lambda_\varepsilon(\mathbf{u})\, d\eta_\varepsilon^{r_\varepsilon} \right| \ge D^*, \quad \varepsilon < \varepsilon^*.$$

Assuming that $\varphi(x)/\varphi(x^i) = 1$, we just come into conflict with (117). So, our supposition was wrong and we get $\lim_{\varepsilon\to 0} I_2 = 0$. Thus, $\mathbf{u} = \mathbf{u}^*$. As a result, we have (see Lemma 6.4)

$$\overline{\alpha}_\varepsilon^* = \left(\alpha^*_{\mathbf{k}_1}, \alpha^*_{\mathbf{k}_2}, ..., \alpha^*_{\mathbf{k}_{J_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u})|_{\partial T_\varepsilon} \xrightarrow{w_a} \mathbf{u}. \tag{118}$$

Using the fact that the weak limit of $\Lambda_\varepsilon(\mathbf{u})$ in $\mathbf{H}^1(\Omega)$ coincides with its weak limit in the variable space $\mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)})$ (see [10]), and closely following the previous arguments, it can be proved that

$$\lim_{\varepsilon\to 0}\int_\Omega |\Lambda_\varepsilon(\mathbf{u})|^2\, d\eta_\varepsilon^{r_\varepsilon} = \int_\Omega |\mathbf{u}^*|^2\, dx. \tag{119}$$

Let us consider the sequence of triplets $\{(\overline{\alpha}_\varepsilon^*, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}\,) \in \Xi_\varepsilon\}_{\varepsilon>0}$, where $(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}\,) = (\widehat{\mathbf{y}}_\varepsilon(\overline{\alpha}_\varepsilon^*), \widehat{p}_\varepsilon(\overline{\alpha}_\varepsilon^*))$ are the corresponding solutions of the boundary value problem (9)–(12). Due to Theorem 6.8, this sequence is relatively compact with respect to the w-convergence. Hence, taking into account Lemma 9.1 and property (119), we deduce: $(\overline{\alpha}_\varepsilon^*, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}) \xrightarrow{\varepsilon\to 0} (\mathbf{u}, \widehat{\mathbf{y}}, \widehat{p}\,)$, where $(\mathbf{u}, \widehat{\mathbf{y}}, \widehat{p}\,)$ and $(\mathbf{u}, \mathbf{y}, p\,)$ belong to the same class of equivalence (here we suppose that the problem (56) may have more than one solution). Thus, to end the proof, it remains only to apply Lemma 9.1 and property (119). □

Remark 7. It is now clear that the structure of the limit problem (56) essentially depends on the parameter C_0. If $C_0 = +\infty$ (that corresponds to the case when the cylinders have smaller cross-size), then the limit minimization problem (56), (107), (108) takes the form of the calculus variation problem (18)–(19). If $0 < C_0 < +\infty$ (that corresponds to the case when the cylinders are of critical cross-size), then the limit minimization problem (56), (107), (108) can be recovered in the form of the optimal control problem (20)–(24).

Remark 8. Following the above approach, we can obtain the explicit mathematical description for the variational w-limit problem (56) in the case when the cost functional to the $\mathbb{P}_\varepsilon$-problem takes the form

$$\mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) = \lambda \int_{\Omega_\varepsilon} |\mathrm{curl}\, \mathbf{y}_\varepsilon|^2\, dx + \frac{\beta\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon}\ \int\limits_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\alpha_{\mathbf{k}_j}\right|^2 d\mathcal{H}^2. \tag{120}$$

Indeed, to do this, we have to use relation (17) and to apply identity (106). As a result, we obtain the following structure for the limit cost functional

$$\mathcal{J}_0(\mathbf{u}, \mathbf{y}) = \lambda \int_\Omega |\mathrm{curl}\, \mathbf{y}|^2\, dx + \beta|\partial Q|_H \int_\Omega |\mathbf{u}|^2\, dx,$$

in which the relaxed term $+\frac{2\pi\lambda}{C_0}\int_\Omega |\mathbf{y}-\mathbf{u}|^2\, dx$ does not appear.

To conclude this section, we combine the results of Theorem 7.3 and Theorem 9.2, and come to the following conclusion concerning the variational properties of the limit problem (56), (107), (108):

Theorem 9.3. *Let* $\left\{(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be the optimal solutions of the problems* $(\mathbb{P}_\varepsilon)$. *Then each w-cluster triplet* $(\mathbf{u}^0, \mathbf{y}^0, p^0)$ *of this sequence satisfies conditions*

$$(\mathbf{u}^0, \mathbf{y}^0, p^0) \in \Xi_0, \qquad \inf_{(\mathbf{u},\mathbf{y},p)\in\,\Xi_0} \mathcal{J}_0(\mathbf{u}, \mathbf{y}) = \mathcal{J}_0\left(\mathbf{u}^0, \mathbf{y}^0\right) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon).$$

10. **Suboptimal controls and their approximation properties.** In this section we deal with the main question of our paper, namely, the construction of suboptimal controls to the original optimal boundary control problem (8)–(12). The main result can be formulated as follows:

Theorem 10.1. *Let $\Lambda_\varepsilon : \mathbf{H}^1_{sol}(\Omega) \mapsto \mathbf{H}^1_{sol}(\Omega_\varepsilon)$ be the linear continuous map such that properties (109)–(110) hold true, and let $(\mathbf{u}^0, \mathbf{y}^0, p^0) \in \Xi_0$ be an optimal solution to the limit problem (56). Then the function*

$$\overline{\alpha}_\varepsilon^{sub} = \left(\alpha^{sub}_{\mathbf{k}_1}, \alpha^{sub}_{\mathbf{k}_2}, ..., \alpha^{sub}_{\mathbf{k}_{J_\varepsilon}}\right) = \Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon} \tag{121}$$

is an asymptotically suboptimal control for the original problem $(\mathbb{P}_\varepsilon)$ in the sense of Definition 7.1.

Proof. To begin with, we note that, following the proof of Theorem 9.2, we can immediately establish the following approximation property: if $(\mathbf{u}^0, \mathbf{y}^0, p^0)$ is an optimal solution to the limit minimization problem (56), then

$$\Lambda_\varepsilon(\mathbf{u}^0) \in \mathbf{L}^2(\Omega, d\eta_\varepsilon^{r(\varepsilon)}), \quad \forall\, \varepsilon > 0, \tag{122}$$

$$\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon} \overset{w_a}{\longrightarrow} \mathbf{u}^0 \text{ as } \varepsilon \to 0,\ \lim_{\varepsilon\to 0} \int_\Omega \left|\Lambda_\varepsilon(\mathbf{u}^0)\right|^2 d\eta_\varepsilon^{r(\varepsilon)} = \int_\Omega \left|\mathbf{u}^0\right|^2 dx. \tag{123}$$

Let us consider now the sequence of triplets $\left\{(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$, where $(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon) = \left(\widehat{\mathbf{y}}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}), \widehat{p}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon})\right)$ are the corresponding solutions of the boundary value problem (9)–(12). Then, following the motivation given in the proof of Theorem 9.2, we conclude that this sequence is relatively compact with respect to the w-convergence, and, extracting, if necessary, a subsequence, satisfies condition

$$(\widehat{\mathbf{y}}_\varepsilon, \widehat{p}_\varepsilon) = \left(\widehat{\mathbf{y}}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}), \widehat{p}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon})\right) \overset{\varepsilon\to 0}{\longrightarrow} (\mathbf{u}^0, \widehat{\mathbf{y}}^0, \widehat{p}^0),$$

where $(\mathbf{u}^0, \widehat{\mathbf{y}}^0, \widehat{p}^0)$ and $(\mathbf{u}^0, \mathbf{y}^0, p^0)$ belong to the same class of equivalence. Consequently, $\mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0) = \mathcal{J}_0(\mathbf{u}^0, \widehat{\mathbf{y}}^0)$.

Let $\left\{(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon, p^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be the optimal solutions to the $\mathbb{P}_\varepsilon$-problems. We observe that

$$\begin{aligned}
&\left|\inf_{(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon, p_\varepsilon)\in\Xi_\varepsilon} \mathcal{J}_\varepsilon(\overline{\alpha}_\varepsilon, \mathbf{y}_\varepsilon) - \mathcal{J}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon)\right| \\
=&\left|\mathcal{J}_\varepsilon(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon) - \mathcal{J}_\varepsilon(\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon}, \widehat{\mathbf{y}}_\varepsilon)\right| \le \left|\mathcal{J}_\varepsilon(\overline{\alpha}^0_\varepsilon, \mathbf{y}^0_\varepsilon) - \mathcal{J}_0(\mathbf{u}^0, \mathbf{y}^0)\right| \\
&+ \lambda\left|\int_\Omega |\nabla \widehat{\mathbf{y}}^0|^2\, dx + \frac{2\pi}{C_0}\int_\Omega \left|\widehat{\mathbf{y}}^0 - \mathbf{u}^0\right|^2 dx - \int_{\Omega_\varepsilon} |\nabla\, \widehat{\mathbf{y}}_\varepsilon|^2\, dx\right| \\
&+ \beta\left| |\partial Q|_H \int_\Omega \left|\mathbf{u}^0\right|^2 dx - \frac{\varepsilon}{r_\varepsilon}\sum_{j=1}^{J_\varepsilon} \int_{\partial T_\varepsilon^{\mathbf{k}_j}} \left|\Lambda_\varepsilon(\mathbf{u}^0)\big|_{\partial T_\varepsilon^{\mathbf{k}_j}}\right|^2 d\mathcal{H}^2\right| \\
=&J_1 + J_2 + J_3.
\end{aligned}$$

To conclude the proof, we note that for a given $\delta > 0$ one can always find: (1) $\varepsilon_1 > 0$ such that $J_1 < \delta/3$ for all $\varepsilon < \varepsilon_1$ by Theorem 7.3; (2) $\varepsilon_2 > 0$ such that $J_2 < \delta/3$ for all $\varepsilon < \varepsilon_2$ by the Lemma 9.1; (3) $\varepsilon_3 > 0$ such that $J_3 < \delta/3$ for all $\varepsilon < \varepsilon_3$ by property (123). Thus, as expected, the estimate of sub-optimality (54) is valid for all $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. ∎

Remark 9. It is worth noticing that for the case when the cylinders $T_\varepsilon^{\mathbf{k}_j}$ have smaller cross-size ($C_0 = +\infty$), the function $\mathbf{u}^0$ and the corresponding suboptimal controls are the same for both problem (8)–(12) and problem (120), (9)–(12).

11. **Conclusion.** To emphasize the contribution of this paper we would like to point out a possible physical application of the above obtained results. It concerns with the control of the flow of an incompressible viscous fluid through a porous medium under the action of an external electric field. In 1802, F. F. Reuss [27] observed, experimentally, the following phenomenon: when an electric field is applied on the boundary of a porous medium in equilibrium, a motion of the fluid appears. Since this motion is a consequence of the electrical field only, it is a good reason to consider this influence as a control action in order to obtain the desired properties of the flow.

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